Original Research Article

A New Approach to Bicomplex Jacobsthal Matrix and Bicomplex Jacobsthal-Lucas Matrix Components

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ABSTRACT

In this present study we define Bicomplex Jacobsthal matrix and Bicomplex Jacobsthal-Lucas matrix. We derive the relations between these matrices. Then, using this matrix representation, we give some identities.

Keywords: Bicomplex Jacobsthal matrix; Jacobsthal matrix; Jacobsthal numbers; Jacobsthal-Lucas numbers.

1 Introduction

Bicomplex numbers are introduced in a book based on multicomplex spaces and function [1] and are given from a number of different points of view of analysis. A bicomplex number is defined by

$$X = a + bi_1 + c\,i_2 + d\,i_3$$

where *a,b,c,d* are real numbers and +1, i_1, i_2, i_3 are units given by rules $i_1^2 = i_2^2 = -1$and, $i_1 i_2 = i_2 i_1 = i_3$. Let X and X' be bicomplex numbers. The addition and subtraction of X and X' are given by

$$X \mp X' = (a \mp a') + (b \mp b')i_1 + (c \mp c')\,i_2 + (d \mp d')i_3$$

and multiplication of these numbers as follows

$$\begin{aligned} XX' &= (a + b\,i_1 + ci_2 + d\,i_3)(a' + b'i_1 + c'i_2 + d'i_3) \\ &= (aa' - bb' + cc' - dd') + (ab' + ba' + cd' + dc')\,i_1 \\ &\quad + (ac' + ca' - bd' - db')\,i_2 + (ad' + da' + cb' + bc')\,i_3. \end{aligned}$$

The conjugates of the bicomplex number X are described by X^{i_1}, X^{i_2} and X^{i_3}. In that case, there are different conjugations as follows, [1]:

$$\begin{aligned} X^{i_1} &= a - bi_1 + ci_2 - di_3, \\ X^{i_2} &= a + bi_1 - ci_2 - di_3, \\ X^{i_3} &= a - bi_1 - ci_2 + di_3. \end{aligned} \qquad (1)$$

The Jacobsthal numbers J_n are defined for all integers $n \geq 0$ by the second order recurrence relation

$$J_{n+2} = J_{n+1} + 2J_n$$

and *initial conditions* $J_0 = 0$, $J_1 = 1$. The Jacobsthal-Lucas numbers j_n are defined for all integers $n \geq 0$ by the same second order recurrence relation as

$$j_{n+2} = j_{n+1} + 2j_n$$

but *initial conditions* $j_0 = 2$ and $j_1 = 1$.

The next Jacobsthal number is also given by the recursion formula:

$$J_{n+1} - 2J_n = (-1)^n \text{ by } J_{n+1} = 2^n - J_n.$$

The first recursion formula above is also satisfied by *the powers of* 2. The Jacobsthal number at a specific point in the sequence may be calculated directly using *the closed-form equation*:

$$J_n = \frac{2^n - (-1)^n}{3}. \quad (2)$$

The Jacobsthal numbers can be extended to *negative indices* using the recurrence relation or *the explicit formula*, giving

$$J_{-n} = \frac{(-1)^{n+1} J_n}{2}.$$

The following identity holds

$$2^n (J_{-n} + J_n) = 3J_n^2.$$

The following Jacobsthal-Lucas number also satisfies:

$$j_{n+1} - 2J_n = -3(-1)^n.$$

The Jacobsthal-Lucas number at a specific point in the sequence may be calculated directly using *the closed-form* equation:

$$j_n = 2^n + (-1)^n. \quad (3)$$

Different applications of Jacobsthal and Jacobsthal-Lucas numbers and their generalization have been studied by many mathematicians in almost all fields of science [2-10]. Jacobsthal $\mathcal{F}$-matrix and Jacobsthal Lucas $\mathcal{F}'$-matrix are presented in [11-14].

