# Packing Problem and Random Coverage in Continuous Domain

**Abstract**

The packing problem is a well known-problem. There are several versions of this problem. In this paper we consider packing or covering of a disc of a given radius $r$ by a number of discs of unit radius. We introduced two types of packing, hexagonal packing and square packing. We show that hexagonal packing is better in the sense that it needs less discs to cover a disc of higher radius. Coverage problem is similar to the packing problem in continuous domain. Coverage is essential in wireless sensor networks. In this paper we also discuss the coverage problem in random deployment scenario.

*Keywords:* Packing problem, Sphere packing problem, Random deployment

## 1. Introduction

The packing problem is a well-known problem. There are several versions of this problem. In this paper we consider packing or covering of a disc of a given radius by a number of discs of unit radius. We introduced two types of packing, hexagonal packing and square packing. We show that hexagonal packing is better in the sense that it needs less discs to cover a disc of higher radius.

The well-known sphere packing problem asks, what fraction of $\mathbb{R}^n$ can be covered by congruent balls that do not intersect except along their boundaries. This is a very important problem as it has many applications in error-correcting codes, spherical codes etc. Analysis of this problem is a very interesting area of research. Linear programming bounds are the most powerful known techniques to produce upper bounds in such problems. In this paper we discuss the optimal packing in two and three dimension and relation between the packing and coverage problem.

The coverage problem has many applications in wireless sensor networks (WSNs). WSNs usually consist of a large number of small sensors equipped with some processing circuit, and a wireless transceiver. The sensors have small size, low battery capacity, small processing power and each sensor contains a low-power radio. They can measure distance, direction, speed, humidity, wind speed, temperature, light, and various other parameters. One of the unique features of a WSN is that they can be randomly deployed in inaccessible terrains.

Because of this surveillance goal, *coverage* is important for any sensor network. In order to fulfill its designated surveillance tasks, a sensor network must cover the Region of Interest (ROI) without leaving any *internal sensing hole*. The sensors can detect events inside a surrounding disc (called sensing disc) of some radius (called sensing radius). The aim of the well-known coverage problem is to place sensors in ROI such a way that they cover the ROI with minimum number of sensors or to cover maximum area of ROI by a fixed number of sensors.

Placement of the sensors may be done in two different ways deterministic placement and random deployment from air. In case of deterministic placement ROI can be fully covered with a sufficient number of sensors. But in case of random deployment some points ROI may be uncovered even if we used large number of sensors. Coverage is main goal for WSN but due shortage of sensors or random deployment of sensors we cannot avoid uncovered region fully. In that case we should calculate the uncovered area. To the best of our knowledge there is no work on uncovered area in random deployment scenario. In this paper our target is to calculate and develop strategies to reduce the uncovered area of ROI.

We consider a completely new problem: how the uncovered area changed with the number of sensors or how the uncovered area depends on the strategy of deployment of the sensors when the sensors are deployed on ROI in a random manner. It is enough to cover each point of ROI by exactly one sensor, so if some portion of the ROI is covered by more than one sensor then we have in some sense 'wastage' of sensing area of sensors. Since the sensing area of a sensor is a circular disc, we cannot avoid the wastage. Our target will be reducing the wastage. One idea is deploying the sensor in some pre-deterministic points such that if they actually placed on that points then the wastage is minimum or in the other words, coverage is maximum. In this paper we try to find the answer of this problem. Now after deployment there will be some uncovered area due to lack of sensors or due to random placement of the sensors. So, to reduce the uncovered area we need some extra sensors. The problem is that, how we deploy this extra sensor such that the wastage is minimize. In this paper we try to give some idea of the solution of this problem.

In this paper, we try to solve the coverage problem in $\mathbb{R}^3$. We have described different coverage criteria and studied the minimum number of sensors to cover an area. We also consider that sensors may not be placed at the required target but may be placed at any point in the plane. We assume that the distance between these two points follows i.i.d. uniform or normal distribution. For uniform we calculate theoretically the uncovered area of ROI. For both the distributions we have done computer simulations. To reduce the uncovered area or volume we have introduced two different strategies using extra sensors and have compared these two strategies. We see that one strategy is better for distributions with higher variance and other strategy is better for distributions with smaller variance.

