Equivalents of Some Ordered Fixed Point Theorems

ABSTRACT. Some ordered fixed point theorems on metric spaces are equivalent to completeness and existences of maximal (or minimal) elements, common fixed points, common stationary points, etc. Some known or new theorems related to the Caristi fixed point theorem can be equivalently formulated. Consequently, dual versions of the Ekeland principle (1972-74), the Caristi theorem (1976, 1979), theorems of Bae-Park (1983), Takahashi (1991), Chen-Cho-Yang (2002), Jachymski (2003), Cobzaş (2022), and others are substantially improved and strengthened.

Key words and phrases. The 2023 Metatheorem, Brøndsted-Jachymski Principle, preorder, metric space, anti-progressive maps, fixed point, stationary point, maximal element.

Mathematics Subject Classification (2020) 03E04, 03E25, 06A06, 06A75, 47H10, 54E35, 54H25, 58E30, 65K10.

1. INTRODUCTION

Since the appearances of the Ekeland variational principle [11, 12] in 1972-74 and the Caristi fixed point theorem [5] in 1976, almost one thousand papers were published on their equivalents, generalizations, modifications, applications, and related topics. Many of them are related to new spaces extending complete metric spaces, new metrics or topologies on them, and new order relations extending the so-called Caristi order.

While working on the same subject in 1983-2000, we found a metatheorem on fixed point theory. It claims that certain order theoretic maximal element statements are equivalent to existences of fixed points of progressive maps, stationary points, common fixed points, common stationary points of families of maps or multimaps. Apparently, their dual to minimal element statements can be possible.

After 22 years have passed, we began again to study our Metatheorem in [23,24] and to apply its extended versions in several works in 2022 [25-29]. Recall that Brøndsted [4] in 1976 observed certain case that maximal elements are fixed points, and that Jachymski [14] in 2003 studied when periodic points are same to fixed points. Recently, in 2022, we obtained extended versions of our Metatheorem with the Brøndsted-Jachymski Principle and applied them to various results related maximality and progressive maps in [25-29]. Later, in the end of 2022, we obtained a more general version of them called the 2023 Metatheorem and its applications in [32]. However, works on their counterparts related minimality and anti-progressive maps are quite a few. We began such study in our recent articles [33,34].

Motivated by such situation, in the present article, we begin with various forms of Maximal (or Minimal) Element Principles and apply them to known or new works related to minimality. Recall that completeness for metric spaces as well as Banach spaces are one of the most important properties of them and a huge number of related papers appeared already; for example, see Cobzaş [8] in 2020. Our another aim in this article is to apply our Principles to equivalent formulations of metric completeness mainly appealing to the Caristi fixed point theorem. Consequently, in this article, we show that some ordered fixed point theorems on metric spaces are equivalent to completeness, existences of maximal (or minimal) elements, common fixed points, common stationary points, and others.

We are based on our 2023 Metatheorem, a prototype of Maximal (or Minimal) Element Principles, and the Brøndsted-Jachymski Principle due to ourselves.

This article is organized as follows: In Section 2, we introduce the 2023 Metatheorem, which is the basis of our ordered fixed point theory. Based on this, we derive several Maximal or Minimal Element Principles in Section 3. From these principles, in Section 4, we introduce the Brøndsted Principle and the Brøndsted-Jachymski Principle. Section 5 devotes to improvements of the Caristi and Zermelo fixed point theorems. In Section 6, we obtain equivalent formulations of the Caristi theorem and its dual. Section 7 is to improve the Bae-Park's generalization [1] of the Caristi theorem and its variants. In Section 8, based on Cobzaş [7], our Metatheorem is applied to equivalencies of the weak Ekeland Principle, the Takahashi Principle, the Caristi theorem, and others. Section 9 is to introduce Jachymski's 2003 Theorem [13] on the equality of periodic point sets and fixed point sets. In Section 10, we derive a consequence of Metatheorem showing the role of the whole space X in the power set $\mathcal{P}(X)$. Finally, Section 11 is for conclusion or epilogue.

In this article all spaces are nonempty and all multimaps are nonempty valued.

2. Basic Principles

In order to obtain some equivalents of the well-known central result of Ivar Ekeland [11, 12] on the variational principle for approximate solutions of minimization problems, we deduced a Metatheorem [20,22] and related works in 1983-2000. Later in 2022 we found an extended version of the Metatheorem [25-27, 29]. Certain other related results will appear in [28,30,31,33-35].

Now the following is the 2023 version in [32], where $\neg$ denotes the negation:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x,y \in X$. Then the following propositions are equivalent:*

(α) *There exists an element $v \in A$ such that $G(v,w)$ holds for any $w \in X\backslash\{v\}$.*

($\beta 1$) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x,y)$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

($\beta 2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x,y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ1) *If* $f : A \to X$ *is a map such that* $\neg G(x, f(x))$ *for any* $x \in A$ *with* $x \neq f(x)$*, then* f *has a fixed element* $v \in A$*, that is,* $v = f(v)$*.*

(γ2) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *satisfying* $\neg G(x, f(x))$ *for all* $x \in A$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(δ1) *If* $F : A \multimap X$ *is a multimap such that, for any* $x \in A \backslash F(x)$ *there exists* $y \in X \backslash \{x\}$ *satisfying* $\neg G(x, y)$*, then* F *has a fixed element* $v \in A$*, that is,* $v \in F(v)$*.*

(δ2) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : A \multimap X$ *such that, for any* $x \in A \backslash F(x)$ *there exists* $y \in X \backslash \{x\}$ *satisfying* $\neg G(x, y)$*. Then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v \in F(v)$ *for all* $F \in \mathfrak{F}$*.*

(ϵ1) *If* $F : A \multimap X$ *is a multimap satisfying* $\neg G(x, y)$ *for any* $x \in A$ *and any* $y \in F(x) \backslash \{x\}$*, then* F *has a stationary element* $v \in A$*, that is,* $\{v\} = F(v)$*.*

(ϵ2) *If* $\mathfrak{F}$ *is a family of multimaps* $F : A \multimap X$ *such that* $\neg G(x, y)$ *holds for any* $x \in A$ *and any* $y \in F(x) \backslash \{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in A$*, that is,* $\{v\} = F(v)$ *for all* $F \in \mathfrak{F}$*.*

(η) *If* Y *is a subset of* X *such that for each* $x \in A \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *satisfying* $\neg G(x, z)$*, then there exists a* $v \in A \cap Y$*.*

From now on, this version will be called the 2023 Metatheorem. This guarantees the truth of all items when one of them is true. Since 1985, we have shown nearly one hundred cases of such situation.