The n^{th}-power of the $\mathcal{F}$-matrix and $\mathcal{F}'$-matrix are

$$\mathcal{F}^n = \mathcal{F}_n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} \text{ and } \mathcal{F}_n' = \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix} \quad (4)$$

where

$$\mathcal{F}^{n+1}\mathcal{F}^n = \mathcal{F}^{2n+1} = \mathcal{F}_{n+1}\mathcal{F}_n = \mathcal{F}_{2n+1} \text{ and also } \det(\mathcal{F}^n) = (-1)^n(-2)^n. \quad (5)$$

In this paper, we define the bicomplex Jacobsthal matrix and the bicomplex Jacobsthal-Lucas matrix by combining Jacobsthal numbers, Jacobsthal-Lucas numbers and bicomplex numbers. We present a matrix representation using some properties of these numbers.

A bicomplex Jacobsthal numbers and bicomplex Jacobsthal-Lucas numbers, respectively, are defined

$$X_n = J_n + J_{n+1}\, i_1 + J_{n+2}\, i_2 + J_{n+3}\, i_3 \quad \text{and} \quad X'_n = j_n + j_{n+1}\, i_1 + j_{n+2}\, i_2 + j_{n+3}\, i_3, \tag{6}$$

where J_n and $\mathrm{j_n}$ are the n^{th} Jacobsthal, Jacobsthal-Lucas numbers and $i_1^2 = i_2^2 = -1$, $i_3^2 = +1$. If we start from $n = 0$, the bicomplex Jacobsthal number and bicomplex Jacobsthal-Lucas number can be written, respectively, as;

$$X_0 = 1i_1 + 1i_2 + 3i_3;\ X_1 = 1 + 1i_1 + 3i_2 + 5i_3$$

and

$$X'_0 = 2 + 1i_1 + 5i_2 + 7i_3;\ \ X'_1 = 1 + 5i_1 + 7i_2 + 17i_3.$$

2 Bicomplex Jacobsthal matrix and Bicomplex Jacobsthal-Lucas matrix

For $n \geq 0$, the n^{th} bicomplex Jacobshtal matrix $\mathcal{B}_n$ and the n^{th} bicomplex Jacobsthal-Lucas matrix $\mathcal{B}'_n$ are de ined as

$$\mathcal{B}_n = \mathcal{F}_n + \mathcal{F}_{n+1}\, i_1 + \mathcal{F}_{n+2}\, i_2 + \mathcal{F}_{n+3}\, i_3 \text{ and } \mathcal{B}'_n = \mathcal{F}'_n + \mathcal{F}'_{n+1} i_1 + \mathcal{F}'_{n+2}\, i_2 + \mathcal{F}'_{n+3}\, i_3 \tag{7}$$

where i_1, i_2 and i_3 are *arbitrary units* which satisfy the relations;

$$i_1^2 = i_2^2 = -1,\ i_3^2 = +1. \tag{8}$$

Starting from $n = 1$, the bicomplex Jacobsthal matrix can be written as;

$$\mathcal{B}_1 = \mathcal{F}_1 + \mathcal{F}_2 i_1 + \mathcal{F}_3\, i_2 + \mathcal{F}_4\, i_3$$

$$= \begin{bmatrix} J_2 & 2J_1 \\ J_1 & 2J_0 \end{bmatrix} + \begin{bmatrix} J_3 & 2J_2 \\ J_2 & 2J_1 \end{bmatrix} i_1 + \begin{bmatrix} J_4 & 2J_3 \\ J_3 & 2J_2 \end{bmatrix} i_2 + \begin{bmatrix} J_5 & 2J_4 \\ J_4 & 2J_3 \end{bmatrix} i_3$$

$$= \begin{bmatrix} 1 + 3i_1 + 5i_2 + 11i_3 & 2 + 2i_1 + 6i_2 + 10i_3 \\ 1 + 1i_1 + 3i_2 + 5i_3 & 2i_1 + 2i_2 + 6i_3 \end{bmatrix}$$

$$= \begin{bmatrix} X_2 & 2X_1 \\ X_1 & 2X_0 \end{bmatrix},$$

where X_0, X_1 and X_2 are the bicomplex Jacobsthal numbers.

