

# Original Research Article

## **Development of Exponential Linear Multistep Methods for the Solution of First-Order Ordinary Differential Equations**

### **Abstract**

Several approaches have been applied by different researchers to produce linear multistep methods (LMMs) for the solution of ordinary differential equations (ODEs). In this paper, some LMMs have been developed via the collocation and interpolation technique using the exponential function as the basis function. The continuous and discrete forms of the methods have been evaluated and tested on some first-order ordinary differential equations. Results are presented in terms of maximum absolute errors and have shown that the proposed methods produce more accurate approximations than the existing LMMs derived using some other polynomial functions. We therefore recommend that the proposed methods should be tested on ODEs of second and higher orders.

**Keywords:** *Collocation, Exponential function, Exponential linear multistep method, Interpolation, Linear multistep method, Ordinary differential equation, optimal order method, Single-step method.*

### **1. Introduction**

Many real-life phenomena are described mathematically in terms of differential equations. They are used to model problems in science, engineering, business, economics, astronomy, and environmental sciences. Differential equations can be used to describe virtually every system undergoing change. The problems often encountered are classified into initial value and boundary value problems depending on the conditions indicated at the endpoints of the domain, Saeed and Alotaibi [1]. In this paper, we will treat initial value problems (IVPs) of ordinary differential equations.

Ordinary differential equations (ODEs) can be solved analytically or numerically. Numerical techniques for solving ODEs are often applied when the analytic methods fail due to the nonlinearity of the differential equations and/or the complex initial/boundary conditions associated with the differential equation, Fadugba [2].

Two common sets of numerical methods are used for approximating solutions to IVPs of ODEs. They are the single-step and the multistep methods. Single-step methods use information from only

one previous step to compute the solution of the present step. They are also self-starting. The commonly applied single-step methods are the Euler's method and the Runge-Kutta family of methods, Ogunrinde and Olubunmi [3]. Multistep methods use information from more than one previous step to obtain the solution of the present step. They are not self-starting hence, need starting values from the single-step methods, Iyorter *et al.* [4]. We shall be deriving single-step methods and multistep methods for the solution of IVPs of ODEs.

The general  $r$ -step linear multistep method (LMM) as presented in Lambert [5] is

$$\sum_{j=0}^r \alpha_j y_{n+j} = h \sum_{j=0}^r \beta_j f_{n+j} \quad (1)$$

where  $\alpha_j$  and  $\beta_j$  are to be determined and  $\alpha_0 + \beta_0 \neq 0$ ,  $\alpha_r = 1$ . Discrete schemes are derived from Equation (1) and applied to solve first-order ODEs.

Several techniques are applied to derive LMMs in discrete form. They include numerical integration, Taylor series expansion and interpolation method. Continuous collocation and interpolation technique is now widely used for the derivation of LMMs, block methods and hybrid methods, Aboiyar *et al.* [6]. The continuous collocation and interpolation technique is applied to derive LMMs of the form:

$$y(x) = \sum_{j=0}^r \alpha_j(x) y_{n+j} + h \sum_{j=0}^r \beta_j(x) f_{n+j} \quad (2)$$

where  $\alpha_j$  and  $\beta_j$  are expressed as continuous functions of  $x$  and are at least one time differentiable.

Many researchers have used collocation and interpolation approach to derive LMMs for the solution of first and higher-order ordinary differential equations. Okunuga and Ehigie [7] derived two-step continuous and discrete LMMs using power series as basis function for the solution of first-order ODEs, Mohammed [8] derived a LMM with continuous coefficients which he used to obtain multiple finite difference methods for the direct solution of first-order ODEs, Odekunle *et al.* [9] developed a continuous LMM with constant step size for the solution of first-order ODE, Akinfewa *et al.* [10]. developed a four-step continuous block hybrid method with four off-step points for the direct solution of first-order ODEs, Aboiyar *et al.* [5] derived LMMs using Hermite polynomials as basis function for the solution of first-order ODEs, Iyorter *et al.* [4] used the Laguerre polynomial as basis function to derive LMMs for the solution of first-order ODEs.