Let X be a set and $G(x, y)$ a sentence formula for $x, y \in X$. A *chain* C in X is defined as follows:
(1) C is a nonempty subset of X;
(2) $G(x, x)$ holds for all $x \in C$;
(3) $G(x, y)$ and $G(y, x)$ imply $x = y$; and
(4) for any $x, y \in C$, $G(x, y)$ or $G(y, x)$ holds.

We need the following as a supplement of Metatheorem:

Metatheorem.* *Let* X *be a set,* $G(x, y)$ *a sentence formula for* $x, y \in X$*, and* $S(x) = \{y \in X : G(x, y)\}$*. Let* $x_0 \in X$ *and* $A = S(x_0)$*.*

Consider the following:

(α) *There exists an element* $v \in A$ *such that* $G(v, w)$ *for any* $w \in X \backslash \{v\}$*.*

(θ1) *For* $v \in A$ *and for each chain* C *in* $S(v)$*, we have* $\bigcap_{x \in C} S(x) \neq \emptyset$*.*

(θ2) *For* $v \in A$ *and a maximal chain* C^* *in* $S(v)$*, we have* $\bigcap_{x \in C^*} S(x) \neq \emptyset$*.*

Then (α) $\Longrightarrow$ (θ1) $\Longrightarrow$ (θ2)*.*

Proof. $(\alpha) \Longrightarrow (\theta 1)$: By (α), $v \in S(x_0)$ implies $G(x_0, v)$ and $G(v, x_0)$. Hence $v = x_0$ by (3). Now $C = \{v\}$ is the unique chain in $S(v)$, and $\bigcap_{x \in C} S(x) = S(v) \neq \emptyset$, which proves $(\theta 1)$.

$(\theta 1) \Longrightarrow (\theta 2)$: Let v be as in $(\theta 1)$. By Hausdorff maximal principle (which can be established), there exists a maximal chain C^* in $S(v)$ and from $(\theta 1)$, we have $\bigcup_{x \in C^*} S(x) \neq \emptyset$. Thus $(\theta 1)$ implies $(\theta 2)$. □

3. Maximal or Minimal Element Principles

Let $(X, \preceq)$ be a *preordered set*; that is, X is a nonempty set and the order $\preceq$ is reflexive and transitive.

From our 2023 Metatheorem, we deduced the following prototype of Maximal (resp. Minimal) Element Principles in [32]:

Theorem 3.1. *Let $(X, \preceq)$ be a preordered set and A be a nonempty subset of X. Then the following statements are equivalent:*

(α) There exists a maximal (resp. minimal) element $v \in A$, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for any $w \in X \backslash \{v\}$.

(β) If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(γ) If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preceq f(x)$ (resp. $f(x) \preceq x$) for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(δ) Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A \backslash F(x)$ there exists $y \in X \backslash \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$). Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.

(ϵ) If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $x \preceq y$ (resp. $y \preceq x$) holds for any $x \in A$ and any $y \in F(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.

(η) If Y is a subset of X such that for each $x \in A \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $x \preceq z$ (resp. $z \preceq x$), then there exists a $v \in A \cap Y$.

Remark 3.2. (1) Note that we claimed that $(\alpha)-(\eta)$ are mutually equivalent in Theorem 3.1 and did not say that the items are true. For a counter-example, consider the real line $\mathbb{R}$ with the usual order.

(2) All the elements v's in Theorem 3.1 are same as we have seen in the proof of the 2023 Metatheorem in [32].

(3) When $\mathfrak{F}$ is a singleton, each of $(\beta) - (\epsilon)$ is denoted by $(\beta 1) - (\epsilon 1)$, respectively, These are also logically equivalent to $(\alpha) - (\eta)$ as shown in the 2023 Metatheorem.

Let $(X, \preceq)$ be a preordered set and $F : X \multimap X$ a multimap. For every $x \in X$, motivated by Jinlu Li [18], we denote

$$S_+F(x) := \{z \in X : u \preceq z \text{ for some } u \in F(x)\},$$
$$S_-F(x) := \{z \in X : z \preceq u \text{ for some } u \in F(x)\}.$$

Especially, we follow Cobzaş [9]: For the identity map $F = 1_X$ and $x \in X$, put

$$S_+(x) = \{z \in X : x \preceq z\}, \quad S_-(x) = \{z \in X : z \preceq x\}.$$

Note that S_+ is denoted by S sometimes.

A partially ordered set (poset) is a preordered set having the anti-symmetric order.

From Theorem 3.1, we have several variants:

Theorem 3.3. *Let $(X, \preceq)$ be a partially ordered set, $F : X \multimap X$ be a multimap, $x_0 \in X$ such that $A = (S_+F(x_0), \preceq)$ has an upper bound (resp. $A = (S_-F(x_0), \preceq)$ has a lower bound) $v \in A$.*

Then the equivalent statements $(\alpha) - (\eta)$ of Theorem 3.1 hold.

Proof. It suffices to show Theorem 3.1(α) for the maximal case: Since $z \preceq v$ for any $z \in A = S_+F(x_0)$, $u \preceq z$ for some $u \in F(x_0)$. Since $u \preceq z \preceq v$, we have $v \in S_+F(x_0)$. Hence v is a maximal element of A. □

For the identity map $F = 1_X$, Theorem 3.3 reduces to the following:

Theorem 3.4. *Let $(X, \preceq)$ be a partially ordered set, $x_0 \in X$ such that $A = (S_+(x_0), \preceq)$ has an upper bound (resp. $A = (S_-(x_0), \preceq)$ has a lower bound) $v \in A$.*

Then the equivalent statements $(\alpha) - (\eta)$ of Theorem 3.1 hold.

From Metatheorem*, we have the following in [34]:

Theorem 3.5. *Let $(X, \preceq)$ be a partially ordered set, $x_0 \in X$, and $A = S_+(x_0)$ (resp. $A = S_-(x_0)$) have an upper bound (resp. a lower bound). Then the following equivalent statements hold:*

(α) *There exists a maximal (resp. minimal) element $v \in A$, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for any $w \in X \backslash \{v\}$.*

($\theta 1$) *There exists $v \in A$ such that, for each chain C in $S_+(v)$ (resp. $S_-(v)$), we have $\bigcap_{x \in C} S_+(x) \neq \emptyset$ (resp. $\bigcap_{x \in C} S_-(x) \neq \emptyset$).*

($\theta 2$) *There exist $v \in A$ and a maximal chain C^* in $S_+(v)$ (resp. $S_-(v)$), we have $\bigcap_{x \in C^*} S_+(x) \neq \emptyset$ (resp. $\bigcap_{x \in C^*} S_-(x) \neq \emptyset$).*

For the motivation of this theorem and its proof, we have a long story as shown in [34]. The conditions ($\theta 1$) and ($\theta 2$) are originated from [13] and Theorem 3.5 extends a part of ([2], Theorem 5.1). See also ([34], Theorem 5.1*).

