Common Fixed Point Theorems Satisfying Generalized (ψ, ϕ)- Weak Contraction in Metric Spaces

Abstract: In this manuscript, we shall prove a common fixed point theorem for four weakly compatible self-maps Z̆, H̆, Ω̆ and K̆ on a metric space (M, d^o) satisfying the following generalized (ψ, ϕ)- weak contraction:

$\psi\left(d^o(\breve{\Omega}\sigma, \breve{K}\omega)\right) \leq \psi(\delta(\sigma,\omega)) - \phi(\delta(\sigma,\omega))$, where

$$\delta(\sigma,\omega) = max\Big\{d^o(\breve{Z}\sigma, \breve{H}\omega), d^o(\breve{Z}\sigma, \breve{\Omega}\sigma), d^o(\breve{H}\omega, \breve{K}\omega), \frac{1}{2}\left[d^o(\breve{Z}\sigma, \breve{K}\omega) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)\right],$$
$$d^o(\breve{Z}\sigma, \breve{\Omega}\sigma)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{H}\omega)}{1 + d^o(\breve{H}\omega, \breve{K}\omega)}\right], d^o(\breve{H}\omega, \breve{K}\omega)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{H}\omega)}{1 + d^o(\breve{Z}\sigma, \breve{\Omega}\sigma)}\right], \frac{{d^o}^2(\breve{Z}\sigma, \breve{\Omega}\sigma)}{1 + d^o(\breve{\Omega}\sigma, \breve{K}\omega)},$$
$$\frac{{d^o}^2(\breve{H}\omega, \breve{K}\omega)}{1 + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}, d^o(\breve{Z}\sigma, \breve{\Omega}\sigma)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{K}\omega) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)}{1 + d^o(\breve{Z}\sigma, \breve{H}\omega) + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}\right],$$
$$d^o(\breve{H}\omega, \breve{K}\omega)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{K}\omega) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)}{1 + d^o(\breve{Z}\sigma, \breve{H}\omega) + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}\right]\Big\}.$$

Also, we have proved common fixed point theorems for the above mentioned weakly compatible self-maps along with E.A. property and (CLR) property. An illustrative example is also provided to support our results.

Keywords: fixed point, weakly compatible maps, E.A. property, (CLR) property, generalized (ψ, ϕ)- weak contraction.

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1. Introduction

Definition 1.1 A coincidence point of a pair of self–maps $P, Q : M \to M$ is a point $u \in M$ for which $Pu = Qu$.

A common fixed point of a pair of self – maps $P, Q : M \to M$ is a point $u \in M$ for which $Pu = Qu = u$.

In 1996, Jungck [2] introduced the concept of weakly compatible maps to study common fixed point theorems:

Definition 1.2 Let (M, d^o) be a metric space. A pair of self – maps $P, Q : M \to M$ is weakly compatible if they commute at their coincidence points, that is, if there exists $u \in M$ such that $PQu = QPu$, where u is coincidence point of P and Q.

In 2002, Aamri and EI Moutawakil [1] introduced the notion of E.A. property as follows:

Definition 1.3 Let (M, d^o) be a metric space. Two self-maps P and Q on M are said to satisfy the E.A. property, if there exists a sequence $\{u_n\}$ in M such that,

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = t, \text{ for some } t \in M.$$

In 2011, Sintunavarat *et.al* [6] introduced the notion of (CLR) property as follows:

Definition 1.4 Let (M, d^o) be a metric space. Two self-maps P and Q on M are said to satisfy the (CLR_P) property, if there exists a sequence $\{u_n\}$ in M such that,

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = P(t), \text{ for some } t \in M.$$

2. Main Results

Theorem 2.1. Let (M, d^o) be a metric space and let $\breve{Z}, \breve{H}, \breve{\Omega}$ and $\breve{K}$ be self-maps on M satisfying the followings:

(2.1) $\breve{\Omega}M \subseteq \breve{H}M, \breve{K}M \subseteq \breve{Z}M$,

for all $\sigma, \omega \in M$, there exists right continuous functions $\psi, \ \phi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\psi(0) = 0 = \phi(0)$ and $\psi(a) < a$ for $a > 0$ such that:

(2.2) $\psi\left(d^o\left(\breve{\Omega}\sigma, \breve{K}\omega\right)\right) \le \psi\left(\delta(\sigma,\omega)\right) - \phi\left(\delta(\sigma,\omega)\right)$, where

$$\begin{aligned}\delta(\sigma,\omega) = max\Big\{ & d^o\left(\breve{Z}\sigma, \breve{H}\omega\right), d^o\left(\breve{Z}\sigma, \breve{\Omega}\sigma\right), d^o\left(\breve{H}\omega, \breve{K}\omega\right), \tfrac{1}{2}\left[d^o\left(\breve{Z}\sigma, \breve{K}\omega\right) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)\right], \\ & d^o\left(\breve{Z}\sigma, \breve{\Omega}\sigma\right)\left[\frac{1 + d^o\left(\breve{Z}\sigma, \breve{H}\omega\right)}{1 + d^o\left(\breve{H}\omega, \breve{K}\omega\right)}\right], d^o\left(\breve{H}\omega, \breve{K}\omega\right)\left[\frac{1 + d^o\left(\breve{Z}\sigma, \breve{H}\omega\right)}{1 + d^o\left(\breve{Z}\sigma, \breve{\Omega}\sigma\right)}\right], \frac{{d^o}^2\left(\breve{Z}\sigma, \breve{\Omega}\sigma\right)}{1 + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}, \\ & \frac{{d^o}^2\left(\breve{H}\omega, \breve{K}\omega\right)}{1 + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}, d^o\left(\breve{Z}\sigma, \breve{\Omega}\sigma\right)\left[\frac{1 + d^o\left(\breve{Z}\sigma, \breve{K}\omega\right) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)}{1 + d^o\left(\breve{Z}\sigma, \breve{H}\omega\right) + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}\right], \\ & d^o\left(\breve{H}\omega, \breve{K}\omega\right)\left[\frac{1 + d^o\left(\breve{Z}\sigma, \breve{K}\omega\right) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)}{1 + d^o\left(\breve{Z}\sigma, \breve{H}\omega\right) + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}\right]\Big\}.\end{aligned}$$

If one of $\breve{Z}M, \breve{H}M, \breve{\Omega}M$ or $\breve{K}M$ is complete subspace of M, then the pair $(\breve{Z}, \breve{\Omega})$ or $(\breve{H}, \breve{K})$ have a coincidence point. Moreover, if the pair $(\breve{Z}, \breve{\Omega})$ or $(\breve{H}, \breve{K})$ is weakly compatible, then $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$ have a unique common fixed point.