Ehigie *et al.* [11] proposed a two-step continuous multistep method of hybrid type for the direct integration of second order ODEs, Anake [12] developed a new class of continuous implicit hybrid one-step methods for solving IVPs of general second order ODEs, using the collocation and interpolation of power series approximate solution, James *et al.* [13] proposed a continuous block

method for the solution of second order IVPs with constant step-size. In this paper, we will use the exponential function as basis function to develop continuous explicit and implicit LMMs via the interpolation and collocation approach, for the solution of first-order ODEs. The corresponding discrete schemes shall also be obtained.

## 2. Methods

Awoyemi *et al.* [14] proposed a basis function of the type

$$y(x) = \sum_{j=0}^k \alpha_j (x - x_k)^j \quad (3)$$

to develop LMM for the solution of third order IVPs. Adeniyi and Alabi [15] developed continuous LMM using the Chebyshev polynomial function of the form:

$$y(x) = \sum_{j=0}^M \alpha_j T_j \left( \frac{x - x_k}{h} \right),$$

where  $T_j(x)$  are some Chebyshev functions.

Aboiyar *et al.* [3] developed continuous LMMs through the collocation and interpolation technique using the Hermite polynomial function of the form, Koornwinder, [16]:

$$y(x) = \sum_{j=0}^k a_j H_j(x - x_k)$$

where  $H_j(x)$  are probabilists' Hermite polynomials, as basis functions.

In this paper, we propose the exponential polynomial functions of the form:

$$y(x) = \sum_{k=0}^n a_k E_k(x - x_j)$$

where  $E_k(x)$  are exponential functions, to develop continuous LMMs for the solution of IVPs of first-order ODEs of the form:

$$y' = f(x, y(x)), \quad y(x_0) = y_0. \quad (4)$$

We will use the exponential function

$$f(x) = \sum_{k=0}^n e^{x^k},$$

where "e" is the Euler's number. The first six exponential functions are:

$$E_0 = e, \quad E_1 = e^x, \quad E_2 = e^{x^2}, \quad E_3 = e^{x^3}, \quad E_4 = e^{x^4}, \quad E_5 = e^{x^5}, \quad E_6 = e^{x^6}.$$

## 2.1 Derivation of the Exponential Linear Multistep Methods

To derive the Exponential Linear Multistep Methods (e-LMMs) we shall consider the function,

$$y(x) = \sum_{k=0}^n a_k E_k(x - x_j), \quad x_j \leq x \leq x_{j+r}. \quad (5)$$

We wish to approximate the exact solution  $y(x)$  to Equation (4) by the function in Equation (5) which satisfies the equations,

$$\left. \begin{aligned} y'(x) &= f(x, y(x)), \quad x_j \leq x \leq x_{j+r} \\ y(x_j) &= y_j. \end{aligned} \right\} \quad (6)$$

### 2.1.1 The one-step explicit exponential linear multistep method

To derive the one-step explicit e-LMM, we shall let  $n = 1$  in Equation (5). This gives:

$$y(x) = a_0 e + a_1 e^{(x-x_j)}. \quad (7)$$

Obtaining the first derivative of Equation (7), we have

$$y'(x) = a_1 e^{(x-x_j)}. \quad (8)$$

Interpolating Equation (7) at  $x = x_j$  and collocating Equation (8) at  $x = x_j$  gives the system of equations:

$$\begin{bmatrix} e & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} y_j \\ f_j \end{bmatrix}.$$

Solution to the system of equations is:

$$\left. \begin{aligned} a_0 &= (y_j - f_j) e^{-1} \\ a_1 &= f_j \end{aligned} \right\} \quad (9)$$

Substituting for  $a_k$ ,  $k = 0, 1$  in Equation (7) yields the continuous form of the one-step explicit e-LMM:

$$y(x) = y_j - f_j + f_j e^{(x-x_j)} \quad (10)$$

Evaluating Equation (10) at  $x = x_{j+1}$  gives the discrete form of the one-step explicit e-LMM:

$$y_{j+1} = y_j + (e^h - 1) f_j \quad (11)$$

### 2.1.2 Two-Step Explicit Exponential Linear Multistep Method

For the two-step explicit e-LMM, we shall let  $n = 2$  in Equation (5). This produces:

$$y(x) = a_0 e + a_1 e^{(x-x_j)} + a_2 e^{(x-x_j)^2}. \quad (12)$$

Differentiating Equation (12) once yields

$$y'(x) = a_1 e^{(x-x_j)} + 2(x-x_j)a_2 e^{(x-x_j)^2}. \quad (13)$$

Interpolating Equation (12) at  $x = x_{j+1}$  and collocating Equation (13) at  $x = x_j, x_{j+1}$  gives the system of equations:

$$\begin{bmatrix} e & e^h & e^{h^2} \\ 0 & 1 & 0 \\ 0 & e^h & 2he^{h^2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_{j+1} \\ f_j \\ f_{j+1} \end{bmatrix}. \quad (14)$$