Application of WSNs are as follows. WSNs are used for measuring distance, direction, speed, humidity, wind speed, temperature, light, etc. They also provide co-operative effort that offers unprecedented opportunities for a broad spectrum of civilian and military applications; such as industrial automation, military tactical surveillance, national security, emergency health care, etc. Sensor networks aim at monitoring their surroundings for event detection and object tracking also.

### *1.1. Related Work*

Many works have been done in packing problem as it has many applications in error-correcting codes, spherical codes etc.[5, 6]. Analysis and application of this problem is also done in various interesting area of research [4]. Linear programming bounds are the most powerful known techniques is producing upper bounds in such problems [7].

Several works have been done to find an efficient algorithm for placing sensors to cover a specific convex region in $\mathbb{R}^2$, like squares and equilateral triangles. When the set is a convex and bounded set, the problem is referred to as *covering problem* in literature. Several variations can be found in [8, 11]. Silva et al. present homological criteria for covering in two-dimensional space [17]. Also, a number of movement-assisted sensor placement algorithms have been proposed. An survey on these topics is presented by Li et al. [14]. Fletcher et al. [9] present randomized algorithms using more than one robot for coverage repair in WSN. They propose two algorithms for grid-based ROI and simulate the path travelled by the robots for different values of parameters (number of sensors, number of robots etc.). There is an extensive literature on sensor positioning and repositioning. Younis and Akkaya [18] provide a survey of models, requirements and strategies that would affect sensor deployment.

Li et al. describe carrier-based sensor relocation by robot to repair sensing holes [15]. They consider grid structure of ROI and use virtual force algorithm. Analysis are done for maximum distance covered and expected distance covered by the robot(s) or by the mobile sensors to achieve the full coverage [12, 16]. There are several works on connectivity and energy saving strategy also. Three-dimensional deployment of sensors using lattice pattern is considered [1]. Almost all the works shows their efficiency in terms of energy, either the consumption battery power of sensors or the length traveled by the robot(s). In almost all previous works on WSNs, the uncovered area is covered either by placing new sensors using an actuator (known as Actuator-Sensor Networks) or by activating passive sensors. Nandi and Li develop an algorithm for actuator to reduce the uncovered area [16].

One variation of the coverage problem with minimum wastage available in literature is the case when centers of the congruent discs are fixed and the objective is to cover a given set of points with minimum number of discs. Stochastic formulations of variations of this problem can be found in [2]. There is another problem on coverage, called $k$-coverage. If every point of the ROI is covered by not less than $k$ many sensors then ROI is called $k$-covered [20]. Sensors may detect the direction of other sensors and the desired event(s), the network consists of these types of sensors are known as direction sensor networks. One can suitably activate some passive sensors and deactivate the active sensor such a way that the life time of the network is maximize [19].

## 2. Assumptions and Definitions

In this Section we define different terms and prove some theoretical results.

**Packing of a disc:** Consider a disc $D$ of radius $r$, centered at origin. A packing or covering of this disc is a set of complex numbers $S \subset \mathbb{C}$ if $D \subset \bigcup_{x \in S} B(x, 1)$, where $B(x, 1)$ be the unit disc centered at $x$.

**Hexagonal packing:** Consider a disc $D$ of radius $r$, centered at origin. Hexagonal packing of this disc is a packing $S^{(n)}$ if $S^{(n)} = \bigcup_{k=0}^{n} S_k$ for some nonnegative integer $n$, where $S_0 = \{0\}$ and for positive integer $k$, $S_k = \bigcup_{t=0}^{5} S_{k,t}$, where $S_{k,0} = \{\frac{3}{2}l + i\frac{\sqrt{3}}{2}(2k - l) : l = 0, 1, \ldots, k - 1\}$ and for $j = 1, 2, \ldots, 5$, $S_{k,j} = exp(i\pi/3)S_{k,j-1} = \left\{exp(i\pi/3)\left(\frac{3}{2}l + i\frac{\sqrt{3}}{2}(2k - l)\right) : l = 0, 1, \ldots, k - 1\right\}$.

Note that, this packing is not unique, if $S^{(n)}$ is a packing of $D$ for some nonnegative integer $n$ then for larger $n$ also $S^{(n)}$ is a packing for the same $D$.