Proof: Let $\sigma_0 \in M$ be an arbitrary point of M. From (2.2), we can construct a sequence $\{\omega_n\}$ in M as follows:

$$\omega_{2n+1} = \breve{\Omega}\sigma_{2n} = \breve{H}\sigma_{2n+1}, \ \ \omega_{2n+2} = \breve{K}\sigma_{2n+1} = \breve{Z}\sigma_{2n+2}, \tag{2.3}$$

for all $n = 0, 1, 2, ...$

Now, we define $d^o{}_{\mathrm{n}} = d^o(\omega_{\mathrm{n}}, \omega_{\mathrm{n+1}})$. If $d^o{}_{2\mathrm{n}} = 0$ for some n, then $d^o(\omega_{2\mathrm{n}}, \omega_{2\mathrm{n+1}}) = 0$.

Then $\omega_{2\mathrm{n}} = \omega_{2\mathrm{n+1}}$, that is, $\breve{K}\sigma_{2\mathrm{n}-1} = \breve{Z}\sigma_{2n} = \breve{\Omega}\sigma_{2n} = \breve{H}\sigma_{2n+1}$.

We can note that $\breve{Z}$ and $\breve{\Omega}$ have a coincidence point. Similarly, if $d^o{}_{2\mathrm{n+1}} = 0$, then $\breve{H}$ and $\breve{K}$ have a coincidence point. Assume that $d^o{}_{\mathrm{n}} \neq 0$ for each n.

On putting, $\sigma = \sigma_{2n}$ and $\omega = \sigma_{2n+1}$ in (2.2), we get

$$\psi\left(d^0(\breve{\Omega}\sigma_{2n}, \breve{K}\sigma_{2n+1})\right) \leq \psi(\delta(\sigma_{2n}, \sigma_{2n+1})) - \phi(\delta(\sigma_{2n}, \sigma_{2n+1})), \tag{2.4}$$

where

$$\begin{aligned}
\delta(\sigma_{2n}, \sigma_{2n+1}) = max\Big\{ & d^o(\breve{Z}\sigma_{2n}, \breve{H}\sigma_{2n+1}), d^o(\breve{Z}\sigma_{2n}, \breve{\Omega}\sigma_{2n}), d^o(\breve{H}\sigma_{2n+1}, \breve{K}\sigma_{2n+1}), \\
& \frac{1}{2}\left[d^o(\breve{Z}\sigma_{2n}, \breve{K}\sigma_{2n+1}) + d^o(\breve{\Omega}\sigma_{2n}, \breve{H}\sigma_{2n+1})\right], \\
& d^o(\breve{Z}\sigma_{2n}, \breve{\Omega}\sigma_{2n})\left[\frac{1 + d^o(\breve{Z}\sigma_{2n}, \breve{H}\sigma_{2n+1})}{1 + d^o(\breve{H}\sigma_{2n+1}, \breve{K}\sigma_{2n+1})}\right], \\
& d^o(\breve{H}\sigma_{2n+1}, \breve{K}\sigma_{2n+1})\left[\frac{1 + d^o(\breve{Z}\sigma_{2n}, \breve{H}\sigma_{2n+1})}{1 + d^o(\breve{Z}\sigma_{2n}, \breve{\Omega}\sigma_{2n})}\right], \\
& \frac{d^{o2}(\breve{Z}\sigma_{2n}, \breve{\Omega}\sigma_{2n})}{1 + d^o(\breve{\Omega}\sigma_{2n}, \breve{K}\sigma_{2n+1})}, \frac{d^{o2}(\breve{H}\sigma_{2n+1}, \breve{K}\sigma_{2n+1})}{1 + d^o(\breve{\Omega}\sigma_{2n}, \breve{K}\sigma_{2n+1})}, \\
& d^o(\breve{Z}\sigma_{2n}, \breve{\Omega}\sigma_{2n})\left[\frac{1 + d^o(\breve{Z}\sigma_{2n}, \breve{K}\sigma_{2n+1}) + d^o(\breve{\Omega}\sigma_{2n}, \breve{H}\sigma_{2n+1})}{1 + d^o(\breve{Z}\sigma_{2n}, \breve{H}\sigma_{2n+1}) + d^o(\breve{\Omega}\sigma_{2n}, \breve{K}\sigma_{2n+1})}\right], \\
& d^o(\breve{H}\sigma_{2n+1}, \breve{K}\sigma_{2n+1})\left[\frac{1 + d^o(\breve{Z}\sigma_{2n}, \breve{K}\sigma_{2n+1}) + d^o(\breve{\Omega}\sigma_{2n}, \breve{H}\sigma_{2n+1})}{1 + d^o(\breve{Z}\sigma_{2n}, \breve{H}\sigma_{2n+1}) + d^o(\breve{\Omega}\sigma_{2n}, \breve{K}\sigma_{2n+1})}\right]\Big\} \\
= max\Big\{ & d^o(\omega_{2n}, \omega_{2n+1}), d^o(\omega_{2n}, \omega_{2n+1}), d^o(\omega_{2n+1}, \omega_{2n+2}), \\
& \frac{1}{2}\left[d^o(\omega_{2n}, \omega_{2n+2}) + d^o(\omega_{2n+1}, \omega_{2n+1})\right],
\end{aligned}$$