4. The Brøndsted-Jachymski Principle

For a preordered set $(X, \preceq)$, a selfmap $f : X \to X$ is said to be *progressive* if $x \preceq f(x)$ for all $x \in X$; and *anti-progressive* if $f(x) \preceq x$ for all $x \in X$. Such maps appear in (γ) of Theorems 3.1, 3.3, and 3.4.

Recently, motivated by Brøndsted [4], we adopted the following in Park [28]:

Brøndsted Principle. *Let $(X, \preceq)$ be a preordered set and $f : X \to X$ be a progressive (resp. anti-progressive) map. Then a maximal (resp. minimal) element $v \in X$ is a fixed point of f.*

Note that this principle is just Theorem 3.1 $(\alpha) \Longrightarrow (\gamma)$ with $X = A$.

Remark 4.1. We noticed that, in most applications of this principle for partially ordered sets $(X, \preceq)$, the existence of a maximal (resp. minimal) element is achieved by the upper (resp. lower) bound of a chain in $(X, \preceq)$ as in Zorn's Lemma.

From now on $\mathrm{Max}(\preceq)$ (resp. $\mathrm{Min}(\preceq)$) denotes the set of maximal (resp. minimal) elements of the order $\preceq$, and $\mathrm{Fix}(f)$ (resp. $\mathrm{Per}(f)$) denotes the set of all fixed points (resp. periodic) points of a map $f : X \to X$, respectively.
The following was given by Jachymski ([14], Proposition 1):

Proposition 4.2. *Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$ be progressive. Then* $\mathrm{Per}(f) = \mathrm{Fix}(f)$.

This also holds for anti-progressive maps. Combining this with the Brøndsted Principle, we obtained the following [30]:

Brøndsted-Jachymski Principle. *Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$ be a progressive map. Then we have*

$$\mathrm{Max}(\preceq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

Similarly, if $f : X \to X$ is a anti-progressive map, then we have

$$\mathrm{Min}(\preceq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

This also follows from Theorem 3.1 for a partially ordered set $X = A$ with

(γ') *If $f : X \to X$ is a map such that $x \preceq f(x)$ (resp. $f(x) \preceq x$) for any $x \in X$, then f has a fixed and periodic element $v \in X$, that is, $v = f(v)$.*

Remark 4.3. Note that this gives a new proof of the Brøndsted-Jachymski Principle. This is not claiming the non-emptiness of the three sets there.

Note that Theorems 3.4 and 3.5 imply the following common generalization of the Zermelo fixed point theorem and Zorn's Lemma:

Theorem 4.4. (Zermelo-Zorn) *If a chain in a partially ordered set $(X, \preceq)$ has an upper bound (resp. a lower bound) $v \in X$, then it is a maximal (resp. minimal) element. Moreover, a selfmap $f : X \to X$ is progressive (resp. anti-progressive) if and only if*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \supset \{v\} \quad (\textit{resp.}\ \mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preceq) \supset \{v\}).$$

PROOF. It is enough to prove the maximal case only. Let A be a chain with an upper bound $v \in X$. Let $x_0 \in A$ and $S_+(x_0) = \{x \in A : x_0 \preceq x\}$. Then $(S_+(x_0), \preceq)$ has an upper bound $v \in X$. Then the conclusion follows from Theorems 3.4 and 3.5. □

Example 4.5. We give an example of Theorem 4.4, for which Zorn's Lemma does not work: Let $C = [0,1] \times \{0\}$ and $D = \mathbb{R} \times \{1\}$ be with their natural orders $\preceq$, and $X = C \cup D \subset \mathbb{R}^2$. Let $f : X \to X$ such that,

$$f(x,y) = \begin{cases} (\frac{1}{2}(x+1), 0) & \text{if } (x,y) = (x,0) \in C, \\ (x+1, 1) & \text{if } (x,y) = (x,1) \in D. \end{cases} \tag{1}$$

Then f is progressive, $S_+(0,0)$ has the upper bound $f(1,0) = (1,0)$, which is maximal. Note that the chain D of X does not have any maximal or minimal element. Hence Zorn's Lemma does not work for X, for which Theorem 4.4 holds.

5. Strengthening the Caristi theorem

A real-valued function $f : X \to \mathbb{R}$ on a topological space X is said to be *lower (resp. upper) semi-continuous* (l.s.c.) (resp. u.s.c.) whenever

$$\{x \in X : f(x) > r\} \quad (\text{resp. } \{x \in X : f(x) < r\})$$

is open for each $r \in \mathbb{R}$.

In 2002, Chen-Cho-Yang [7] introduced the following concept of lower semicontinuity *from above*:

Definition 5.1. [7] Let X be a topological space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous from above* at a point $x \in X$ if $x_n \to x$ as $n \to \infty$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply that $\lim_{n\to\infty} f(x_n) \geq f(x)$.

Obviously, the usual lower semicontinuity implies lower semicontinuity from above, but the converse does not hold. In fact, Chen-Cho-Yang [7] gave an example of a function which is lower semicontinuous from above at a point, but not lower semicontinuous at that point.

Similarly, we define the following motivated by Lin-Du [19]:

Definition 5.1.* Let X be a topological space. A function $f : X \to \overline{\mathbb{R}}$ is said to be *upper semicontinuous from below* at a point $x \in X$ if $x_n \to x$ as $n \to \infty$ and $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n) \leq \cdots$ imply that $\lim_{n\to\infty} f(x_n) \leq f(x)$.

Chen-Cho-Yang [7] showed that the Weierstrass theorem, Ekeland's variational principle, and Caristi's fixed point theorem hold for lower semicontinuity from above. See also [35]. In fact, the following is obtained by Chen-Cho-Yang [7]:

Theorem 5.2. (Caristi) *Let (X,d) be a complete metric space and $f : X \to X$ be a map such that for all $x \in X$,*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)) \tag{2}$$

where a function $\phi : X \to \mathbb{R}^+$ is lower semicontinuous from above. Then f has a fixed point.