Consider the coverage problem in WSNs which are composed of static sensors (equivalently, sensing discs) dropped stochastically in a region of interest (ROI). The ROI is partitioned into several identical regular hexagons of side $a$. Although the topology of the ROI may so that partitioning into hexagons is not possible, if we consider the ROI to be the whole of $\mathbb{R}^2$, we do not have such a problem. To cover each hexagon by one sensor one should take $a \leq r$, where $r$ is the sensing radius. If $r = a$ each regular hexagon is covered by the sensor (also known as node) at its center if the sensor is placed exactly at the center of that hexagon. We assume that the ROI is $\mathbb{R}^2$ or a convex and bounded subset of $\mathbb{R}^2$. Consider that the sensors are so small, that, we can think of a sensor as a point.

Now we define some useful terms. *Node* is the point where a sensor or the center of a disc is placed after deployment. In this paper we use the term node to mean the point as well as the corresponding sensor. *Vertex* is the point where the center of a disc is targeted to be placed. $N(V)$ is the node correspond to the vertex $V$, i.e., the center of a disc is placed at $N(V)$ when the target was to drop at $V$. Similarly, $V(N)$ is the corresponding vertex of a node $N$. *Sensing Disc* $S_N$ of a node $N$ is a closed disc of radius $r$ and center $N$, which is covered by the disc or sensor placed at that node. The radius $r$ is known as *Sensing Radius*, which is assumed to be same for all discs. More generally one can consider discs with different radius. Throughout the paper, by the word 'disc' we consider closed discs only. In higher dimensions we call this *Sensing Ball*.

*Adjacent vertex* of a particular vertex means the vertex which is at the distance not more than $2r$ from that particular vertex. Therefore, the sensing disc of a vertex has nonempty intersection with the sensing disc of its adjacent vertex and empty intersection with the sensing disc of a non-adjacent vertex (which is not an adjacent vertex).

$\mathcal{V}$ is the set of all vertices and $Adj_V$ is the set of all the adjacent vertices of a vertex $V$ (see Figure **??**). Similar definitions and notations apply for nodes also, and the respective notations are $\mathcal{N}$ and $Adj_N$ for $N \in \mathcal{N}$. Denote distance between two points $A$ and $B$ in $\mathbb{R}^n$ as $d(A, B)$. A point $P \in \mathbb{R}^n$ is said to be covered by a node $N$ if $d(P, N) \leq r$ and the point $P$ is said to be covered by a set of nodes $\mathcal{N}$ if $P$ is covered by at least one node in $\mathcal{N}$. A point $P \in \mathbb{R}^n$ is said to be uncovered by a node $N$ if it not covered by $N$ and the point $P$ is said to be uncovered by $\mathcal{N}$ if $P$ is uncovered by all the nodes in $\mathcal{N}$. Note that when there is no randomness, then the vertex and the corresponding node is same, i.e., $N(V) = V$ and $V(N) = N$. *Sensing hole* in $\mathbb{R}^n$ (resp. ROI) is a connected subset of $\mathbb{R}^n$ (resp. ROI) whose elements are uncovered by $\mathcal{N}$. *Adjacent sensing hole* of a particular node means the sensing hole whose boundary intersects with the boundary of the sensor disc of that node. $\mathbb{R}^n$ (resp. ROI) will be called *covered by a set of nodes of sensing radius* $r$ if every point of $\mathbb{R}^n$ (resp. ROI) is covered by at least one node. Volume of a set $B$ will be denoted as $\mathrm{Vol}(B)$.

Now we will define an important term say, wastage. Let $S$ be a bounded subset of $\mathbb{R}^n$, which is covered by a set of finite nodes $\mathcal{N}$. Define the wastage in $S$ for $\mathcal{N}$ as

$$W_{\mathcal{N}}(S) = \frac{\sum_{N \in \mathcal{N}} \text{Vol}(S \cap S_N) - \text{Vol}(S)}{\sum_{N \in \mathcal{N}} \text{Vol}(S \cap S_N)}.$$