3 Some identities on Bicomplex Jacobsthal Matrix

Identity 3.1. Let $n \geq 1$ be integer. Then, from the equality (1), we can give the following relation between bicomplex Jacobsthal matrices

$$\mathcal{B}_n - \mathcal{B}_{n+1} i_1 - \mathcal{B}_{n+2}\ i_2 + \mathcal{B}_{n+3} i_3 = 45\left(\mathcal{F}_{n-1} + 2(-1)^n\, \mathcal{F}'_{n-1} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right).$$

Proof. We will give the proof of identity

$$\mathcal{B}_n - \mathcal{B}_{n+1} i_1 - \mathcal{B}_{n+2}\ i_2 + \mathcal{B}_{n+3} i_3.$$

We have,

$$\mathcal{B}_n - \mathcal{B}_{n+1} i_1 - \mathcal{B}_{n+2}\ i_2 + \mathcal{B}_{n+3} i_3 = \mathcal{F}_n + \mathcal{F}_{n+1}\ i_1 + \mathcal{F}_{n+2}\ i_2 + \mathcal{F}_{n+3}\ i_3$$

$$- (\mathcal{F}_{n+1} + \mathcal{F}_{n+2}\ i_1 + \mathcal{F}_{n+3} i_2 + \mathcal{F}_{n+4}\ i_3\ ) i_1$$

$$-(\mathcal{F}_{n+2} + \mathcal{F}_{n+3}\ i_1 + \mathcal{F}_{n+4} i_2 + \mathcal{F}_{n+5}\ i_3\ )$$

$$+(\mathcal{F}_{n+3} + \mathcal{F}_{n+4}\ i_1 + \mathcal{F}_{n+5}\ i_2 + \mathcal{F}_{n+6}\ i_3) i_3$$

$$= \mathcal{F}_n + \mathcal{F}_{n+2} - \mathcal{F}_{n+4} - \mathcal{F}_{n+6}$$

If we use the equalities (1) and (4), then we can write as

$$= \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} - \begin{bmatrix} J_{n+3} & 2J_{n+2} \\ J_{n+2} & 2J_{n+1} \end{bmatrix} - \begin{bmatrix} J_{n+5} & 2J_{n+4} \\ J_{n+4} & 2J_{n+3} \end{bmatrix} + \begin{bmatrix} J_{n+7} & 2J_{n+6} \\ J_{n+6} & 2J_{n+5} \end{bmatrix}$$

$$= \begin{bmatrix} J_{n+1} - J_{n+3} - J_{n+5} + J_{n+7} & 2(J_n - J_{n+2} - J_{n+4} + J_{n+6}) \\ J_n - J_{n+2} - J_{n+4} + J_{n+6} & 2(J_{n-1} - J_{n+1} - J_{n+3} + J_{n+5}) \end{bmatrix},$$

from the equalities (2) and (3), we get;

$$= 15 \begin{bmatrix} 2^{n+1} & 2^{n+1} \\ 2^n & 2^n \end{bmatrix}$$

$$= 15 \left(\begin{bmatrix} 3J_n & 3J_n \\ 3J_{n-1} & 3J_{n-1} \end{bmatrix} + \begin{bmatrix} j_n & j_n \\ j_{n-1} & j_{n-1} \end{bmatrix} \right)$$

$$= 15 \left(3 \begin{bmatrix} J_n & J_n + 2J_{n-1} - 2J_{n-1} \\ J_{n-1} & J_{n-1} + 2J_{n-2} - 2J_{n-2} \end{bmatrix} + \begin{bmatrix} j_n & j_n \\ j_{n-1} & j_{n-1} \end{bmatrix} \right)$$

$$= 15 \left(3 \begin{bmatrix} J_n & 2J_{n-1} \\ J_{n-1} & 2J_{n-2} \end{bmatrix} + 3 \begin{bmatrix} 0 & J_n - 2J_{n-1} \\ 0 & J_{n-1} - 2J_{n-2} \end{bmatrix} + \begin{bmatrix} j_n & 2j_{n-1} \\ j_{n-1} & 2j_{n-2} \end{bmatrix} + \begin{bmatrix} 0 & j_n - 2j_{n-1} \\ 0 & j_{n-1} - 2j_{n-2} \end{bmatrix} \right)$$

$$= 15 \left(3\mathcal{F}_{n-1} + 3 \begin{bmatrix} 0 & J_n - 2J_{n-1} \\ 0 & J_{n-1} - 2J_{n-2} \end{bmatrix} + \mathcal{F}'_{n-1} + \begin{bmatrix} 0 & j_n - 2j_{n-1} \\ 0 & j_{n-1} - 2j_{n-2} \end{bmatrix} \right)$$