Note that (X, d) can be made into a partially ordered set by defining

$$x \preceq y \iff \phi(y) \leq \phi(x)$$

for $x,\ y \in D$.

Here we give a new proof of the Caristi Theorem 5.2:

PROOF. Since $\phi : X \to \mathbb{R}^+$ is l.s.c. from above at any $z \in X$, for any $\{x_n\}$ converging to z such that

$$\phi(x_1) \geq \phi(x_2) \geq \cdots \geq \phi(x_n) \geq \cdots \Longrightarrow \lim_{n\to\infty} \phi(x_n) \geq \phi(z)$$

and hence $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq z$. Note that $C = \{z\} \subset S_+(x_1)$ is a chain in $S_+(z)$. Let $v = z \in C$. Then $C = \{v\} \subset \bigcup_{x\in C} S_+(x) \neq \emptyset$. Hence, Theorem 3.5($\theta$1) holds, v is maximal by (α), and our Caristi Theorem 3.4(γ1) holds. □

For the early history of various proofs of the Caristi theorem, see Kirk [17]. Our above proof seems to be elementary.

Recall that Kirk [16] in 1976 showed that a metric space X is complete if and only if the Caristi theorem on X holds. Moreover, Park [22] in 1984 extended to seven equivalent statements for completeness, which showed certain basic proper properties of complete metric spaces. Now the new Caristi theorem 5.2 is equivalent to completeness. Recently in 2022, Cobzaş [8] collected a large number of results on fixed points in ordered structures and their completeness properties.

A map satisfying (2) is called the Caristi map. Now, Theorem 5.2 simply tells that *any Caristi map on a metric space X has a fixed point if and only if X is complete.*

A Caristi map f is progressive by defining $x \preceq f(x)$ if and only if (2) holds for any $x \in X$. Based on the characterization of completeness by Kirk [16] in 1976, by improving it, we have the following:

Theorem 5.3. *A metric space (X, d) is complete if and only if*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset$$

for any Caristi map $f : X \to X$.

We also improved Zermelo's fixed point theorem given implicitly in 1908 [39] as follows in [32]:

Theorem 5.4. *Let $(X, \preceq)$ be a partially ordered set in which a nonempty well-ordered subset has a least upper bound. Then every progressive map $f : X \to X$ has*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset.$$

6. Dual formulation of Caristi theorem

In our previous work [35], we gave equivalent formulations of the Caristi theorem. Now we can add some more equivalent propositions.

From Theorems 3.4, 3.5 and 5.3, we have the following:

Theorem 6.1. *Let $(X, \preceq)$ be a partially ordered metric space, and a function $\varphi : X \to \mathbb{R}^+$ be lower semicontinuous from above such that*

$$x \preceq y \quad \textit{iff} \quad d(x,y) \leq \varphi(x) - \varphi(y) \quad \textit{for} \quad x, y \in X.$$

Then the following statements are equivalent:

(0) *(X, d) is complete.*

(α) *There exists a maximal element $v \in X$; that is, $v \not\preceq w$ for any $w \in X \backslash \{v\}$.*

($\gamma 1$) *If $f : X \to X$ is a map such that $x \preceq f(x)$ for any $x \in X$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

($\gamma 2$) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $x \preceq f(x)$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

($\theta 1$) *There exists $v \in X$ such that, for each chain C in $S(v)$, we have $\bigcap_{x \in C} S(x) \neq \emptyset$.*

($\theta 2$) *There exist $v \in X$ and a maximal chain C^* in $S(v)$, we have $\bigcap_{x \in C^*} S(x) \neq \emptyset$.*

Remark 6.2. (1) Note that (0) $\Longleftrightarrow$ ($\gamma 1$) extends the characterization of completeness in Kirk [16].

(2) Recall that (α) is originated from Ekeland [11,12] where the maximal element $v \in X$ in (α) is called a d-point.

(3) Recall that ($\gamma 1$) extends the one in Caristi [5]. The equivalence of (α) and ($\gamma 1$) extends the one of Brézis-Browder [3]. Moreover, ($\gamma 2$) extends the one given by Kasahara [15] and Siegel [34].

(4) We already gave seven equivalent conditions for metric completeness in [28]. Theorem 6.1 and our Metatheorem give additional properties of complete metric spaces.

The following is a dual of Theorem 6.1; see also [35].

Theorem 6.3. *Let $(X, \preceq)$ be a partially ordered metric space, and a function $\varphi : X \to \mathbb{R}^+$ be upper semicontinuous from below and bounded from above such that*

$$y \preceq x \quad \textit{iff} \quad d(x,y) \leq \varphi(y) - \varphi(x) \quad \textit{for} \quad x, y \in X.$$

Then the following statements are equivalent:

(0) *(X, d) is complete.*

(α) *There exists a minimal element $v \in X$; that is, $w \not\preceq v$ for any $w \in X \backslash \{v\}$.*

(β) If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that for any $x \in X$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $y \preceq x$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(γ) If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $f(x) \preceq x$ for all $x \in X$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(δ) If $\mathfrak{F}$ is a family of multimaps $T : X \multimap X$ such that, for any $x \in X\backslash T(x)$ there exists $y \in X\backslash\{x\}$ satisfying $y \preceq x$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v \in T(v)$ for all $T \in \mathfrak{F}$.

(ϵ) If $\mathfrak{F}$ is a family of multimaps $T : X \multimap X$ such that $y \preceq x$ holds for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.

(η) If Y is a subset of X such that for each $x \in X\backslash Y$ there exists a $z \in X\backslash\{x\}$ such that $z \preceq x$, then there exists an element $v \in Y$.

Proof. Clear from ([35], Theorem D) and Theorem 3.1, □

Remark 6.4. (1) All the elements v's in Theorem 6.3 are same as we have seen in the proofs of Metatheorem or Theorem 3.1.

(2) Theorem 6.3(γ) implies a dual to the Caristi fixed point theorem 5.2 and can be stated as follows:

Theorem 6.5. *Let $(X, \preceq)$ be a partially ordered metric space, and a function $\varphi : X \to \mathbb{R}^+$ be upper semicontinuous from below and bounded from above such that*

$$y \preceq x \quad \text{iff} \quad d(x, y) \leq \varphi(y) - \varphi(x) \quad \text{for} \quad x, y \in X.$$

Then (X, d) is complete if and only if every anti-progressive map $f : X \to X$ has

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preceq) \neq \emptyset.$$

Note that there are nearly one thousand papers related to the Caristi theorem for its extensions, modifications, and applications. However, the contents of this article have something different from them.