$$d^o\big(\omega_{2n,}\omega_{2n+1}\big)\left[\frac{1+d^o\big(\omega_{2n,}\omega_{2n+1}\big)}{1+d^o\big(\omega_{2n+1,}\omega_{2n+2}\big)}\right], d^o\big(\omega_{2n+1,}\omega_{2n+2}\big)\left[\frac{1+d^o\big(\omega_{2n,}\omega_{2n+1}\big)}{1+d^o\big(\omega_{2n,}\omega_{2n+1}\big)}\right],$$

$$\frac{{d^o}^2(\omega_{2n},\omega_{2n+1})}{1+d^o(\omega_{2n+1},\omega_{2n+2})}, \frac{{d^o}^2(\omega_{2n+1},\omega_{2n+2})}{1+d^o(\omega_{2n+1},\omega_{2n+2})},$$

$$d^o\big(\omega_{2n,}\omega_{2n+1}\big)\left[\frac{1+d^o\big(\omega_{2n,}\omega_{2n+2}\big)+d^o\big(\omega_{2n+1,}\omega_{2n+1}\big)}{1+d^o\big(\omega_{2n,}\omega_{2n+1}\big)+d^o\big(\omega_{2n+1,}\omega_{2n+2}\big)}\right]$$

$$d^o\big(\omega_{2n+1,}\omega_{2n+2}\big)\left[\frac{1+d^o\big(\omega_{2n,}\omega_{2n+2}\big)+d^o\big(\omega_{2n+1,}\omega_{2n+1}\big)}{1+d^o\big(\omega_{2n,}\omega_{2n+1}\big)+d^o\big(\omega_{2n+1,}\omega_{2n+2}\big)}\right]\Bigg\}$$

$$= max\Bigg\{{d^o}_{2n}, \;\; {d^o}_{2n}, {d^o}_{2n+1}, \frac{1}{2}[{d^o}_{2n}+{d^o}_{2n+1}+0],$$

$${d^o}_{2n}\,\frac{1+{d^o}_{2n}}{1+{d^o}_{2n+1}}, {d^o}_{2n+1}, \frac{{{d^o}^2}_{2n}}{1+{d^o}_{2n+1}}, \frac{{{d^o}^2}_{2n+1}}{1+{d^o}_{2n+1}},$$

$${d^o}_{2n}\frac{1+{d^o}_{2n}+{d^o}_{2n+1}}{1+{d^o}_{2n}+{d^o}_{2n+1}}, \;\; {d^o}_{2n+1}\frac{1+{d^o}_{2n}+{d^o}_{2n+1}}{1+{d^o}_{2n}+{d^o}_{2n+1}}\Bigg\}$$

$$= max\{{d^o}_{2n}, {d^o}_{2n+1}\}. \tag{2.5}$$

Now, from (2.4), we have

$$\psi\big(d^o(\omega_{2n+1},\omega_{2n+2})\big) \le \psi(\max\{{d^o}_{2n}, {d^o}_{2n+1}\}) - \phi(\max\{{d^o}_{2n}, {d^o}_{2n+1}\}). \tag{2.6}$$

Now, if ${d^o}_{2n+1} \ge {d^o}_{2n}$ for some n, then from (2.6), we get

$$\begin{aligned}\psi({d^o}_{2n+1}) &\le \psi({d^o}_{2n+1}) - \phi({d^o}_{2n+1})\\ &< \psi({d^o}_{2n+1}),\end{aligned} \tag{2.7}$$

which is a contradiction. Thus, ${d^o}_{2n} > {d^o}_{2n+1}$ for all n, and so, from (2.6), we have

$$\psi({d^o}_{2n+1}) \le \psi({d^o}_{2n}) - \phi({d^o}_{2n}) \text{ for all } n \in N. \tag{2.8}$$

Similarly,

$$\psi({d^o}_{2n}) \le \psi({d^o}_{2n-1}) - \phi({d^o}_{2n-1})$$

$$\psi({d^o}_{2n-1}) \le \psi({d^o}_{2n-2}) - \phi({d^o}_{2n-2}).$$

In general, we have for all $n = 1, 2, 3$ ....

$$\begin{aligned}\psi({d^o}_{n}) &\le \psi({d^o}_{n-1}) - \phi({d^o}_{n-1}) \qquad (2.9)\\ &< \psi({d^o}_{n-1}).\end{aligned}$$

Hence sequence $\{\psi({d^o}_{n})\}$ is monotonically decreasing and bounded below. Thus, there exists $s \ge 0$, such that

$$\lim_{n\to\infty} \psi({d^o}_{n}) = s. \tag{2.10}$$

From (2.9), we deduce that

$$0 \leq \phi({d^o}_{n-1}) \leq \psi({d^o}_{\mathrm{n-1}}) - \psi({d^o}_{\mathrm{n}}).$$

Taking limit as $n \to \infty$ and using (2.10), we get

$$\lim_{n\to\infty} \phi({d^o}_{n-1}) = 0,$$

this implies that

$$\lim_{n\to\infty} \phi({d^o}_{n-1}) = \lim_{n\to\infty}\phi\big(d^o(\omega_{n-1}, \omega_n)\big) = 0 \tag{2.11}$$

$$or \lim_{n\to\infty} {d^o}_n = \lim_{n\to\infty} d^o(\omega_n, \omega_{n+1}) = 0. \tag{2.12}$$

Now, we claim that $\{\omega_{\mathrm{n}}\}$ is a Cauchy sequence. For this, it is sufficient to show that $\{\omega_{2\mathrm{n}}\}$ is a Cauchy sequence. Let, if possible, $\{\omega_{2\mathrm{n}}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$, such that for each even integer $2a$ there exists even integers $2m(a) > 2n(a) > 2a$ such that

$$d^o(\omega_{2\mathrm{n}(a)}, \omega_{2\mathrm{m}(a)}) \geq \epsilon \tag{2.13}$$

for every even integer $2a$, suppose that $2m(a)$ be the least positive integer exceeding $2n(a)$ satisfying (2.13), such that

$$d^o\,(\omega_{2\mathrm{n}(a)}, \omega_{2\mathrm{m}(a)-2}) < \epsilon. \tag{2.14}$$