If $\mathcal{N}$ be such that $|S_{N_1} \cap S_{N_2} \cap S_{N_3}| \leq 1$ for distinct $N_1, N_2, N_3 \in \mathcal{N}$ (see figure 1a), then

$$W_{\mathcal{N}}(S) = \frac{\sum_{N_1 \neq N_2 \in \mathcal{N}} \text{Vol}(S \cap S_{N_1} \cap S_{N_2})}{\sum_{N \in \mathcal{N}} \text{Vol}(S \cap S_N)}.$$

Intuitively, the denominator represents the sum of the volume (common with $S$) of all spheres. Numerator denotes the difference between the previous volume and the volume what we cover by these spheres, i.e., the volume of the sets whose points are covered by exactly two sensors, which can be thought as the wastage (in layman sense) of volume. Hence wastage the represent the proportion of wastage to the total volume.

Let $\mathcal{N}$ be a set of nodes which cover $\mathbb{R}^n$ such that $\mathcal{N} \cap S$ is finite for any bounded subset $S$ of $\mathbb{R}^n$. Then wastage in $\mathbb{R}^n$ for $\mathcal{N}$ is defined by

$$W_{\mathcal{N}}(\mathbb{R}^n) = \lim_{x \to \infty} W_{\mathcal{N} \cap B_x}(B_x),$$

where $B_x$ be the ball in $\mathbb{R}^n$ of radius $x$ and centered at origin (equivalently, at any point).

Intuitively, wastage in $\mathbb{R}^n$ is the proportion of wastage volume in $\mathbb{R}^n$. Note that we can take any increasing sequence of sets whose union is $\mathbb{R}^n$ other than $B_x$, e.g., for $n = 2$ partitioned $\mathbb{R}^2$ into hexagons or octagons and then take an increasing sequence of union of finitely many such polygons with the property that limit of this sequence is $\mathbb{R}^2$. In that case we can similarly define wastage. It can be proved that these two definitions are equivalent.

## 3. A Result on Coverage Problem

**Theorem 1.** *Let ROI be a bounded and convex subset of $\mathbb{R}^n$ and number of nodes in ROI is finite. Then the ROI is covered by the set of nodes $\mathcal{N}$ if and only if any interior point of the ROI which also belongs to boundary (i.e., circumference) of a sensing ball belongs to another sensing ball.*

*Moreover, if ROI $\subset \mathbb{R}^2$ then ROI is covered by a set of nodes if and only if the set of interior points of ROI which is on the boundary of a sensing ball of a node and does not belong to the interior of any other sensing ball, is finite.*

*Proof.* Let the ROI be covered by a set of nodes $\mathcal{N}$. Suppose there is point $A$ which belongs to the intersection of the boundary of sensing ball $S_N$ of node $N$ (denoted as $Bd(S_N)$) and interior of the ROI, but does not belong to the any other sensing ball. Then $d(A, N) = r$ and $d(A, N') > r$ for all $N' \in \mathcal{N} \setminus \{N\}$. Let $d = \min\{d(A, N') : N' \in \mathcal{N} \setminus \{N\}\}$. Therefore, $d > r$ as $\mathcal{N}$ is finite. Hence the ball $B_{(d-r)/2}(A)$ with center $A$ and radius $(d-r)/2$ has no intersection with the sensing ball of any node except one node $N$. Since $A \in Bd(S_N), B_{(d-r)/2}(A) \not\subset S_N$, hence ROI cannot be covered by $\mathcal{N}$.

Moreover, if ROI $\subset \mathbb{R}^2$, the set of interior points of ROI which is on the boundary of the sensing disc of a node and does not belong to the interior of any other sensing disc is finite because there are finitely many points which belong to the intersection of boundaries of more than one sensing discs.

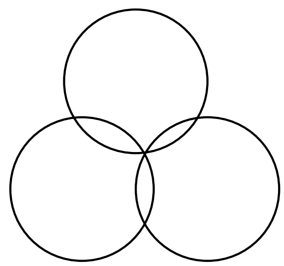
Conversely, let ROI be not covered by a set of nodes $\mathcal{N}$. Then there is a point $A$ such that $d(A, N) > r$ for all $N \in \mathcal{N}$. Let, $d = \min\{d(A, N') : N' \in \mathcal{N}\}$, then the boundary of the ball $B_{d-r}(A)$ intersects at most one point with boundary of each sensing ball and there is a sensing ball whose boundary, say $Bd$, such that $Bd \cap B_{d-r}(A)$ is a singleton set, say, $\{B\}$. Then $B$ is a point which belongs to the intersection of the boundary of sensing ball of a node and interior of the ROI but neither belongs to the interior nor on the boundary of any other sensing ball.