7. Revisit to Bae-Park in 1983

In attempting to improve the Caristi fixed point theorem, Kirk has raised the question of whether f continues to have a fixed point if we replace $d(x, f(x))$ by $d(x, f(x))^p$ where $p > 1$ in the original Caristi theorem (cf. Caristi [5]).

In Bae-Park [1], after giving an example showing that Kirk's problem is not affirmative, it is recalled that the Caristi theorem and Theorem 6.1(α) are equivalent (Brézis-Browder [3]). This could be expressed more explicitly as follows by combining Theorems 6.1(α) and (γ1); see Bae-Park [1, Theorem 1]:

Theorem 7.1. *Let (M,d) be a metric space, $\phi : M \to \mathbb{R}^+$ an arbitrary function. Let $\mathfrak{F}$ be the family of all selfmaps of M such that for each $x \in M$, we have*

$$(*) \qquad d(x, f(x))^p \leq \phi(x) - \phi(f(x)) \text{ where } p > 0.$$

Then $v \in M$ is a common fixed point of $\mathfrak{F}$ if and only if v satisfies $\phi(v) - \phi(x) < d(v,x)^p$ for every other point $x \in M$.

In Theorem 7.1, if M is complete and ϕ is lower semicontinuous *from above*, and if $0 < p < 1$, then ϕ has a point $v \in M$ satisfying $\phi(v) - \phi(x) < d(v,x)^p$ for each other point $x \in M$. This extends Ekeland's Theorem 6.1(α). To prove this, take a new metric ρ on M with $\rho(x,y) = d(x,y)/(1 + d(x,y)$ which is equivalent to the original metric d.

Moreover, we obtain the following extension of [1, Theorem 2] and other related results.

Theorem 7.2. *Let (M,d) be a complete metric space and $\phi : M \to \mathbb{R}^+$ a lower semicontinuous function from above. Let $\mathfrak{F}$ be the family of all selfmaps f of M such that for each $x \in M$, we have*

$$d(x, f(x))^p \leq \phi(x) - \phi(f(x)).$$

(1) *If $0 < p \leq 1$, then $\mathfrak{F}$ has a common fixed point.*
(2) *If ϕ has a minimal d-point $v \in M$, then it is a common fixed point of $\mathfrak{F}$.*

This extends corresponding parts of Theorem 6.1.
Furthermore, Theorems 7.1 and 7.2 also can be reformulated by adopting our method in Theorem 3.1. For Theorem 7.2, we have the following:

Theorem 7.3. *Let (X,d) be a partially ordered metric space, a function $\varphi : X \to \mathbb{R}^+$ be lower semicontinuous from above, and $0 < p \leq 1$ such that*

$$x \preceq y \quad \textit{iff} \quad d(x,y)^p \leq \varphi(x) - \varphi(y) \quad \textit{for} \quad x, y \in X.$$

Then the statements in Theorem 6.1 are equivalent. For example,

(0) *(X,d) is complete.*

(α) *There exists a maximal element $v \in X$; that is, $v \not\preceq w$ (that is, $d(v,w)^p > \varphi(v) - \varphi(w)$) for any $w \in X \backslash \{v\}$.*

(γ) *Let $\mathfrak{F}$ be the family of all selfmaps f of X such that for each $x \in M$, we have $x \preceq f(x)$ or*

$$d(x, f(x))^p \leq \phi(x) - \phi(f(x)).$$

Then $\mathfrak{F}$ has a common fixed point $v \in X$.

($\theta 1$) *There exists $v \in X$ such that, for each chain C in $S(v)$, we have $\bigcap_{x \in C} S(x) \neq \emptyset$.*
($\theta 2$) *There exist $v \in X$ and a maximal chain C^* in $S(v)$, we have $\bigcap_{x \in C^*} S(x) \neq \emptyset$.*
Here $S(x) = \{y \in X : x \preceq y\}$ for $x \in X$.

Proof. Note that (γ) is just Theorem 7.2. From (γ), others (α) – ($\theta 2$) routinely follow. □

Note that Theorem 7.3 for $p = 1$ reduces to Theorem 6.1 and that the dual version of them holds; that is, Theorem 6.3 with $d(x, y)^p$ instead of $d(x, y)$.

Moreover, by applying the Brøndsted-Jachymski Principle to Theorem 7.3, we have the following:

Theorem 7.4. *Under the hypothesis of Theorem 6.5, if $f : X \to X$ is progressive (that is, $d(x, f(x)^p \leq \phi(x) - \phi(f(x)$ for each $x \in X$) for $0 < p \leq 1$, then we have*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset.$$

Similarly, if $f : X \to X$ is a anti-progressive map, then we have

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preceq) \neq \emptyset.$$

8. Equivalency due to Cobzaş

Early in 1985, Daneš [10] proved that the Daneš drop theorem, Krasnoselskii-Zabreiko renorming theorem, Browder's generalization of the Bishop-Phelps theorem, Caristi's fixed point theorem, and Ekeland's variational principle are all equivalent. Some others also mentioned some equivalences among extended Ekelend's variational principle, extended Takahashi's minimization theorem, Caristi-Kirk fixed point theorem for set-valued maps, and Oettli-Théra theorem.

Our Metatheorem was originated from the Ekeland Principle which has equivalent forms like the Caristi fixed point theorem, Takahashi's minimization theorem, and many others. Our recent applications of Metatheorem to those theorems were given in [19, 21-24,26-32].

Recently, Cobzaş [9] in 2022 gave versions of Ekeland, Takahashi, Caristi Principles in preordered quasi-metric spaces, and the equivalence to some completeness results for the underlying quasi-metric spaces.

For convenience, Cobzaş [9] formulated these three principles as follows:

Theorem 8.1. (Ekeland, Takahashi and Caristi principles) *Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ a proper bounded below l.s.c. function. Then the following statements hold:*

[wEk] *There exists $z \in \operatorname{dom} \varphi$ such that $\varphi(z) < \varphi(x) + d(x, z)$ for all $x \in X \backslash \{z\}$.*

[Tak] *If for every $x \in \operatorname{dom} \varphi$ with $\varphi(x) > \inf \varphi(X)$ there exists an element $y \in \operatorname{dom} \varphi \backslash \{\mathrm{x}\}$ such that $\varphi(y) + d(x, y) \leq \varphi(x)$, then φ attains its minimum on X, i.e., there exists $z \in \operatorname{dom} \varphi$ such that $\varphi(z) = \inf \varphi(X)$.*

[Car] *If the mapping $f : X \to X$ satisfies $d(f(x), x) + \varphi(f(x)) \leq \varphi(x)$ for all $x \in \operatorname{dom} \varphi$, then f has a fixed point in $\operatorname{dom} \varphi$, i.e., there exists $z \in \operatorname{dom} \varphi$ such that $f(z) = z$.*

Here [wEk] means the weak Ekeland principle, [Tak] the Takahashi principle, and [Car] the Caristi fixed point theorem. Moreover, following our way in the present article. the lsc function can be replaced by the function lower semicontinuous from above.