From (2.13), we get

$$\epsilon \leq d^o\left(\omega_{2\mathrm{n}(a)}, \omega_{2\mathrm{m}(a)}\right)$$

$$\leq d^o\big(\omega_{2\mathrm{n}(a)}, \omega_{2\mathrm{m}(a)-2}\big) + d^o\big(\omega_{2\mathrm{m}(a)-2}, \omega_{2\mathrm{m}(a)-1}\big) + d^o\big(\omega_{2\mathrm{m}(a)-1}, \omega_{2\mathrm{m}(a)}\big).$$

Using (2.12) and (2.14) in the above inequality, we get

$$\lim_{n\to\infty} d^o(\omega_{2\mathrm{n}(a)}, \omega_{2\mathrm{m}(a)}) = \epsilon \tag{2.15}$$

Also by the triangular inequality,

$$\left|d^o\big(\omega_{2n(a)}, \omega_{2m(a)-1}\big) - d^o(\omega_{2n(a)}, \omega_{2m(a)})\right| \leq d^o\big(\omega_{2m(a)-1}, \omega_{2m(a)}\big) = {d^o}_{2m(a)-1},$$

$$\left|d^o\big(\omega_{2n(a)+1}, \omega_{2m(a)-1}\big) - d^o(\omega_{2n(a)}, \omega_{2m(a)})\right| \leq {d^o}_{2m(a)-1} + {d^o}_{2m(a).} \tag{2.16}$$

Using (2.12), we have

$$\lim_{n\to\infty} d^o(\omega_{2n(a)}, \omega_{2m(a)-1}) = \lim_{n\to\infty} d^o\big(\omega_{2\mathrm{n}(\mathrm{a})+1}, \omega_{2\mathrm{m}(\mathrm{a})-1}\big) = \epsilon \tag{2.17}$$

Now, from (2.2), we have

$$\psi\left(d^o\big(\breve{\Omega}\sigma_{2n(a)}, \breve{\mathrm{K}}\sigma_{2m(a)-1}\big)\right) \leq \psi\left(\delta\big(\sigma_{2n(a)}, \sigma_{2m(a)-1}\big)\right) - \phi\left(\delta\big(\sigma_{2n(a)}, \sigma_{(2m(a)-1)}\big)\right), \tag{2.18}$$

where

$$
\begin{aligned}
&\delta(\sigma_{2n(a)}, \sigma_{2m(a)-1}) \\
&= max\{d^o(\breve{Z}\sigma_{2n(a)}, \breve{H}\sigma_{2m(a)-1}), d^o(\breve{Z}\sigma_{2n(a)}, \breve{\Omega}\sigma_{2n(a)}), d^o(\breve{H}\sigma_{2m(a)-1}, \breve{K}\sigma_{2m(a)-1}), \\
&\quad \frac{1}{2}[d^o(\breve{Z}\sigma_{2n(a)}, \breve{K}\sigma_{2m(a)-1}) + d^o(\breve{\Omega}\sigma_{2n(a)}, \breve{H}\sigma_{2m(a)-1})], \\
&\quad d^o(\breve{Z}\sigma_{2n(a)}, \breve{\Omega}\sigma_{2n(a)})\left[\frac{1 + d^o(\breve{Z}\sigma_{2n(a)}, \breve{H}\sigma_{2m(a)-1})}{1 + d^o(\breve{H}\sigma_{2m(a)-1}, \breve{K}\sigma_{2m(a)-1})}\right], \\
&\quad d^o(\breve{H}\sigma_{2m(a)-1}, \breve{K}\sigma_{2m(a)-1})\left[\frac{1 + d^o(\breve{Z}\sigma_{2n(a)}, \breve{H}\sigma_{2m(a)-1})}{1 + d^o(\breve{Z}\sigma_{2n(a)}, \breve{\Omega}\sigma_{2n(a)})}\right], \\
&\quad \frac{{d^o}^2(\breve{Z}\sigma_{2n(a)}, \breve{\Omega}\sigma_{2n(a)})}{1 + d^o(\breve{\Omega}\sigma_{2n(a)}, \breve{K}\sigma_{2m(a)-1})}, \frac{{d^o}^2(\breve{H}\sigma_{2m(a)-1}, \breve{K}\sigma_{2m(a)-1})}{1 + d^o(\breve{\Omega}\sigma_{2n(a)}, \breve{K}\sigma_{2m(a)-1})}, \\
&\quad d^o(\breve{Z}\sigma_{2n(a)}, \breve{\Omega}\sigma_{2n(a)})\left[\frac{1 + d^o(\breve{Z}\sigma_{2n(a)}, \breve{K}\sigma_{2m(a)-1}) + d^o(\breve{\Omega}\sigma_{2n(a)}, \breve{H}\sigma_{2m(a)-1})}{1 + d^o(\breve{Z}\sigma_{2n(a)}, \breve{H}\sigma_{2m(a)-1}) + d^o(\breve{\Omega}\sigma_{2n(a)}, \breve{K}\sigma_{2m(a)-1})}\right], \\
&\quad d^o(\breve{H}\sigma_{2m(a)-1}, \breve{K}\sigma_{2m(a)-1})\left[\frac{1 + d^o(\breve{Z}\sigma_{2n(a)}, \breve{K}\sigma_{2m(a)-1}) + d^o(\breve{\Omega}\sigma_{2n(a)}, \breve{H}\sigma_{2m(a)-1})}{1 + d^o(\breve{Z}\sigma_{2n(a)}, \breve{H}\sigma_{2m(a)-1}) + d^o(\breve{\Omega}\sigma_{2n(a)}, \breve{K}\sigma_{2m(a)-1})}\right]\} \\
&= max\{d^o(\omega_{2n(a)}, \omega_{2m(a)-1}), d^o(\omega_{2n(a)}, \omega_{2n(a)+1}), d^o(\omega_{2m(a)-1}, \omega_{2m(a)}), \\
&\quad \frac{1}{2}[d^o(\omega_{2n(a)}, \omega_{2m(a)}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)-1})], \\
&\quad d^o(\omega_{2n(a)}, \omega_{2n(a)+1})\left[\frac{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)-1})}{1 + d^o(\omega_{2m(a)-1}, \omega_{2m(a)})}\right], \\
&\quad d^o(\omega_{2m(a)-1}, \omega_{2m(a)})\left[\frac{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)-1})}{1 + d^o(\omega_{2n(a)}, \omega_{2n(a)+1})}\right], \\
&\quad \frac{{d^o}^2(\omega_{2n(a)}, \omega_{2n(a)+1})}{1 + d^o(\omega_{2n(a)+1}, \omega_{2m(a)})}, \frac{{d^o}^2(\omega_{2m(a)-1}, \omega_{2m(a)})}{1 + d^o(\omega_{2n(a)+1}, \omega_{2m(a)})}, \\
&\quad d^o(\omega_{2n(a)}, \omega_{2n(a)+1})\left[\frac{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)-1})}{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)-1}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)})}\right] \\
&\quad d^o(\omega_{2m(a)-1}, \omega_{2m(a)})\left[\frac{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)-1})}{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)-1}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)})}\right]\}
\end{aligned}
$$