Solving the system of equations by the Gaussian elimination method, we get:

$$\left. \begin{aligned} a_0 &= y_{j+1} e^{-1} - f_j e^{(h-1)} - \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-1}, \\ a_1 &= f_j, \\ a_2 &= \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-h^2} \end{aligned} \right\} \quad (15)$$

Substituting for  $a_k$ ,  $k = 0, 1, 2$  in Equation (12), we have the two-step continuous explicit e-LMM:

$$y(x) = y_{j+1} + f_j \left( \frac{1}{2h} e^h - e^h \right) - f_{j+1} \left( \frac{1}{2h} \right) + e^{(x-x_j)} f_j + e^{(x-x_j)^2} \left[ \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-h^2} \right] \quad (16)$$

Evaluating Equation (16) at  $x = x_{j+2}$ , we obtain the discrete form of the two-step explicit e-LMM:

$$y_{j+2} = y_{j+1} + f_j \left( \frac{1}{2h} e^h - e^h + e^{2h} - \frac{1}{2h} e^{(3h^2+h)} \right) + f_{j+1} \left( \frac{1}{2h} e^{3h^2} - \frac{1}{2h} \right) \quad (17)$$

### 2.1.3 Three-Step Explicit Exponential Linear Multistep Method

To derive the three-step explicit e-LMM, we set  $n = 3$  in Equation (5). This gives:

$$y(x) = a_0 e + a_1 e^{(x-x_j)} + a_2 e^{(x-x_j)^2} + a_3 e^{(x-x_j)^3}. \quad (18)$$

Differentiating Equation (18) once gives

$$y'(x) = a_1 e^{(x-x_j)} + 2(x-x_j)a_2 e^{(x-x_j)^2} + 3(x-x_j)^2 a_3 e^{(x-x_j)^3}. \quad (19)$$

Interpolating Equation (18) at  $x = x_{j+2}$  and collocating Equation (19) at  $x = x_j, x_{j+1}, x_{j+2}$  gives the system of equations:

$$\begin{bmatrix} e & e^{2h} & e^{4h^2} & e^{8h^3} \\ 0 & 1 & 0 & 0 \\ 0 & e^h & 2he^{h^2} & 3h^2e^{h^3} \\ 0 & e^{2h} & 4he^{4h^2} & 12h^2e^{8h^3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_{j+2} \\ f_j \\ f_{j+1} \\ f_{j+2} \end{bmatrix}. \quad (20)$$

Solving the system of equations using the Gaussian elimination method, we have:

$$\left. \begin{aligned} a_3 &= \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}}, \\ a_2 &= f_{j+1} \left( \frac{1}{2h} e^{-h^2} \right) - f_j \left( \frac{1}{2h} e^{(h-h^2)} \right) - \left[ \left( \frac{3h}{2} e^{(h^3-h^2)} \right) \cdot a_3 \right], \\ a_1 &= f_j, \\ a_0 &= y_{j+2} e^{-1} + f_j \left( \frac{1}{2h} e^{(3h^2+h-1)} - e^{(2h-1)} \right) \\ &\quad - f_{j+1} \left( \frac{1}{2h} e^{(3h^2-1)} \right) + \left[ \left( \frac{3h}{2} e^{(h^3+3h^2-1)} - e^{(8h^3-1)} \right) \cdot a_3 \right]. \end{aligned} \right\} \quad (21)$$

Substitute for  $a_k$ ,  $k = 0, 1, 2, 3$  in Equation (18). This yields the three-step continuous explicit e-LMM:

$$\begin{aligned} y(x) &= y_{j+2} + f_j \left( \frac{1}{2h} e^{(3h^2+h)} - e^{2h} \right) - f_{j+1} \left( \frac{1}{2h} e^{3h^2} \right) \\ &\quad + \left[ \left( \frac{3h}{2} e^{(h^3+3h^2)} - e^{8h^3} \right) \cdot \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \right] + e^{(x-x_j)} f_j \\ &\quad + e^{(x-x_j)^2} \left[ - \left( \frac{3h}{2} e^{(h^3-h^2)} \cdot \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \right) \right] \\ &\quad + e^{(x-x_j)^3} \left[ \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right]. \end{aligned} \quad (22)$$