Our Metatheorem can be applied to give equivalencies for various situations as we have shown in our previous works. Motivated by this, we derive the following; see also [35]:

Theorem 8.2. *Let (X, d) be a metric space and $\varphi : X \to \overline{\mathbb{R}}$ a proper l.s.c. function from above and bounded from below (resp. u.s.c. function from below and bounded from above). Let $A = \operatorname{dom}\varphi = \{x \in X : -\infty < \varphi(x) < \infty\}$.*

Then the following statements are equivalent:

(0) (X, d) *is complete.*

(α) *There exists a maximal (resp. minimal) element $v \in A$ such that*

$$d(v,w) > \varphi(v) - \varphi(w) \quad (resp.\ \ d(v,w) > \varphi(w) - \varphi(v))$$

for any $w \in X\backslash\{v\}$. [wEk]

(β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying*

$$d(x,y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x,y) \leq \varphi(y) - \varphi(x)),$$

then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \quad (resp.\ \ d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for all $x \in A\backslash\{f(x)\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$. [Car]

(δ) *Let $\mathfrak{F}$ be a family of multimaps $T : A \multimap X$ such that, for any $x \in A\backslash T(x)$, there exists $y \in X\backslash\{x\}$ satisfying*

$$d(x,y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x,y) \leq \varphi(y) - \varphi(x)),$$

then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in T(v)$ for all $T \in \mathfrak{F}$.

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ such that*

$$d(x,y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x,y) \leq \varphi(y) - \varphi(x))$$

holds for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.

(η) *If Y is a subset of X such that for each $x \in A\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying*

$$d(x,z) \leq \varphi(x) - \varphi(z) \quad (resp.\ \ d(x,z) \leq \varphi(z) - \varphi(x)),$$

then there exists a $v \in A \cap Y$.

($\theta1$) *There exists $v \in A$ such that, for each chain C in $S(v)$, we have $\bigcap_{x\in C} S(x) \neq \emptyset$.*

($\theta2$) *There exist $v \in A$ and a maximal chain C^* in $S(v)$, we have $\bigcap_{x\in C^*} S(x) \neq \emptyset$.*

Here $S(x) = \{y \in X : d(x, y) \leq \varphi(y) - \varphi(x)\}$.

This theorem includes various earlier related results and is very useful as follows:

(1) (0) $\Longleftrightarrow$ (γ) extends Kirk's characterization [16] of completeness.

(2) Recall that (α) implies the variational principle of Ekeland (1979) and also given by Brunner (1987), ($\delta 1$) essentially due to Tuy (1981), (γ) to Kasahara (1975), ($\epsilon 1$) to Maschler-Peleg (1976), and ($\gamma 1$) to Caristi (1976), which implies the Banach contraction principle. See [32].

(3) Note that "(α) $\Longleftrightarrow$ [wEk] with its dual form."

(4) Consider the following particular form of (γ) :

($\gamma 1$) *If* $f : A \to X$ *is a map such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \quad (\textit{resp.} \ \ d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for any $x \in A$*, then* f *has a fixed element* $v \in A$*, that is,* $v = f(v)$.

Then "($\gamma 1$) $\Longleftrightarrow$ [Car] with its dual form."

(5) From Caristi's theorem, Mizoguchi-Takahashi [20] in 1989 deduced a particular form of ($\delta 1$) and applied it to obtain Ekeland's ε-variational principle, generalizations of Nadler's and Reich's theorems. This can be done also from our version of Caristi's theorem.

(6) By using Ekeland's variational principle, Mizoguchi-Takahashi [20] derived the following Caristi-Kirk's theorem [6], which is the set-valued version of the Caristi fixed point theorem:

Theorem 8.3. (Caristi-Kirk) *Let* (X, d) *be a complete metric space and* $T : X \multimap X$ *be a multimap with nonempty values such that for each* $x \in X$*, there exists* $y \in T(x)$ *satisfying* $d(x, y) + \varphi(y) \leq \varphi(x)$*, where* $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *is a proper, lower semicontinuous and bounded below functional. Then,* T *has a fixed point, that is, there exists* $\bar{x} \in X$ *such that* $\bar{x} \in T(\bar{x})$.

Here the lower semicontinuous function can be replaced by the one *from above.*

(7) From Theorem 8.2(α), (γ) and the Brøndsted-Jachymski Principle, we have the following:

Theorem 8.5. *Let* (X, d) *be a complete metric space and* $\varphi : X \to \mathbb{R} \cup \{\infty\}$ *a proper function l.s.c. from above and bounded below (resp. u.s.c. from below and bounded above). If* $f : X \to X$ *is a map such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \ \ (\textit{resp.}\ d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for any $x \in X$*. Then we have*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset \ \ (\textit{resp.}\ \mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preceq) \neq \emptyset).$$

Consequently, this section demonstrates the usefulness of our Metatheorem. Until now, we gave more than one hundred examples or applications of our Metatheorem, and each of them might have useful consequences.

9. Jachymski Type Equivalents

In this article, we introduced many examples of maps $f : X \to X$ satisfying $\mathrm{Per}(f) = \mathrm{Fix}(f) \neq \emptyset$. Such sets X can have more rich properties by the following main theorem of Jachymski ([14], Theorem 2):

Theorem 9.1. [14] *Let X be a nonempty abstract set and $T : X \to X$. Then the following statements are equivalent:*

(a) $\mathrm{Per}(T) = \mathrm{Fix}(T) \neq \emptyset$.