Assuming $a \to \infty$ and using (2.14), (2.15) and (2.17), we have

$$= max\left\{\epsilon, 0, 0, \frac{1}{2}(\epsilon + \epsilon), 0, 0, 0, 0, 0, 0\right\}$$

$$= \epsilon.$$

Hence, $\delta(\sigma_{2n(a)}, \sigma_{2m(a)-1}) = \epsilon$.

Now, by (2.18), we have

$$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction, since $\epsilon > 0$.

Thus, $\{\omega_{2n}\}$ is a Cauchy sequence. So, $\{\omega_n\}$ is a Cauchy sequence.

Now, suppose that $\breve{Z}M$ is complete. Since $\{\omega_{2n}\}$ is contained in $\breve{Z}M$ and has limit in $\breve{Z}M$ say μ, that is, $\lim_{n\to\infty} \omega_{2n} = \mu$. let $\nu \in \breve{Z}^{-1}(\mu)$ then $\breve{Z}\nu = \mu$.

Now, we shall prove that $\breve{\Omega}\nu = \mu$.

Let, if possible, $\breve{\Omega}\nu \neq \mu$ that is, $d^o(\breve{\Omega}\nu, \mu) = b > 0$.

Now, on putting, $\sigma = \nu, \omega = \sigma_{2n-1}$ in (2.2), we have

$$\psi\left(d^o\left(\breve{\Omega}\nu, \breve{K}\sigma_{2n-1}\right)\right) \le \psi\left(\delta(\nu, \sigma_{2n-1})\right) - \phi\left(\delta(\nu, \sigma_{2n-1})\right), \tag{2.19}$$

where

$$\begin{aligned}\delta(\nu, \sigma_{2n-1}) = max\Big\{ & d^o\left(\breve{Z}\nu, \breve{H}\sigma_{2n-1}\right), d^o\left(\breve{Z}\nu, \breve{\Omega}\nu\right), d^o\left(\breve{H}\sigma_{2n-1}, \breve{K}\sigma_{2n-1}\right), \\ & \frac{1}{2}\left[d^o\left(\breve{Z}\nu, \breve{K}\sigma_{2n-1}\right) + d^o(\breve{\Omega}\nu, \breve{H}\sigma_{2n-1})\right], \\ & d^o\left(\breve{Z}\nu, \breve{\Omega}\nu\right)\left[\frac{1 + d^o\left(\breve{Z}\nu, \breve{H}\sigma_{2n-1}\right)}{1 + d^o\left(\breve{H}\sigma_{2n-1}, \breve{K}\sigma_{2n-1}\right)}\right], \\ & d^o\left(\breve{H}\sigma_{2n-1}, \breve{K}\sigma_{2n-1}\right)\left[\frac{1 + d^o\left(\breve{Z}\nu, \breve{H}\sigma_{2n-1}\right)}{1 + d^o\left(\breve{Z}\nu, \breve{\Omega}\nu\right)}\right], \frac{{d^o}^2\left(\breve{Z}\nu, \breve{\Omega}\nu\right)}{1 + d^o(\breve{\Omega}\nu, \breve{K}\sigma_{2n-1})}, \\ & \frac{{d^o}^2\left(\breve{H}\sigma_{2n-1}, \breve{K}\sigma_{2n-1}\right)}{1 + d^o(\breve{\Omega}\nu, \breve{K}\sigma_{2n-1})}, d^o\left(\breve{Z}\nu, \breve{\Omega}\nu\right)\left[\frac{1 + d^o\left(\breve{Z}\nu, \breve{K}\sigma_{2n-1}\right) + d^o(\breve{\Omega}\nu, \breve{H}\sigma_{2n-1})}{1 + d^o\left(\breve{Z}\nu, \breve{H}\sigma_{2n-1}\right) + d^o(\breve{\Omega}\nu, \breve{K}\sigma_{2n-1})}\right], \\ & d^o\left(\breve{H}\sigma_{2n-1}, \breve{K}\sigma_{2n-1}\right)\left[\frac{1 + d^o\left(\breve{Z}\nu, \breve{K}\sigma_{2n-1}\right) + d^o(\breve{\Omega}\nu, \breve{H}\sigma_{2n-1})}{1 + d^o\left(\breve{Z}\nu, \breve{H}\sigma_{2n-1}\right) + d^o(\breve{\Omega}\nu, \breve{K}\sigma_{2n-1})}\right]\Big\}.\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} \delta(\nu, \sigma_{2n-1}) = max\Big\{ & d^o(\mu,\mu), d^o(\mu,b), d^o(\mu,\mu), \tfrac{1}{2}\left[d^o(\mu,\mu) + d^o(b,\mu)\right], \\ & d^o(\mu,b)\left[\frac{1 + d^o(\mu,\mu)}{1 + d^o(\mu,\mu)}\right], d^o(\mu,\mu)\left[\frac{1 + d^o(\mu,\mu)}{1 + d^o(\mu,b)}\right], \\ & \frac{{d^o}^2(\mu,b)}{1 + d^o(b,\mu)}, \frac{{d^o}^2(\mu,\mu)}{1 + d^o(b,\mu)},\end{aligned}$$