Moreover, if ROI $\subset \mathbb{R}^2$, in that case, the set of interior points of ROI which is on the circumference of sensing disc of a node and does not belong to the interior of any other sensing disc is infinite (a suitable arc containing $B$). $\square$

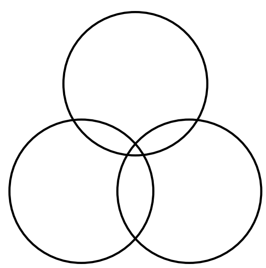
**Remark 1.** *It seems that when $\mathbb{R}^n$ is covered by a set of nodes with minimum wastage then intersection of interior of three balls centered at three distinct nodes is empty. In this paper we consider only the situations like Figure 1a and 1c but not like Figure 1b.*

**Remark 2.** *Consider the regular unit hexagons centered at elements of $S^{(n)}$ then interior these hexagons are disjoint and two different hexagons have at most one side common and three different hexagon intersect at most at a single point. If union of these hexagons cover a disc then their centers form a packing of that disc. Throughout the paper by the word hexagon we mean hexagon with its interior.*

**Remark 3.** *A hexagon $H(x)$ centered at $x \in \mathbb{C}$ is a subset of $B(x,1)$ and $\bigcup_{x\in S^{(n)}} H(x)$ is simply connected subset of $\mathbb{C}$. Hence, if $D \subset \bigcup_{x\in S} H(x)$ for some $S \subset S^{(n)}$ then $S$ is a packing of $D$.*



(a)



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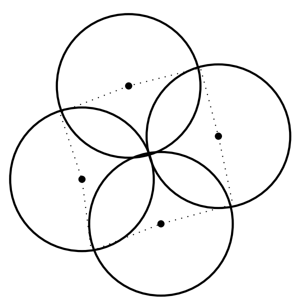

(c)

Figure 1: Sensing discs in different situations

**$k$-th class hexagon:** Define $H_k = \bigcup_{x\in S_k} H(x)$ and we say any hexagon whose center is a point of $S_k$ as a $k$-th class hexagon.

Also define $H^{(n)} = \bigcup_{k=0}^{n} H_k$ and $H_{k_i}$ is the set of all points of $i$-th hexagon in the $k$-th class.

**Remark 4.** *Total number of hexagon of $k$-th class is $6k$.*

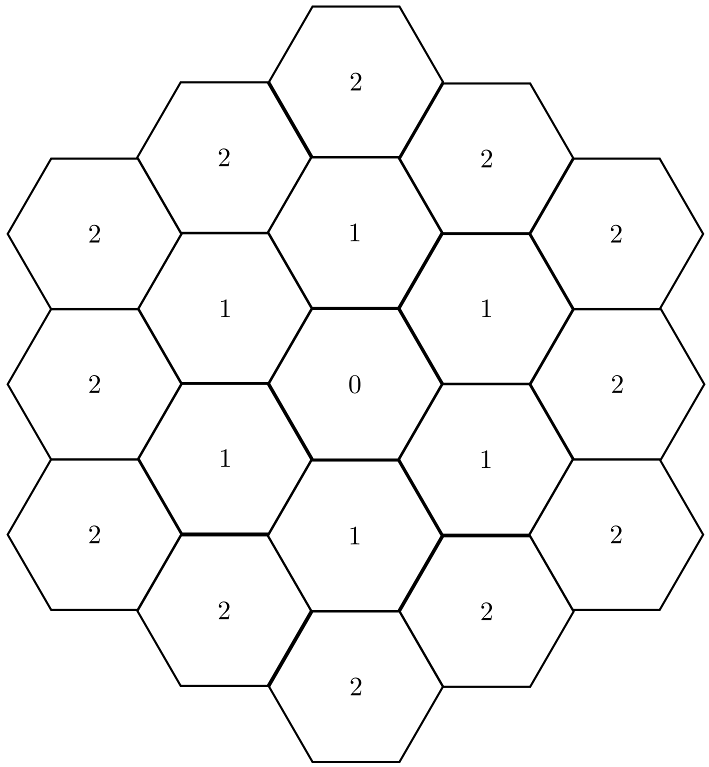


Figure 2: Different classes of hexagons (center of 0-th class hexagon is the origin)