(b) (Zermelo) *There exists a partial ordering $\preceq$ such that every chain in $(X, \preceq)$ has a supremum and T is progressive with respect to $\preceq$.*

(c) (Caristi) *There exists a complete metric d and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that T satisfies Caristi's condition.*

(d) *There exists a complete metric d and a d-Lipschitzian function $\varphi : X \to \mathbb{R}^+$ such that T satisfies Caristi's condition and T is nonexpansive with respect to d; i.e.*

$$d(Tx, Ty) \leq d(x, y) \quad \textit{for all} \quad x, y \in X.$$

(e) (Hicks-Rhoades) *For each $\alpha \in (0, 1)$, there exists a complete metric d such that T is nonexpansive with respect to d and*

$$d(Tx, T^2x) \leq \alpha d(x, Tx) \quad \textit{for all} \quad x \in X.$$

(f) *There exists a complete metric d such that T is continuous with respect to d and for each $x \in X$, the sequence $(T^n x)_{n=1}^{\infty}$ is convergent (the limit may depend on x).*

(g) *There exists a partition of X, $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, such that all the sets X_γ are nonempty, T-invariant and pairwise disjoint, and for all $\gamma \in \Gamma$, $T|_{X_\gamma}$ has a unique periodic point.*

(h) *For each $\alpha \in (0, 1)$, there exists a partition of X, $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, and complete metrics d_γ on X_γ such that all the sets X_γ are nonempty; T-invariant and pairwise disjoint; and*

$$d_\gamma(Tx, Ty) \leq \alpha d_\gamma(x, y) \quad \textit{for all } x, y \in X.$$

Remark 9.2. [14] Implication (a) $\Longrightarrow$ (b) is a converse to Zermelo's theorem. Implication (a) $\Longrightarrow$ (c) is a reciprocal to Caristi's theorem; in fact, a stronger result, (a) $\Longrightarrow$ (d) can be obtained here. Implication (a) $\Longrightarrow$ (e) is a converse to a fixed point theorem of Hicks-Rhoades. Finally (a) $\Longrightarrow$ (f) answers a question posed by Matkowski.

Remark 9.3. Each of (a)–(h) seems to be order theoretic fixed point theorems. For them, we state our own comments.

(a) This could be $\mathrm{Fix}(T) = \mathrm{Per}(T) \supset \mathrm{Max}(\preceq) \neq \emptyset$ by defining $\preceq$ on X.

(b) Zermelo's theorem is improved by Theorem C, Theorem D(iii), Section 10(I) in [29] and its equivalents there. Note that its conclusion should be as above (a).

(c) Caristi's theorem is improved by Section 5, especially in Theorems 5.3 and 5.4. Their conclusions imply (a).

(d) This is a variant of Caristi's theorem and its conclusion should be as in (a).

(e) Here nonexpansiveness is redundant in view of Theorem H(δ) in [32]. Moreover, the continuity in the following is also redundant:

Proposition 9.4. (Rus [36]) *Let f be a continuous selfmap of a complete metric space X satisfying*

$$d(fx, f^2x) \leq \alpha d(x, fx) \quad \text{for every } x \in X,$$

where $0 < \alpha < 1$. Then f has a fixed point.

We also improve Zermelo's fixed point theorem 5.5 by applying Theorem 9.1(b) as follows:

Theorem 9.5. *Let $(X, \preceq)$ be a any nonempty set. There exists a partial ordering $\preceq$ such that every chain in $(X, \preceq)$ has a supremum if and only if every progressive map $f : X \to X$ has*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset.$$

Similarly, we have a new theorem from Theorem 9.1(c):

Theorem 9.6. *For any nonempty set X and a map $f : X \to X$. Then there exists a complete metric d and a lower semicontinuous (from above) function $\varphi : X \to \mathbb{R}^+$ such that $f : X \to X$ satisfies Caristi's condition if and only if*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset.$$

This gives a new proof of the Caristi theorem.

10. Metatheorem for Power Sets

As an application of Metatheorem, we consider the power set $\mathcal{P}(X)$ of a set X. Then $(\mathcal{P}(X), \subseteq)$ is a partially ordered set. Let $G(A, B)$ denote $A \not\subseteq B$ for $A, B \in \mathcal{P}(X)$ in Metatheorem. The following consequence of Metatheorem shows the role of the whole space X in the power set $\mathcal{P}(X)$:

Theorem 10.1. *Let X be a set and $\mathcal{P}(X)$ its power set. Then the following equivalent propositions hold:*

(α) *The set X is maximal in $\mathcal{P}(X)$; that is, $X \not\subseteq B$ for all $B \in \mathcal{P}(X) \backslash \{X\}$.*

($\beta 1$) *The set X is a fixed element of a map $f : \mathcal{P}(X) \to \mathcal{P}(X)$, that is, $X = f(X)$, if, for any $A \in \mathcal{P}(X)$ with $A \neq f(A)$, there exists a $B \in \mathcal{P}(X) \backslash \{A\}$ satisfying $A \subseteq B$.*

(γ1) *The set* X *is a fixed element of a map* $f : \mathcal{P}(X) \to \mathcal{P}(X)$ *if* $A \subseteq f(A))$ *for any* $A \in \mathcal{P}(X)$ *with* $A \neq f(A)$.

(δ1) *The set* X *is a fixed element of a multimap* $F : \mathcal{P}(X) \multimap \mathcal{P}(X)$, *that is,* $X \in F(X)$, *if, for any,* $A \in \mathcal{P}(X)\backslash F(A)$ *there exists* $B \in \mathcal{P}(X)\backslash\{A\}$ *satisfying* $A \subseteq B$.

(ϵ1) *The set* X *is a stationary element of a multimap* $F : \mathcal{P}(X) \multimap \mathcal{P}(X)$, *that is,* $\{X\} = F(X)$, *if* $A \subseteq B$ *for any* $A \in \mathcal{P}(X)$ *and any* $B \in \mathcal{P}(X)\backslash\{A\}$.

(η) *The set* $X \in Y$ *for a subset* Y *of* $\mathcal{P}(X)$ *if, for each* $C \in \mathcal{P}(X)\backslash Y$, *there exists a* $D \in \mathcal{P}(X)\backslash\{X\}$ *satisfying* $C \subseteq D$.

11. Conclusion

Since the Ekeland variational principle in 1972 and the Caristi fixed point theorem in 1976 appeared, more than one thousand related papers were published. Most of them are related to certain maximum principles in Nonlinear Analysis and belong to Ordered Fixed Point Theory [32].

In 1985-86, we obtained Metatheorem on equivalents of maximal element theorems and various types of fixed point theorems. In 2022, we improved Metatheorem several times and its consequence like the Brøndsted-Jachymski principle. They were applied to a large number of existing or new results [25]–[35]. Finally, in the end of 2022, we obtained the reorganized 2023 Metatheorem and applied it to the new foundations of Ordered Fixed Point Theory [33].