$$d^o(\mu,b)\left[\frac{1+d^o(\mu,\mu)+d^o(b,\mu)}{1+d^o(\mu,\mu)+d^o(b,\mu)}\right],$$

$$d^o(\mu,\mu)\left[\frac{1+d^o(\mu,\mu)+d^o(b,\mu)}{1+d^o(\mu,\mu)+d^o(b,\mu)}\right]\Bigg\}$$

$$= max\left\{0, d^o(\mu,b), 0, \frac{1}{2}\left(0+d^o(\mu,b)\right), d^o(\mu,b), 0, \frac{d^{o\,2}(\mu,b)}{1+d^o(b,\mu)}, 0, d^o(\mu,b), 0\right\}$$

$$= d^o(\mu,b).$$

Thus, from (2.19), we have

$$\psi\left(d^o(\mu,b)\right) \le \psi\left(d^o(\mu,b)\right) - \phi\left(d^o(\mu,b)\right)$$
$$< \psi\left(d^o(\mu,b)\right),$$

which is a contradiction, since $\phi\left(d^o(\mu,b)\right) > 0$.

Thus, $\mu = b$. Hence $\breve{Z}\nu = \breve{\Omega}\nu = \mu$, which implies that ν is coincidence point of the pair $(\breve{Z},\breve{\Omega})$. Since $\breve{\Omega}M \subseteq \breve{H}M$, $\breve{\Omega}\nu = \mu$ implies that, $\mu \in \breve{H}M$.

Let $\theta \in \breve{H}^{-1}\mu$. Then $\breve{H}\theta = \mu$. By using the same arguments as above, we can easily verify that $\breve{K}\,\theta = \mu = \breve{H}\theta$, that is, θ is the coincidence point of the pair $(\breve{H},\breve{K})$.

Similarly, we can prove the result if $\breve{H}M$ is complete subspace of M instead of $\breve{Z}M$. Now, if $\breve{K}M$ is complete then by (2.1), $\mu \in \breve{K}M \subseteq \breve{Z}M$.

In the same manner, if $\breve{\Omega}M$ is complete then $\mu \in \breve{\Omega}M \subseteq \breve{H}M$. Now, since the pair $(\breve{Z},\breve{\Omega})$ and $(\breve{H},\breve{K})$ are weakly compatible, so

$$\mu = \breve{\Omega}\nu = \breve{Z}\nu = \breve{K}\theta = \breve{H}\theta$$
$$\breve{Z}\mu = \breve{Z}\breve{\Omega}\nu = \breve{\Omega}\breve{Z}\nu = \breve{\Omega}\mu$$
$$\breve{H}\mu = \breve{H}\breve{K}\theta = \breve{K}\breve{H}\theta = \breve{K}\,\mu. \tag{2.20}$$

Now, we shall prove that $\breve{K}\mu = \mu$.

Let, if possible, $\breve{K}\mu \ne \mu$.

From (2.2), we have

$$\psi\left(d^o\left(\mu,\breve{K}\mu\right)\right) = \psi\left(d^o\left(\breve{\Omega}\nu,\breve{K}\mu\right)\right) \le \psi\left(\delta(\nu,\mu)\right) - \phi\left(\delta(\nu,\mu)\right),$$

where

$$\delta(\nu,\mu) = max\{d^o\left(\breve{Z}\nu,\breve{H}\mu\right), d^o\left(\breve{Z}\nu,\breve{\Omega}\nu\right), d^o\left(\breve{H}\mu,\breve{K}\mu\right),$$

$$\frac{1}{2}\left[d^o\left(\breve{Z}\nu,\breve{K}\mu\right) + d^o(\breve{\Omega}\nu,\breve{H}\mu)\right],$$

$$d^o(\breve{Z}\nu, \breve{\Omega}\nu)\left[\frac{1 + d^o(\breve{Z}\nu, \breve{H}\mu)}{1 + d^o(\breve{H}\mu, \breve{K}\mu)}\right],$$

$$d^o(\breve{H}\mu, \breve{K}\mu)\left[\frac{1 + d^o(\breve{Z}\nu, \breve{H}\mu)}{1 + d^o(\breve{Z}\nu, \breve{\Omega}\nu)}\right],$$

$$\frac{d^{o2}(\breve{Z}\nu, \breve{\Omega}\nu)}{1 + d^o(\breve{\Omega}\nu, \breve{K}\mu)}, \ \frac{d^{o2}(\breve{H}\mu, \breve{K}\mu)}{1 + d^o(\breve{\Omega}\nu, \breve{K}\mu)},$$

$$d^o(\breve{Z}\nu, \breve{\Omega}\nu)\left[\frac{1 + d^o(\breve{Z}\nu, \breve{K}\mu) + d^o(\breve{\Omega}\nu, \breve{H}\mu)}{1 + d^o(\breve{Z}\nu, \breve{H}\mu) + d^o(\breve{\Omega}\nu, \breve{K}\mu)}\right],$$

$$d^o(\breve{H}\mu, \breve{K}\mu)\left[\frac{1 + d^o(\breve{Z}\nu, \breve{K}\mu) + d^o(\breve{\Omega}\nu, \breve{H}\mu)}{1 + d^o(\breve{Z}\nu, \breve{H}\mu) + d^o(\breve{\Omega}\nu, \breve{K}\mu)}\right]\Bigg\}.$$