Evaluating Equation (22) at  $x = x_{j+3}$ , we obtain the discrete scheme of the three-step explicit e-LMM:

$$y_{j+3} = y_{j+2} + f_j \left[ \frac{1}{2h} e^{(3h^2+h)} - e^{2h} + e^{3h} - \frac{1}{2h} e^{(8h^2+h)} \right] + f_{j+1} \left[ \frac{1}{2h} e^{8h^2} - \frac{1}{2h} e^{3h^2} \right] \\ \left[ \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2 e^{8h^3} - 6h^2 e^{(3h^2+h^3)}} \right] \cdot \left[ \begin{array}{c} \frac{3h}{2} e^{(h^3+3h^2)} - e^{8h^3} \\ -\frac{3h}{2} e^{(h^3+8h^2)} + e^{27h^3} \end{array} \right]. \quad (23)$$

#### 2.1.4 The two-step explicit optimal order exponential linear multistep method

To derive the two-step explicit optimal order e-LMM, we shall consider the system of equations in Equation (14) except for the first row. Evaluating Equation (12) at  $x = x_j$ , we have:

$$[e \quad 1 \quad 1][a_0] = [y_j]. \quad (24)$$

The first row of Equation (14) is therefore replaced with Equation (24). The system of equations is solved, and the result is as obtained in Equation (15) except for  $a_0$ , which we have:

$$a_0 = y_j e^{-1} - f_j e^{-1} - \left[ \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-(h^2+1)} \right].$$

Substituting for  $a_k$ ,  $k = 0, 1, 2, 3$  in Equation (12), we have the two-step continuous explicit optimal order e-LMM:

$$y(x) = y_j - f_j - \left[ \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-h^2} \right] + e^{(x-x_j)} f_j + e^{(x-x_j)^2} \left[ \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-h^2} \right]. \quad (25)$$

Evaluating Equation (25) at  $x = x_{j+2}$ , we obtain the discrete scheme of the two-step explicit optimal order e-LMM:

$$y_{j+2} = y_j - f_j - \left[ \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-h^2} \right] + e^{2h} f_j + e^{3h^2} \left[ \frac{1}{2h} (f_{j+1} - e^h f_j) \right]. \quad (26)$$

#### 2.1.5 The three-step explicit optimal order exponential linear multistep method

To derive the three-step explicit optimal order e-LMM, we shall consider the system of equations in Equation (20) except for the first row. Evaluating Equation (18) at  $x = x_{j+1}$ , we have

$$[e \quad e^h \quad e^{h^2} \quad e^{h^3}][a_0] = [y_{j+1}]. \quad (27)$$

The first row of Equation (20) is now replaced with Equation (27). The system of equations is solved, and the result is as obtained in Equation (21) except for  $a_0$ , which we have:

$$a_0 = y_{j+1} e^{-1} + f_j \left( \frac{1}{2h} e^{(h-1)} - e^{(h-1)} \right) - f_{j+1} \left( \frac{1}{2h} e^{-1} \right) + \left[ \left( \frac{3h}{2} e^{(h^3-1)} - e^{(h^3-1)} \right) \cdot a_3 \right].$$

Substituting for  $a_k$ ,  $k = 0, 1, 2, 3$  in Equation (18), we have the three-step continuous explicit optimal order e-LMM:

$$\begin{aligned} y(x) &= y_{j+1} + f_j \left( \frac{1}{2h} e^h - e^h \right) - f_{j+1} \left( \frac{1}{2h} \right) \\ &+ \left[ \left( \frac{3h}{2} e^{h^3} - e^{h^3} \right) \cdot \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \right] + e^{(x-x_j)} f_j \\ &+ e^{(x-x_j)^2} \left[ - \left( \frac{3h}{2} e^{(h^3-h^2)} \cdot \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \right) \right] \\ &+ e^{(x-x_j)^3} \left[ \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right]. \end{aligned} \quad (28)$$