$$= 45\mathcal{F}_{n-1} + 15\mathcal{F}'_{n-1} \left(3 \begin{bmatrix} 0 & -3(-1)^n \\ 0 & -3(-1)^{n-1} \end{bmatrix} + \begin{bmatrix} 0 & 3(-1)^n \\ 0 & 3(-1)^{n-1} \end{bmatrix} \right)$$

$$= 45 \left(\mathcal{F}_{n-1} + 2(-1)^n\ \mathcal{F}'_{n-1} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \right).$$

Identity 3.2. For $n \geq 1$

$$\mathcal{B}_n \mathcal{B}_n^{i_3} + \mathcal{B}_{n-1} \mathcal{B}_{n-1}^{i_3} = -93 \left(\mathcal{F}'_{2n-2} + \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \right)$$

where $\mathcal{B}_n^{i_3}$ is the conjugation with respect to *the imaginary unit* i_3.

Proof. Now we will prove the identity $\mathcal{B}_n \times \mathcal{B}_n^{i_3} + \mathcal{B}_{n-1} \times \mathcal{B}_{n-1}^{i_3}$.
By using the equalities (1), (4) and (6), we get

$$=\mathcal{F}_{n-1}^2 + 2\mathcal{F}_n^2 - 2\mathcal{F}_{n+2}^2 - \mathcal{F}_{n+3}^2$$

$$= \mathcal{F}_{2n-2} + 2\mathcal{F}_{2n} - 2\mathcal{F}_{2n+4} - \mathcal{F}_{2n+6}$$

$$= \begin{bmatrix} J_{2n-1} & 2J_{2n-2} \\ J_{2n-2} & 2J_{2n-3} \end{bmatrix} + 2\begin{bmatrix} J_{2n+1} & 2J_{2n} \\ J_{2n} & 2J_{2n-1} \end{bmatrix} - 2\begin{bmatrix} J_{2n+5} & 2J_{2n+4} \\ J_{2n+4} & 2J_{2n+3} \end{bmatrix} - \begin{bmatrix} J_{2n+7} & 2J_{2n+6} \\ J_{2n+6} & 2J_{2n+5} \end{bmatrix}$$

$$= \begin{bmatrix} J_{2n-1} + 2J_{2n+1} - 2J_{2n+5} + J_{2n+7} & 2(J_{2n-2} + 2J_{2n} - 2J_{2n+4} + J_{2n+6}) \\ J_{2n-2} + 2J_{2n} - 2J_{2n+4} + J_{2n+6} & 2(J_{2n-3} + 2J_{2n-1} - 2J_{2n+3} + J_{2n+5}) \end{bmatrix}.$$

If we use the equalities $J_{n+r} - J_{n-r} = \frac{1}{3}\left(2^{n-r}(2^{2r} - 1)\right)$, $j_{n+1} - 2j_n = 3(-1)^{n+1}$ and $j_n = 2^n + (-1)^n$ in [4], we have;

$$= -93\begin{bmatrix} 2^{2n-1} & 2^{2n-1} \\ 2^{2n-2} & 2^{2n-2} \end{bmatrix}$$

$$= -93\begin{bmatrix} j_{2n-1} - (-1)^{2n-1} & j_{2n-1} - (-1)^{2n-1} \\ j_{2n-2} - (-1)^{2n-2} & j_{2n-2} - (-1)^{2n-2} \end{bmatrix}$$

$$= -93\begin{bmatrix} j_{2n-1} - (-1)^{2n-1} & j_{2n-1} + 2j_{2n-2} - 2j_{2n-2} - (-1)^{2n-1} \\ j_{2n-2} - (-1)^{2n-2} & j_{2n-2} + 2j_{2n-3} - 2j_{2n-3} - (-1)^{2n-2} \end{bmatrix}$$

$$= -93\left(\begin{bmatrix} j_{2n-1} & 2j_{2n-2} \\ j_{2n-2} & 2j_{2n-3} \end{bmatrix} + \begin{bmatrix} -(-1)^{2n-1} & 3(-1)^{2n-1} - (-1)^{2n-1} \\ -(-1)^{2n-2} & 3(-1)^{2n-2} - (-1)^{2n-2} \end{bmatrix}\right)$$

$$= -93\left(\mathcal{F}'_{2n-2} + \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\right).$$