Using (2.20), we have

$$= max\{d^o(\mu, \breve{K}\mu), d^o(\mu, \mu), d^o(\breve{K}\mu, \breve{K}\mu),$$

$$\tfrac{1}{2}\left[d^o(\mu, \breve{K}\mu) + d^o(\mu, \breve{K}\mu)\right],$$

$$d^o(\mu, \mu)\left[\frac{1 + d^o(\mu, \breve{K}\mu)}{1 + d^o(\breve{K}\mu, \breve{K}\mu)}\right], d^o(\breve{K}\mu, \breve{K}\mu)\left[\frac{1 + d^o(\mu, \breve{K}\mu)}{1 + d^o(\mu, \mu)}\right],$$

$$\frac{d^{o2}(\mu, \mu)}{1 + d^o(\mu, \breve{K}\mu)}, \frac{d^{o2}(\breve{K}\mu, \breve{K}\mu)}{1 + d^o(\mu, \breve{K}\mu)},$$

$$d^o(\mu, \mu)\left[\frac{1 + d^o(\mu, \breve{K}\mu) + d^o(\mu, \breve{K}\mu)}{1 + d^o(\mu, \breve{K}\mu) + d^o(\mu, \breve{K}\mu)}\right],$$

$$d^o(\breve{K}\mu, \breve{K}\mu)\left[\frac{1 + d^o(\mu, \breve{K}\mu) + d^o(\mu, \breve{K}\mu)}{1 + d^o(\mu, \breve{K}\mu) + d^o(\mu, \breve{K}\mu)}\right]\Bigg\}$$

$$= max\{d^o(\mu, \breve{K}\mu), 0, 0, d^o(\mu, \breve{K}\mu), 0, 0, 0, 0, 0, 0, 0\}$$

$$= d^o(\mu, \breve{K}\mu).$$

Thus, we have

$$\psi\left(d^o(\mu, \breve{K}\mu)\right) \leq \psi\left(d^o(\mu, \breve{K}\mu)\right) - \ \phi\left(d^o(\mu, \breve{K}\mu)\right)$$

$$< \psi\left(d^o(\mu, \breve{K}\mu)\right),$$

which is a contradiction. So, $\mu = \breve{K}\mu$. Similarly, $\breve{\Omega}\mu = \mu$.

Thus, we get $\breve{K}\mu = \breve{H}\mu = \breve{Z}\mu = \breve{\Omega}\mu = \mu$.

Hence μ is the common fixed point of $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$.

For the uniqueness, let κ be another common fixed point of $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$. Now, we claim that $\kappa = \mu$. Let, if possible $\kappa \neq \mu$.

From (2.2), we have

$$\begin{aligned}\psi(d^o(\mu,\kappa)) &= \psi\left(d^o(\breve{\Omega}\mu, \breve{K}\kappa)\right)\\ &\leq \psi(\delta(\mu,\kappa)) - \phi(\delta(\mu,\kappa))\\ &= \psi(d^o(p,\kappa)) - \phi(d^o(p,\kappa))\\ &< \psi(d^o(p,\kappa)),\end{aligned}$$

which is a contradiction. Thus, $\kappa = \mu$ and hence the uniqueness follows. This completes the proof of the theorem.

Theorem 2.2. Let (M, d^o) be a metric space and let $\breve{Z}, \breve{H}, \breve{\Omega}$ and $\breve{K}$ be self-maps on M satisfying (2.1), (2.2) and the followings:

(2.21) The pairs ($\breve{Z}$, $\breve{\Omega}$) and ($\breve{H}$, $\breve{K}$) are weakly compatible.

(2.22) The pair ($\breve{Z}$, $\breve{\Omega}$) or ($\breve{H}$, $\breve{K}$) satisfy the E.A. property.

If one of $\breve{Z}M, \breve{H}M, \breve{\Omega}M$ or $\breve{K}M$ is complete subspace of M, then $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$ have a unique common fixed point.

Proof: Suppose that the pair ($\breve{Z}$, $\breve{\Omega}$) satisfies the E.A. property. Then, there exists a sequence $\{\sigma_n\}$ in M, such that $\lim_{n\to\infty} \breve{Z}\sigma_n = \lim_{n\to\infty} \breve{\Omega}\sigma_n = \mu$ for some μ in M.

Suppose that $\breve{H}M$ is a complete subspace of M. Then $\mu = \breve{H}\alpha$ for some $\alpha \in M$.

Now, we shall show that $\breve{K}\alpha = \breve{H}\alpha$. Let, if possible $\breve{K}\alpha \neq \breve{H}\alpha$.

From (2.2), we have

$$\psi\left(d^o(\breve{\Omega}\sigma_n, \breve{K}\alpha)\right) \leq \psi(\delta(\sigma_n,\alpha)) - \phi(\delta(\sigma_n,\alpha))$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi\left(d^0(\breve{\Omega}\sigma_n, \breve{K}\alpha)\right) \leq \lim_{n\to\infty} \psi(\delta(\sigma_n,\alpha)) - \lim_{n\to\infty} \phi(\delta(\sigma_n,\alpha)), \tag{2.23}$$

where

$$\begin{aligned}\delta(\sigma_n,\alpha) = max\Big\{&d^o(\breve{Z}\sigma_n, \breve{H}\alpha), d^o(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n), d^o(\breve{H}\alpha, \breve{K}\alpha),\\ &\frac{1}{2}\left[d^o(\breve{Z}\sigma_n, \breve{K}\alpha) + d^o(\breve{\Omega}\sigma_n, \breve{H}\alpha)\right],\\ &d^o(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n)\left[\frac{1 + d^o(\breve{Z}\sigma_n, \breve{H}\alpha)}{1 + d^o(\breve{H}\alpha, \breve{K}\alpha)}\right],\end{aligned}$$

$$d^o(\breve{H}\alpha, \breve{K}\alpha)\left[\frac{1 + d^o(\breve{Z}\sigma_n, \breve{H}\alpha)}{1 + d^o(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n)}\right],$$

$$\frac{d^{o2}(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n)}{1 + d^o(\breve{\Omega}\sigma_n, \breve{K}\alpha)}, \frac{d^{o2}(\breve{H}\alpha, \breve{K}\alpha)}{1 + d^o(\breve{\Omega}\sigma_n, \breve{K}\alpha)},$$

$$d^o(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n)\left[\frac{1 + d^o(\breve{Z}\sigma_n, \breve{K}\alpha) + d^o(\breve{\Omega}\sigma_n, \breve{H}\alpha)}{1 + d^o(\breve{Z}\sigma_n, \breve{H}\alpha) + d^o(\breve{\Omega}\sigma_n, \breve{K}\alpha)}\right],$$