**Theorem 2.** *Marked vertices in Figure3 ($C, E$ etc.) are nearest points in $Bd(S_n) \cap H_{n_i}$ from origin.*

*Proof.* The line $CD$ is parallel to horizontal axis, hence $\angle OCD > \pi/2$. So, any point on $CD$ has a greater distance than $C$ from $O$. Similar thing happens for $DE$ also. ☐

**Theorem 3.** *Between two rays from origin and perpendicular to two consecutive sides of class 0 hexagon, there are $n-1$ class $n$ hexagon.*

*Proof.* Centre of $A$ is $(0, \sqrt{3}n + \frac{\sqrt{3}}{2})$. Centre of $B$ is $(\frac{3n}{2}, \frac{\sqrt{3}n}{2})$. Centre of $n_i$-th hexagon is $(\frac{3i}{2}, \frac{\sqrt{3}}{2}i)$. Coordinate of marked corner is $(\frac{3i-1}{2}, \frac{\sqrt{3}}{2}(2n+1-i))$ for $i = 1, 2, \cdots, n$. Distance from origin is

$$\begin{aligned} t_{n,i} &= \frac{1}{2}\sqrt{(3i-1)^2 + 3(2n+1-i)^2} \\ &= \sqrt{3i^2 - 3i(n+1) + (3n^2+3n+1)}. \end{aligned}$$

Minimum value is

$$t_n = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd }, \\ \frac{\sqrt{(3n+1)^2+3}}{2} & \text{if } n \text{ is even }, \end{cases} \quad (1)$$

☐

**Theorem 4.** *Centers of $H_{k_i}$ form a hexagon of for each $n$.*

*Proof.* Starting from the center lies on the positive part of the vertical axis $k+1$ consecutive centers are collinear. Then starting from the $k$-th centered $k+1$ consecutive centers are collinear and so on. Since there are $6k$ many center we have the theorem. ☐

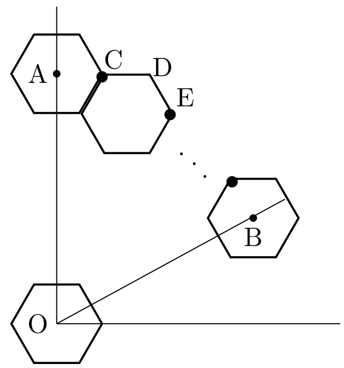


Figure 3: Nearest point on the boundary from origin

**Input**: $R$.

$n = \lceil \frac{R}{\sqrt{2}} \rceil$;
$s = 0$;
**for** $i = 1, 2, \ldots, n$ **do**
    $r = \lceil \frac{R^2 - 2(i-1)^2}{\sqrt{2}} \rceil$;
**end**
$s = s + 4r$;
Report $s$;

**Algorithm 1:** Square Tiling Algorithm.

**Input**: $R$.

Find $N$ s.t. $t_{(N-1)} < R <= t_N$;
$s = 1$;
**for** $n = 1, \ldots, N$ **do**
    $t[n, 0] = \frac{\sqrt{3}(2n+1)}{2}$ ;
    **for** $i = 1, \ldots, n$ **do**
        $t[n, i] = \frac{\sqrt{(3i-1)^2 + 3(2n+1-i)^2}}{2}$;
    **end**
**end**
**for** $n = 2, \ldots, N$ **do**
    **for** $i = 0, \ldots, n-1$ **do**
        **if** $t[n-1, i] < R$ **then**
            $s = s + 1$;
        **end**
    **end**
**end**
Report $1 + 6s$;

**Algorithm 2:** Hexagonal Tiling Algorithm.

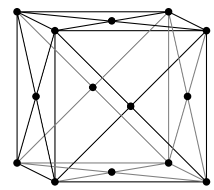

Figure 4: Face-Centered Cube (dots are center of spheres)