In this article we are strictly restricted ourselves in the category of metric spaces. Since there have been appeared nearly one hundred artificial extensions or modifications of complete metric spaces, the contents of the present article can be extended or applied to them. The author sometimes doubts the necessity or usefulness or real applicability of many of them.

The present article is based on certain maximal (or minimal) element principles particular to the 2023 Metatheorem and applied them to several existing related results as a continuation of [34]. Further true application would be possible and readers are encouraged to find further study on such topics.

References

[1] J.S. Bae and S. Park, Remarks on the Caristi-Kirk fixed point theorem, Bull. Korean Math. Soc. 19 (1983), 57–60.

[2] Z. Boros, M. Iqbal, A. Száz, A relational improvement of a true particular case of Fierro's maximality theorem, (2022), manuscript.

[3] H. Brézis, F.E. Browder, A general principle on ordered sets in nonlinear functional analysis, Advances in Math. 21 (1976), 355–364.

[4] A. Brøndsted, Fixed point and partial orders, Shorter Notes, Proc. Amer. Math. Soc. 60 (1976), 365–366.

[5] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215 (1976), 241–251.

[6] J. Caristi, Fixed point theory and inwardness conditions, Applied Nonlinear Analysis, Academic Press (1979), 479–483.

[7] Y. Chen, Y.J. Cho, L. Yang, Note on the results with lower semi-continuity, Bull. Korean Math. Soc. 39(4) (2002), 535–541.

[8] S. Cobzaş, Fixed points and completeness in metric and generalized metric spaces, J. Math. Sciences, 250(3) (2020), 475–535. DOI 10.1007/s10958-020-05027-1

[9] S. Cobzaş, Ekeland, Takahashi and Caristi principles in preordered quasi-metric spaces, Quaestiones Mathematicae, (2022), 1–22. DOI: 10.2989/16073606.2022.2042417

[10] J. Daneš, Equivalence of some geometric and related results of nonlinear functional analysis, Comment. Math. Univ. Carolinae 26 (1985), 443–454.

[11] I. Ekeland, Sur les problèmes variationnels, C.R. Acad. Sci. Paris 275 (1972), 1057–1059; 276 (1973), 1347–1348.

[12] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974), 324–353.

[13] R. Fierro, Maximality, fixed points and variational principles for mappings on quasi-uniform spaces, Filomat 31:16 (2017), 5345–5355. https://doi.org/10.2298/FIL1716345F

[14] J. Jachymski, Converses to fixed point theorems of Zermelo and Caristi, Nonlinear Analysis 52 (2003), 1455–1463.

[15] S. Kasahara, On fixed points in partially ordered sets and Kirk-Caristi theorem, Math. Seminar Notes 3 (1975), 229–232.

[16] W.A. Kirk, Caristi's fixed point theorem and metric convexity, Colloq. Math. XXXVI (1976), 81–86.

[17] W.A. Kirk, Contraction mappings and extensions, Chapter 1, Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims, ed.), Kluwer Academic Publ. (2001), 1–34.

[18] J. Li, Inductive properties of fixed point sets of mappings on posets and on partially ordered topological spaces, Fixed Point Theory Appl. (2015) 2015:211 DOI 10.1186/s13663-015-0461-8

[19] L.-J. Lin, W.-S. Du, Systems of equilibrium problems with applications to new variants of Ekeland's variational principle, fixed point theorems and parametric optimization problems, J. Glob. Optim. 40 (2008), 663–677. DOI 10.1007/s10898-007-9146-0

[20] N. Mizoguchi, W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989), 177–188.

[21] S. Park, On extensions of the Caristi-Kirk fixed point theorem, J. Korean Math. Soc. 19 (1983), 143–151.

[22] S. Park, Characterizations of metric completeness, Colloq. Math. XLIX (1984), 21–26.

[23] S. Park, Some applications of Ekeland's variational principle to fixed point theory, in Approximation Theory and Applications (S.P. Singh, ed.), Pitman Res. Notes Math. 133 (1985), 159–172.

[24] S. Park, Equivalent formulations of Ekeland's variational principle for approximate solutions of minimization problems and their applications, Operator Equations and Fixed Point Theorems (S.P. Singh et al., eds.), MSRI-Korea Publ. 1 (1986), 55–68.

[25] S. Park, Equivalents of various maximum principles, Results in Nonlinear Analysis 5(2) (2022), 169–174.

[26] S. Park, Applications of various maximum principles, J. Fixed Point Theory (2022) 2022-3, 1–23. ISSN:2052–5338.

[27] S. Park, Equivalents of maximum principles for several spaces, Top. Algebra Appl. 10 (2022), 68–76. 10.1515/taa-2022-0113

[28] S. Park, Equivalents of generalized Brøndsted principle, J. Informatics Math. Sci. DOI: 10.13140/RG.2.2.34752.61440

[29] S. Park, Equivalents of ordered fixed point theorems of Kirk, Caristi, Nadler, Banach, and others, Adv. Th. Nonlinear Anal. Appl. 6(4) (2022), 420–432.

[30] S. Park, Extensions of ordered fixed point theorems, DOI: 10.13140/RG.2.2.21699.48160

[31] S. Park, Equivalents of the Caristi fixed point theorem, DOI: 10.13140/RG.2.2.11043.32804

[32] S. Park, Foundations of ordered fixed point theory, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 61(2) (2022), 1–51.

[33] S. Park, Applications of several minimum principles, Adv. Th. Nonlinear Anal. Appl. 7(1) (2023), 52–60. DOI:10.13140/RG.2.2.25495.65442

[34] S. Park, Remarks on the metatheorem in ordered fixed point theory, Manuscript.

[35] S. Park, New generalizations of the Caristi fixed point theorem, Manuscript.

[36] I.A. Rus, Weakly Picard mappings, Comment. Math. Univ. Carolinae 34 (1973), 769–773.

[37] J. Siegel, A new proof of Caristi's fixed point theorem, Proc. Amer. Math. Soc. 66 (1977), 54–56.

[38] W. Takahashi, Existence theorems generalizing fixed point theorems for multivalued mappings, Fixed point theory and applications (Marseille, 1989), Pitman Res. Notes Math. Ser. 252, Longman Sci. Tech., Harlow, (1991), 397–406.

[39] E. Zermelo, Neuer Beweis für die Möglichkeit einer Wohlordnung, Math. Ann. 65 (1908), 107–128.