Evaluating Equation (28) at  $x = x_{j+3}$ , we obtain the discrete scheme of the three-step explicit optimal order e-LMM:

$$\begin{aligned} y_{j+3} &= y_{j+1} + f_j \left( \frac{1}{2h} e^h - e^h + e^{3h} - \frac{1}{2h} e^{(8h^2+h)} \right) + f_{j+1} \left( \frac{1}{2h} e^{8h^2} - \frac{1}{2h} \right) \\ &\left[ \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right] \cdot \left( \frac{3h}{2} e^{h^3} - e^{h^3} - \frac{3h}{2} e^{(h^3+8h^2)} + e^{27h^3} \right). \end{aligned} \quad (29)$$

### 2.1.6 One-Step Implicit Exponential Linear Multistep Method

To derive the one-step implicit e-LMM, consider the continuous equation, Equation (25):

$$y(x) = y_j - f_j - \left[ \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-h^2} \right] + e^{(x-x_j)} f_j + e^{(x-x_j)^2} \left[ \frac{1}{2h} (f_{j+1} - e^h f_j) e^{-h^2} \right].$$

Evaluating at  $x = x_{j+1}$  gives the discrete form of the one-step implicit e-LMM:

$$y_{j+1} = y_j + f_j \left( e^h - \frac{1}{2h} e^h + \frac{1}{2h} e^{(h-h^2)} - 1 \right) + f_{j+1} \left( \frac{1}{2h} - \frac{1}{2h} e^{-h^2} \right). \quad (30)$$

### 2.1.7 Two-Step Implicit Exponential Linear Multistep Method

To derive the two-step implicit e-LMM, consider the continuous equation, Equation (28):

$$\begin{aligned}
y(x) &= y_{j+1} + f_j \left( \frac{1}{2h} e^h - e^h \right) - f_{j+1} \left( \frac{1}{2h} \right) \\
&+ \left[ \left( \frac{3h}{2} e^{h^3} - e^{h^3} \right) \cdot \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \right] + e^{(x-x_j)} f_j \\
&+ e^{(x-x_j)^2} \left[ \left( \frac{3h}{2} e^{(h^3-h^2)} \cdot \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \right) \right] \\
&+ e^{(x-x_j)^3} \left[ \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right].
\end{aligned}$$

Evaluating at  $x = x_{j+2}$  gives the discrete form of the two-step implicit e-LMM:

$$\begin{aligned}
y_{j+2} &= y_{j+1} + f_j \left[ \frac{1}{2h} e^h - e^h + e^{2h} - \frac{1}{2h} e^{(3h^2+h)} \right] + f_{j+1} \left[ \frac{1}{2h} e^{3h^2} - \frac{1}{2h} \right] \\
&\left[ \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right] \cdot \left( \frac{3h}{2} e^{h^3} - e^{h^3} - \frac{3h}{2} e^{(h^3+3h^2)} + e^{8h^3} \right) \quad (31)
\end{aligned}$$

### 2.1.8 Two-Step Implicit Optimal Order Exponential Linear Multistep Method

To derive the two-step implicit optimal order e-LMM, we shall interpolate Equation (18) at  $x = x_j$  and collocate Equation (19) at  $x = x_j, x_{j+1}, x_{j+2}$ . This gives the system of equations:

$$\begin{bmatrix} e & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & e^h & 2he^{h^2} & 3h^2e^{h^3} \\ 0 & e^{2h} & 4he^{4h^2} & 12h^2e^{8h^3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_j \\ f_j \\ f_{j+1} \\ f_{j+2} \end{bmatrix}.$$

Solving the system of equations using the Gaussian elimination method produces:

$$a_3 = \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}},$$

$$a_2 = f_{j+1} \left( \frac{1}{2h} e^{-h^2} \right) - f_j \left( \frac{1}{2h} e^{(h-h^2)} \right) - \left[ \left( \frac{3h}{2} e^{(h^3-h^2)} \right) \cdot a_3 \right],$$

$$a_1 = f_j,$$

$$a_0 = y_j e^{-1} + f_j \left( \frac{1}{2h} e^{(h-h^2-1)} - e^{-1} \right) - f_{j+1} \left( \frac{1}{2h} e^{-(h^2+1)} \right) - \left[ \left( \frac{3h}{2} e^{(h^3-h^2-1)} \right) \cdot a_3 \right].$$