Identity 3.3.

$$\mathcal{B}_{n+1}^2 - \mathcal{B}_n^2 = 4(X_3 + X'_3 + 3 - 14i_1 - 12i_2)\mathcal{F}_{2n}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

Proof. By using the equalities (7) and (8),

$$\mathcal{B}_{n+1}^2 = \mathcal{F}_{2n+2} - \mathcal{F}_{2n+4} - \mathcal{F}_{2n+6} + \mathcal{F}_{2n+8} + 2(\mathcal{F}_{n+1}\mathcal{F}_{n+2} - \mathcal{F}_{n+3}\mathcal{F}_{n+4})i_1$$
$$+2(\mathcal{F}_{n+1}\mathcal{F}_{n+3} - \mathcal{F}_{n+2}\mathcal{F}_{n+4})i_2 + 2(\mathcal{F}_{n+1}\mathcal{F}_{n+4} + \mathcal{F}_{n+2}\mathcal{F}_{n+3})i_3$$

$$= \mathcal{F}_{2n+2} - \mathcal{F}_{2n+4} - \mathcal{F}_{2n+6} + \mathcal{F}_{2n+8} + 2(\mathcal{F}_{2n+3} - \mathcal{F}_{2n+7})i_1$$
$$+2(\mathcal{F}_{2n+4} - \mathcal{F}_{2n+6})i_2 + 4i_3\,\mathcal{F}_{2n+5},$$

$$= \mathcal{F}_{2n}(\mathcal{F}_2 - \mathcal{F}_4 - \mathcal{F}_6 + \mathcal{F}_8) + 2\mathcal{F}_{2n}(\mathcal{F}_3 - \mathcal{F}_7)i_1$$

$$+2\mathcal{F}_{2n}(\mathcal{F}_4 - \mathcal{F}_6)i_2 + 4\mathcal{F}_{2n+5}i_3$$

$$= \mathcal{F}_{2n}\left(\begin{bmatrix} 120 & 120 \\ 60 & 60 \end{bmatrix} + 2\begin{bmatrix} -80 & -80 \\ -40 & -40 \end{bmatrix}i_1 + \begin{bmatrix} -64 & -64 \\ -32 & -32 \end{bmatrix}i_2 + \begin{bmatrix} 84 & 88 \\ 44 & 40 \end{bmatrix}i_3\right)$$

$$= \mathcal{F}_{2n}\begin{bmatrix} 120 - 160i_1 - 64i_2 + 84i_3 & 120 - 160i_1 - 64i_2 + 88i_3 \\ 60 - 80i_1 - 32i_2 + 44i_3 & 60 - 80i_1 - 32i_2 + 40i_3 \end{bmatrix}.$$

Similarly, we can get

$$\mathcal{B}_n^2 = \mathcal{F}_{2n}\begin{bmatrix} 80-40i_1-16i_2+4i_3 & 80-40i_1-16i_2+8i_3 \\ 40-20i_1-8i_2+4i_3 & 40-20i_1-8i_2 \end{bmatrix}$$

$$\mathcal{B}_{n+1}^2 - \mathcal{B}_n^2 = (20-60i_1-24i_2+40i_3)\mathcal{F}_{2n}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= 4(X_3 + X'_3 + 3 - 14i_1 - 12i_2)\mathcal{F}_{2\mathrm{n}}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

Identity 3.4. For $n \geq 0$, The n^{th} negabicomplex Jacobsthal matrix is

$$\mathcal{B}_{-n} = \left(\frac{\begin{bmatrix} 2J_{n-1} & -2J_n \\ -J_n & J_{n+1} \end{bmatrix}}{2^n(-1)^n}\right)\begin{bmatrix} X_1 & 2X_1 \\ X_0 & X_1-(1+i_2+3i_3) \end{bmatrix}$$