## 4. Sphere Packing and Coverage Problem with minimum Wastage

The sphere packing problem in $\mathbb{R}^n$ is trivial for $n = 1$. For $n = 2$, the standard hexagonal packing is optimal. For $n = 3$ it was a long-time open problem and for $n \geq 4$ the problem remains unsolved. For $n = 3$, Hales has proved that the face-centered cube packing is optimal [10]. Some basic background on sphere packing problem may be found in [3]. In two dimensional case, sphere packing problem known as circle packing problem. Circle packing problem is to arrange circles (of equal or varying radii) on a given surface such that no overlapping occurs and so that all circles touch another. A circle packing algorithm is presented in [13]. We use the idea of sphere packing problem to find the answer of the first problem. We use exactly the same patterns for sphere packing in our covering in two and three dimensional space.

The hexagonal placement is optimal for sphere packing problem in $\mathbb{R}^2$. We discuss the hexagonal placement of nodes for coverage problem in $\mathbb{R}^2$ in previous two sections. It is well known that face-centered cube packing (see Figure 4) is optimal for sphere packing problem in $\mathbb{R}^3$ and for $n > 3$ optimal placement is not known [10]. In various situations WSNs may be three dimensional. In this section we discuss a similar type of placement of nodes (similar to face-centered cube packing) to cover $\mathbb{R}^3$. Consider the set $\mathcal{N} = \{(2k, 2l, 2m) : k, l, m \in \mathbb{Z}\} \cup \{(2k + 1, 2l + 1, 2m) : k, l, m \in \mathbb{Z}\} \cup \{(2k + 1, 2l, 2m + 1) : k, l, m \in \mathbb{Z}\} \cup \{(2k, 2l + 1, 2m + 1) : k, l, m \in \mathbb{Z}\}$.

We partition $\mathbb{R}^3$ as a cube grid and take the nodes at the 8 corners and the center of the 6 faces of all the cubes. If $r$ be the sensing radius then we consider set of nodes $\{rN : N \in \mathcal{N}\}$. The placement of nodes in our case is similar to the choice of center of spheres in face centered cube packing, only difference being that the distance between the two nodes is less in our case which confirms the covering.

**Theorem 5.** *Consider a partition of cube $C$ of side $2nr$ unit into $n^3$ many cubes of side $2r$ unit. Then number of nodes required to cover the cube $C$ is $4n^3 + 6n^2 + 3n + 1$, where the nodes are placed as discussed above (similar to face-centered cube). The proportion of wastage volume is $1 - \frac{8n^3}{(4n^3+6n^2+3n+1)\times\frac{4}{3}\pi}$, which is approximately $1 - \frac{3}{2\pi}$ for large $n$.*

*Proof.* Clearly there are $(n+1)^3$ many corner nodes and $n^2(n+1)$ many nodes at the center of faces parallel to one of the three coordinate planes. Hence number of nodes is $(n+1)^3 + 3n^2(n+1) = 4n^3 + 6n^2 + 3n + 1$.

We need $4n^3 + 6n^2 + 3n + 1$ spheres of radius $r$ to cover the cube of side $2nr$. Total volume of the spheres is $(4n^3 + 6n^2 + 3n + 1)\frac{4}{3}\pi r^3$ and they cover volume of $8n^3r^3$ units. Hence the proportion of wastage volume is $\frac{(4n^3+6n^2+3n+1)\times\frac{4}{3}\pi r^3 - 8n^3r^3}{(4n^3+6n^2+3n+1)\times\frac{4}{3}\pi r^3}$. Hence the result. □

### *4.1. Simulation Results*

We consider $n = 13$ and $r = 1$. So, we have 9842 many nodes. Hence the volume of ROI is $13^3 \times 2^3$ unit. We simulate the proportion for covered volume using two strategies, Strategy 1 (St. 1) and Strategy 2 (St. 2), which are exactly same as in case of $\mathbb{R}^2$. If we use $p$% extra nodes then for Strategy 2, we have to partition ROI of volume $13^3 \times 2^3$ unit into $m^3$ many cubes where $4m^3 + 6m^2 + 3m + 1 = 9842 \times (1 + \frac{p}{100})$.