Substitute for  $a_k$ ,  $k = 0, 1, 2, 3$  in Equation (18). This yields the two-step continuous implicit optimal order e-LMM:

$$\begin{aligned}
y(x) = & y_j + f_j \left[ \left( \frac{1}{2h} e^{(h-h^2)} \right) - 1 \right] - f_{j+1} \left( \frac{1}{2h} e^{-h^2} \right) - \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \\
& + \left[ \left( \frac{3h}{2} e^{(h^3-h^2)} \right) \cdot \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \right] + e^{(x-x_j)} f_j \\
& + e^{(x-x_j)^2} \left[ - \left( \frac{3h}{2} e^{(h^3-h^2)} \cdot \left( \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right) \right) \right] \\
& + e^{(x-x_j)^3} \left[ \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right]. \tag{32}
\end{aligned}$$

Evaluating Equation (32) at  $x = x_{j+2}$ , we obtain the discrete scheme of the two-step implicit optimal order e-LMM:

$$\begin{aligned}
y_{j+2} = & y_j + f_j \left[ \left( \frac{1}{2h} e^{(h-h^2)} \right) + e^{2h} - \frac{1}{2h} e^{(3h^2+h)} - 1 \right] + f_{j+1} \left( \frac{1}{2h} e^{3h^2} - \frac{1}{2h} e^{-h^2} \right) \\
& \left[ \frac{f_{j+2} - f_{j+1}(2e^{3h^2}) + f_j(2e^{(3h^2+h)} - e^{2h})}{12h^2e^{8h^3} - 6h^2e^{(3h^2+h^3)}} \right] \cdot \left( \frac{3h}{2} e^{(h^3-h^2)} - \frac{3h}{2} e^{(h^3+3h^2)} + e^{8h^3} - 1 \right). \tag{33}
\end{aligned}$$

### 3. Results

Eight LMMs have been derived for the solution of ODEs – the explicit exponential linear multistep methods of step number  $k = 1, 2, 3$  represented respectively by Equations (11), (17) and (23); the explicit optimal order e-LMMs of step number  $k = 2, 3$  represented by Equations (26) and (28); the implicit e-LMMs of step number  $k = 1, 2$  represented by Equations (30) and (31), and the implicit optimal order e-LMMs of step number  $k = 2$  represented by Equation (33). Six of the developed methods are tested on two IVPs of ODEs. They are Equations (17), (23), (26), (28), (31), and (33). Since the methods are not self-starting, the Runge-Kutta methods are used to obtain the starting values. For the implicit schemes, the derived explicit methods of corresponding order of accuracy are used as predictor methods. Results by the proposed methods are compared with already existing methods like the Adams-Bashforth, Adams-Moulton and the optimal order methods

developed using other polynomial functions. The maximum absolute errors associated with the methods are tabulated for easy understanding and comparison with the other existing methods.

The maximum absolute error is defined as

$$\text{Maximum Error} = \text{Max}(|y(x) - y_n(x)|)$$

where  $y(x)$  is the exact solution and  $y_n(x)$  is the approximate solution.

**Example 1.**

Consider the first-order stiff ordinary differential equation,

$$y' = \frac{y(1-y)}{2y-1}, \quad y(0) = \frac{5}{6}, \quad 0 \leq x \leq 1,$$

whose exact solution is

$$y(x) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{5}{36} \exp(-x)}.$$

Table 1. Maximum absolute errors of some existing LMMs and the proposed methods for Example 1.