Proof. We will give proof of identity $\mathcal{B}_{-n}$. We have

$$\mathcal{B}_{-n} = \mathcal{F}_{-n} + \mathcal{F}_{-n+1}\, i_1 + \mathcal{F}_{-n+2}\, i_2 + \mathcal{F}_{-n+3}\, i_3$$

$$\mathcal{B}_{-n} = \mathcal{F}_{-n}(\mathcal{F}_0 + \mathcal{F}_1\, i_1 + \mathcal{F}_2\, i_2 + \mathcal{F}_3\, i_3)$$

$$= (\mathcal{F}_n)^{-1}(\mathcal{F}_0 + \mathcal{F}_1\, i_1 + \mathcal{F}_2\, i_2 + \mathcal{F}_3\, i_3)$$

$$= \left(\begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}\right)^{-1}\begin{bmatrix} 1+i_1+2i_2+3i_3 & i_1+i_2+2i_3 \\ i_1+i_2+2i_3 & 1+i_2+i_3 \end{bmatrix}$$

$$= \left(\frac{\begin{bmatrix} 2J_{n-1} & -2J_n \\ -J_n & J_{n+1} \end{bmatrix}}{|\mathcal{F}_n|}\right)\begin{bmatrix} X_1 & 2X_1 \\ X_0 & X_1-(1+i_2+3i_3) \end{bmatrix}$$

$$= \left(\frac{\begin{bmatrix} 2J_{n-1} & -2J_n \\ -J_n & J_{n+1} \end{bmatrix}}{(-1)^n 2^n}\right)\begin{bmatrix} X_1 & 2X_1 \\ X_0 & X_1-(1+i_2+3i_3) \end{bmatrix}$$

where X_1 and X_0 are bicomplex Jacobsthal numbers.

4 Some Applications on Bicomplex Jacobstal Matrix

Let $\mathcal{B}_n$ be the n^{th} bicomplex Jacobsthal matrix, for $n \geq 0$, these number is 2^{th} linear recurrence sequence. Then, we suppose the sets of $\mathbb{C}_2$ and $\mathbb{C}_2'$ are

$$\mathbb{C}_2 = \{\, \mathcal{B}_n \mid \mathcal{B}_n = \mathcal{F}_n + \mathcal{F}_{n+1}\, i_1 + \mathcal{F}_{n+2}\, i_2 + \mathcal{F}_{n+3}\, i_3\,,\ \ \mathcal{F}_n \text{ is } n^{th}\ \mathcal{F}\text{-matrix} \},$$

and

$$\mathbb{C}_2' = \{\, \mathcal{B}_n \mid \mathcal{B}_n = \begin{bmatrix} \alpha_n & \beta_n \\ \beta_n & \alpha_n \end{bmatrix}; \alpha_n, \beta_n \epsilon \mathbb{C} \ \}$$

Then there is an *isomorphism* between $\mathbb{C}_2$ and $\mathbb{C}_2'$, in that case, we can write

$$\mathcal{B}_n = (\mathcal{F}_n, \mathcal{F}_{n+1}, \mathcal{F}_{n+2}, \mathcal{F}_{n+3}) \rightarrow \mathcal{B}_n = \begin{bmatrix} \mathcal{F}_n + \mathcal{F}_{n+1}i_1 & \mathcal{F}_{n+2} + \mathcal{F}_{n+3}i_1 \\ \mathcal{F}_{n+2} + \mathcal{F}_{n+3}i_1 & \mathcal{F}_n + \mathcal{F}_{n+1}i_1 \end{bmatrix}.$$

Thus, we can write

$$\mathbb{C}_2' = \{ \mathcal{B}_n \mid \mathcal{B}_n = \alpha_n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \beta_n \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \alpha_n, \beta_n \ \textit{is } n^{th} \textit{ complex Jacobsthal matrix} \}$$

and

$$\mathcal{B}_n = \mathcal{F}_n U_1 + \mathcal{F}_{n+1} U_2 + \mathcal{F}_{n+2} U_3 + \mathcal{F}_{n+3} U_4,$$

where

$$\mathcal{B}_n = \mathcal{F}_n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{F}_{n+1} \begin{bmatrix} i_1 & 0 \\ 0 & i_1 \end{bmatrix} + \mathcal{F}_{n+2} \begin{bmatrix} 0 & i_2 \\ i_2 & 0 \end{bmatrix} + \mathcal{F}_{n+3} \begin{bmatrix} 0 & i_3 \\ i_3 & 0 \end{bmatrix}.$$

Since $det\mathcal{B}_n \neq 0$, there is the inverse of matrix $\mathcal{B}_n$ and it is in $\mathbb{C}_2'$.