We simulate the proportion of coverage for two different strategies, for five different distributions as described in the previous section (in case of $\mathbb{R}^2$) and for nine different values of $p$ (see Table 1). Uniform distribution with parameter $t$ has the density function $f(x) = \frac{3x^2}{t^3} I_{(0,t)}$. It is noted from the simulation results that, St. 1 is better than St. 2 in higher variance cases. For lower variance cases St. 2 is better for most values of $p$. This observation is almost same as in the two-dimensional case. So, we can conclude that Strategy 1 is better for distributions with higher variance and Strategy 2 is better for distributions with lower variance.

Table 1: Simulation for proportion of coverage area for two strategies in $\mathbb{R}^3$

| | $U(0.5)$ | | $U(1)$ | | $N(0, 0.10)$ | | $N(0, 0.25)$ | | $N(0, 0.50)$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | St. 1 | St. 2 | St. 1 | St. 2 | St. 1 | St. 2 | St. 1 | St. 2 | St. 1 | St. 2 |
| 0.00 | 0.97091 | 0.97090 | 0.92352 | 0.92331 | 0.96579 | 0.96419 | 0.93854 | 0.93470 | 0.93480 | 0.93301 |
| 0.05 | 0.96900 | 0.97104 | 0.92905 | 0.92918 | 0.96853 | 0.96876 | 0.94403 | 0.94552 | 0.94084 | 0.93266 |
| 0.10 | 0.97590 | 0.97374 | 0.93385 | 0.93346 | 0.96949 | 0.97082 | 0.94702 | 0.94751 | 0.94383 | 0.94559 |
| 0.15 | 0.97752 | 0.97814 | 0.94230 | 0.93677 | 0.97233 | 0.97790 | 0.95402 | 0.95051 | 0.95284 | 0.94584 |
| 0.20 | 0.98108 | 0.98248 | 0.94602 | 0.94501 | 0.97968 | 0.97653 | 0.96180 | 0.95500 | 0.95301 | 0.95184 |
| 0.25 | 0.98531 | 0.98503 | 0.95684 | 0.94392 | 0.98232 | 0.97869 | 0.96454 | 0.95771 | 0.96202 | 0.96002 |
| 0.50 | 0.99179 | 0.99148 | 0.97301 | 0.96253 | 0.99134 | 0.98588 | 0.97873 | 0.97053 | 0.98028 | 0.96689 |
| 0.75 | 0.99552 | 0.99451 | 0.99033 | 0.97067 | 0.99579 | 0.99357 | 0.99302 | 0.98172 | 0.98701 | 0.97846 |
| 1.00 | 0.99847 | 0.99753 | 0.99450 | 0.97950 | 0.99802 | 0.99518 | 0.99530 | 0.98653 | 0.99428 | 0.98272 |

## 5. Conclusion

In this paper, we try to solve the coverage problem in $\mathbb{R}^i$ for $i = 2, 3$ which is similar to packing problem. We show the connection between the coverage and packing problem in continuous domain. We have described different coverage criteria and studied the minimum number of sensors to cover an area. We also consider that sensors may not be placed at the required target but may be placed at any point in the plane. We assume that the distance between these two points follows i.i.d. uniform or normal distribution. For uniform we calculate theoretically the uncovered area of ROI. For both the distributions we have done computer simulations. To reduce the uncovered area or volume we have introduced two different strategies using extra sensors and have compared these two strategies. We see that strategy 1 is better for distributions with higher variance and strategy 2 is better for distributions with smaller variance.

In future, we will try to find the theoretical results for normal distribution and will consider the coverage problem for higher dimensions to find optimal placement of sensors. We will consider ROI as a square grid in two dimensions and the other interesting distributions. Here we consider only two strategies but there may be other strategies which may be better for some specific distributions. We will try to classify them in future with respect to uncovered volume for different distributions and different type of partitions. In future, we will try to solve coverage problem of deployment of sensor randomly, and dropping of extra sensors and use of actuators.

We expect in future work that strategy 1 may be better for any arbitrary probability distributions with higher variance and strategy 2 may be better for any arbitrary probability distributions with smaller variance. May be proper mixing of these two strategies is better in many situations. By mixing we mean that some of the extra sensor are used for increasing the side length and rest are used for double dyploying.

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