| Stepsize $h$ | Method                               | Number of Steps | Maximum Error      |
|--------------|--------------------------------------|-----------------|--------------------|
| $10^{-1}$    | 2-Step Adams-Bashforth Method        | 10              | 5.3444576725 E-004 |
|              | 2-Step explicit e-LMM                | 10              | 5.7723348711 E-004 |
|              | 2-Step explicit Optimal Order Method | 10              | 3.8002546364 E-004 |
|              | 2-Step explicit Optimal Order e-LMM  | 10              | 4.3928959548 E-004 |
|              | 3-Step Adams-Bashforth Method        | 10              | 1.1304306662 E-004 |
|              | 3-Step explicit e-LMM                | 10              | 2.1311968307 E-004 |
|              | 3-Step explicit Optimal Order Method | 10              | 1.0735253231 E-004 |
|              | 3-Step explicit Optimal Order e-LMM  | 10              | 1.3825727024 E-004 |
|              | 2-Step Adams-Moulton Method          | 10              | 4.6740289726 E-005 |
|              | 2-Step implicit e-LMM                | 10              | 7.8530249726 E-005 |
|              | 2-Step implicit Optimal Order Method | 10              | 1.1075823148 E-005 |
|              | 2-Step implicit Optimal Order e-LMM  | 10              | 1.5948970098 E-005 |
| $10^{-2}$    | 2-Step Adams-Bashforth Method        | 100             | 5.8284133604 E-006 |
|              | 2-Step explicit e-LMM                | 100             | 3.8802372648 E-006 |
|              | 2-Step explicit Optimal Order Method | 100             | 2.4790925597 E-006 |
|              | 2-Step explicit Optimal Order e-LMM  | 100             | 1.6622102665 E-006 |
|              | 3-Step Adams-Bashforth Method        | 100             | 1.5151706500 E-007 |
|              | 3-Step explicit e-LMM                | 100             | 1.6111703327 E-007 |
|              | 3-Step explicit Optimal Order Method | 100             | 7.4785721393 E-008 |
|              | 3-Step explicit Optimal Order e-LMM  | 100             | 7.1801623314 E-008 |
|              | 2-Step Adams-Moulton Method          | 100             | 6.6061233372 E-008 |
|              | 2-Step implicit e-LMM                | 100             | 6.0575596983 E-008 |
|              | 2-Step implicit Optimal Order Method | 100             | 1.6880070897 E-008 |
|              | 2-Step implicit Optimal Order e-LMM  | 100             | 6.5316876352 E-009 |
| $10^{-3}$    | 2-Step Adams-Bashforth Method        | 1000            | 5.8797610225 E-008 |
|              | 2-Step explicit e-LMM                | 1000            | 3.7202811942 E-008 |
|              | 2-Step explicit Optimal Order Method | 1000            | 2.3665440940 E-008 |
|              | 2-Step explicit Optimal Order e-LMM  | 1000            | 1.4987933561 E-008 |
|              | 3-Step Adams-Bashforth Method        | 1000            | 1.5660250874 E-010 |
|              | 3-Step explicit e-LMM                | 1000            | 1.5073498005 E-010 |
|              | 3-Step explicit Optimal Order Method | 1000            | 7.0337513591 E-011 |
|              | 3-Step explicit Optimal Order e-LMM  | 1000            | 6.7731709130 E-011 |
|              | 2-Step Adams-Moulton Method          | 1000            | 6.8476335713 E-011 |
|              | 2-Step implicit e-LMM                | 1000            | 5.5000004551 E-011 |
|              | 2-Step implicit Optimal Order Method | 1000            | 1.7505996652 E-011 |
|              | 2-Step implicit Optimal Order e-LMM  | 1000            | 3.0706548415 E-012 |