Definition 4.1. The n^{th} *bicomple Jacobsthal vectors* $\vec{\mathcal{B}}_n$ and the n^{th} *bicomplex Jacobsthal-Lucas vectors* $\vec{\mathcal{B}}_n'$ with matrix components are defined as

$$\vec{\mathcal{B}}_n = \mathcal{F}_{n+1}\, i_1 + \mathcal{F}_{n+2}\, i_2 + \mathcal{F}_{n+3}\, i_3$$

and

$$\vec{\mathcal{B}}_n' = \mathcal{F}_{n+1}' i_1 + \mathcal{F}_{n+2}'\, i_2 + \mathcal{F}_{n+3}' i_3,$$

respectively.

Theorem 4.1. Let $\vec{\mathcal{B}}_n$ and $\vec{\mathcal{B}}_{n+1}$ be bicomplex Jacobsthal vectors. *The cross product* is defined as

$$\vec{\mathcal{B}}_n \times \vec{\mathcal{B}}_{n+1} = det \begin{bmatrix} i_1 & i_2 & i_3 \\ \mathcal{F}_{n+1} & \mathcal{F}_{n+2} & \mathcal{F}_{n+3} \\ \mathcal{F}_{n+2} & \mathcal{F}_{n+3} & \mathcal{F}_{n+4} \end{bmatrix}$$

where in *the permanent* of $\vec{\mathcal{B}}_\text{n} \times \vec{\mathcal{B}}_{\text{n}+1}$, *the signatures of the permutations* are not taken into account, [15].

Proof. Now, if we calculate $\vec{\mathcal{B}}_n \times \vec{\mathcal{B}}_{n+1}$, we obtain

$$=(\mathcal{F}_{n+2}\mathcal{F}_{n+4} + \mathcal{F}_{n+3}\mathcal{F}_{n+3})i_1 - (\mathcal{F}_{n+1}\mathcal{F}_{n+4} + \mathcal{F}_{n+2}\mathcal{F}_{n+3})i_2 + (\mathcal{F}_{n+1}\mathcal{F}_{n+3} + \mathcal{F}_{n+2}\mathcal{F}_{n+2})i_3$$

$$=2\mathcal{F}_{2n+6}i_1 - 2\mathcal{F}_{n+5}i_2 + 2\mathcal{F}_{n+4}i_3$$

$$=2(\mathcal{F}_{2n+6}i_1 - \mathcal{F}_{2n+5}i_2 + \mathcal{F}_{2n+4}i_3 + \mathcal{F}_{2n+3}i_2 + \mathcal{F}_{2n+2}i_1 - \mathcal{F}_{2n+3}i_2 - \mathcal{F}_{2n+2}i_1)$$

$$=2(\mathcal{F}_{2n+2}i_1 + \mathcal{F}_{n+3}i_2 + \mathcal{F}_{n+4}i_3) + 2(\mathcal{F}_{2n+6} - \mathcal{F}_{2n+2})i_1 - 2(\mathcal{F}_{n+5} + \mathcal{F}_{n+3})i_2$$

Finally, we have

$$= \mathcal{F}_{2n}\left(\begin{bmatrix}40 & 40\\ 20 & 20\end{bmatrix} i_1 - \begin{bmatrix}25 & 28\\ 14 & 12\end{bmatrix} i_2\right) + \vec{\mathcal{B}}_{2n+1},$$

where $\vec{\mathcal{B}}_{2n+1}$ is bicomplex Jacobsthal vector.

5 Conclusion

In this study, we firstly introduced bicomplex Jacobsthal numbers and bicomplex Jacobsthal-Lucas numbers and matrices. Then, we give some identities that hold an important place in the literature on these matrices. We have also given bicomplex Jacobsthal vector $\vec{\mathcal{B}}_n$ with $\mathcal{F}$-matrix component to exert in geometry.

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