## Example 2

Solve the IVP

$$y' = xe^{3x} - 2y, \quad y(0) = 0, \quad 0 \leq x \leq 1,$$

with exact solution

$$y(x) = \frac{1}{5}xe^{3x} - \frac{1}{25}e^{3x} + \frac{1}{25}e^{-2x}.$$

**Table 2. Maximum absolute errors of some existing LMMs and the proposed methods for Example 2.**

| Stepsize<br>$h$ | Method                               | Number of<br>Steps | Maximum Error       |
|-----------------|--------------------------------------|--------------------|---------------------|
| $10^{-1}$       | 2-Step Adams-Bashforth Method        | 10                 | 1.1778198813 E-001  |
|                 | 2-Step explicit e-LMM                | 10                 | 1.0043721109 E-001  |
|                 | 2-Step explicit Optimal Order Method | 10                 | 5.9842389199 E-002  |
|                 | 2-Step explicit Optimal Order e-LMM  | 10                 | 5.0808397909 E-002  |
|                 | 3-Step Adams-Bashforth Method        | 10                 | 3.2428666318 E-002  |
|                 | 3-Step explicit e-LMM                | 10                 | 2.1140401080 E-002  |
|                 | 3-Step explicit Optimal Order Method | 10                 | 1.47960778409 E-002 |
|                 | 3-Step explicit Optimal Order e-LMM  | 10                 | 9.5157045396 E-003  |
|                 | 2-Step Adams-Moulton Method          | 10                 | 1.2160744259 E-002  |
|                 | 2-Step implicit e-LMM                | 10                 | 8.1167810802 E-003  |
|                 | 2-Step implicit Optimal Order Method | 10                 | 3.1257910269 E-003  |
|                 | 2-Step implicit Optimal Order e-LMM  | 10                 | 9.7890082533 E-004  |
| $10^{-2}$       | 2-Step Adams-Bashforth Method        | 100                | 1.4097396721 E-003  |
|                 | 2-Step explicit e-LMM                | 100                | 1.3060522381 E-003  |
|                 | 2-Step explicit Optimal Order Method | 100                | 5.7868338326 E-004  |
|                 | 2-Step explicit Optimal Order e-LMM  | 100                | 5.3562629225 E-004  |
|                 | 3-Step Adams-Bashforth Method        | 100                | 4.5307998120 E-005  |
|                 | 3-Step explicit e-LMM                | 100                | 3.0238026843 E-005  |
|                 | 3-Step explicit Optimal Order Method | 100                | 2.0185635016 E-005  |
|                 | 3-Step explicit Optimal Order e-LMM  | 100                | 1.3489489944 E-005  |
|                 | 2-Step Adams-Moulton Method          | 100                | 1.3936187653 E-005  |
|                 | 2-Step implicit e-LMM                | 100                | 9.3816802451 E-006  |
|                 | 2-Step implicit Optimal Order Method | 100                | 2.9537732478 E-006  |
|                 | 2-Step implicit Optimal Order e-LMM  | 100                | 9.2314098810 E-007  |
| 1000            | 2-Step Adams-Bashforth Method        | 1000               | 1.4375467650 E-005  |
|                 | 2-Step explicit e-LMM                | 1000               | 1.34315803049 E-005 |
|                 | 2-Step explicit Optimal Order Method | 1000               | 5.7652170473 E-006  |
|                 | 2-Step explicit Optimal Order e-LMM  | 1000               | 5.3862592946 E-006  |

|                  |   |      |                    |
|------------------|---|------|--------------------|
| 10 <sup>-3</sup> | <b>3-Step Adams-Bashforth Method</b>        | 1000 | 4.6881527105 E-008 |
|                  | <b>3-Step explicit e-LMM</b>                | 1000 | 3.0406068774 E-008 |
|                  | <b>3-Step explicit Optimal Order Method</b> | 1000 | 2.0843756943 E-008 |
|                  | <b>3-Step explicit Optimal Order e-LMM</b>  | 1000 | 1.3537275301 E-008 |
|                  | <b>2-Step Adams-Moulton Method</b>          | 1000 | 1.4130341341 E-008 |
|                  | <b>2-Step implicit e-LMM</b>                | 1000 | 9.2376506444 E-009 |
|                  | <b>2-Step implicit Optimal Order Method</b> | 1000 | 2.9325022410 E-009 |
|                  | <b>2-Step implicit Optimal Order e-LMM</b>  | 1000 | 7.7494544115 E-010 |

#### 4. Discussion

Six of the eight derived e-LMMs have been applied to solve two IVPs of first-order ODEs using three different stepsizes. Table 1 and Table 2 show the maximum absolute errors associated with these methods as compared to similar methods already in existence but derived using some basis functions aside the exponential function. Particularly, results by the two-step and three-step Adams-Bashforth methods are compared with those of the two-step and three-step explicit e-LMMs. This has also been done for the results of the corresponding optimal methods and those of the implicit schemes. It has been observed generally that, the smaller the stepsize, the smaller the error hence, the more accurate the approximations.

The proposed methods have performed better than the existing ones for the explicit, optimal order and implicit schemes. The implicit optimal order schemes have performed better than the rest of the methods. We therefore recommend that the proposed methods be used as preferred methods for the solution of first-order ODEs and should be tested for higher order ODEs. This shows that LMMs can be derived using many different functions.

#### 5. Conclusion

Solution of differential equations is in turn, solution to some world problems as differential equations represent real world phenomena. In this paper, some methods have been developed for solving first-order ordinary differential equations using the approach of collocation and interpolation of functions. The exponential function has been used as the basis function. The proposed methods have been tested on some randomly selected ODEs and the results were compared with similar existing methods. The proposed methods performed wonderfully well, even better than the existing methods. We therefore conclude that the proposed e-LMMs be used for the solution of first-order ODEs.

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