$$d^o(\breve{H}\alpha, \breve{K}\alpha)\left[\frac{1 + d^o(\breve{Z}\sigma_n, \breve{K}\alpha) + d^o(\breve{\Omega}\sigma_n, \breve{H}\alpha)}{1 + d^o(\breve{Z}\sigma_n, \breve{H}\alpha) + d^o(\breve{\Omega}\sigma_n, \breve{K}\alpha)}\right]\Bigg\}.$$

$$= max\Bigg\{d^o(\mu,\mu), d^o(\mu,\mu), d^o(\mu, \breve{K}\alpha), \frac{1}{2}[d^o(\mu, \breve{K}\alpha) + d^o(\mu,\mu)],$$

$$d^o(\mu,\mu)\left[\frac{1 + d^o(\mu,\mu)}{1 + d^o(\mu, \breve{K}\alpha)}\right], d^o(\mu, \breve{K}\alpha)\left[\frac{1 + d^o(\mu,\mu)}{1 + d^o(\mu,\mu)}\right],$$

$$\frac{d^{o2}(\mu,\mu)}{1 + d^o(\mu, \breve{K}\alpha)}, \frac{d^{o2}(\mu, \breve{K}\alpha)}{1 + d^o(\mu, \breve{K}\alpha)},$$

$$d^o(\mu,\mu)\left[\frac{1 + d^o(\mu, \breve{K}\alpha) + d^o(\mu,\mu)}{1 + d^o(\mu,\mu) + d^o(\mu, \breve{K}\alpha)}\right],$$

$$d^o(\mu, \breve{K}\alpha)\left[\frac{1 + d^o(\mu, \breve{K}\alpha) + d^o(\mu,\mu)}{1 + d^o(\mu,\ \mu) + d^o(\mu, \breve{K}\alpha)}\right]\Bigg\}$$

$$= max\left\{0, 0, d^o(\mu, \breve{K}\alpha), \tfrac{1}{2}d^o(\mu, \breve{K}\alpha), 0, d^o(\mu, \breve{K}\alpha), 0,\ d^o(\mu, \breve{K}\alpha), 0, d^o(\mu, \breve{K}\alpha)\right\}$$

$$= max\{d^o(\mu, \breve{K}\alpha)\}.$$

Thus, from (2.23), we have

$$\psi\left(d^o(\mu, \breve{K}\alpha)\right) \le \psi\left(d^o(\mu, \breve{K}\alpha)\right) - \phi\left(d^o(\mu, \breve{K}\alpha)\right)$$
$$< \psi\left(d^o(\mu, \breve{K}\alpha)\right),$$

which is possible only when $\breve{K}\alpha = \mu = \breve{H}\alpha$. Since $\breve{H}$ and $\breve{K}$ are weakly compatible, therefore, $\breve{K}\breve{H}\alpha = \breve{H}\breve{K}\alpha$, implies that, $\breve{K}\breve{K}\alpha = \breve{K}\breve{H}\alpha = \breve{H}\breve{K}\alpha = \breve{H}\breve{H}\alpha$. Since $\breve{K}M \subseteq \breve{Z}M$, there exists $\beta \in M$, such that, $\breve{K}\alpha = \breve{Z}\beta$.

Now, we claim that $\breve{Z}\beta = \breve{\Omega}\beta$. Let, if possible, $\breve{Z}\beta \ne \breve{\Omega}\beta$.

From (2.2), we have

$$\psi\left(\mathrm{d}^0(\breve{\Omega}\beta, \breve{K}\alpha)\right) \le \psi(\delta(\beta,\alpha)) - \phi(\delta(\beta,\alpha)), \tag{2.24}$$

where

$$\delta(\beta,\alpha) = max\{d^o(\breve{Z}\beta,\breve{H}\alpha), d^o(\breve{Z}\beta,\breve{\Omega}\beta), d^o(\breve{H}\alpha,\breve{K}\alpha),$$

$$\frac{1}{2}[d^o(\breve{Z}\beta,\breve{K}\alpha) + d^o(\breve{\Omega}\beta,\breve{H}\alpha)],$$

$$d^o(\breve{Z}\beta,\breve{\Omega}\beta)\left[\frac{1+d^o(\breve{Z}\beta,\breve{H}\alpha)}{1+d^o(\breve{H}\alpha,\breve{K}\alpha)}\right],$$

$$d^o(\breve{H}\alpha,\breve{K}\alpha)\left[\frac{1+d^o(\breve{Z}\beta,\breve{H}\alpha)}{1+d^o(\breve{Z}\beta,\breve{\Omega}\beta)}\right],$$

$$\frac{{d^o}^2(\breve{Z}\beta,\breve{\Omega}\beta)}{1+d^o(\breve{\Omega}\beta,\breve{K}\alpha)}, \frac{{d^o}^2(\breve{H}\alpha,\breve{K}\alpha)}{1+d^o(\breve{\Omega}\beta,\breve{K}\alpha)},$$

$$d^o(\breve{Z}\beta,\breve{\Omega}\beta)\left[\frac{1+d^o(\breve{Z}\beta,\breve{K}\alpha)+d^o(\breve{\Omega}\beta,\breve{H}\alpha)}{1+d^o(\breve{Z}\beta,\breve{H}\alpha)+d^o(\breve{\Omega}\beta,\breve{K}\alpha)}\right],$$

$$d^o(\breve{H}\alpha,\breve{K}\alpha)\left[\frac{1+d^o(\breve{Z}\beta,\breve{K}\alpha)+d^o(\breve{\Omega}\beta,\breve{H}\alpha)}{1+d^o(\breve{Z}\beta,\breve{H}\alpha)+d^o(\breve{\Omega}\beta,\breve{K}\alpha)}\right]\}$$

$$= max\{d^o(\breve{K}\alpha,\breve{H}\alpha), d^o(\breve{K}\alpha,\breve{\Omega}\beta), d^o(\breve{H}\alpha,\breve{K}\alpha),$$

$$\frac{1}{2}[d^o(\breve{K}\alpha,\breve{K}\alpha) + d^o(\breve{\Omega}\beta,\breve{K}\alpha)],$$

$$d^o(\breve{K}\alpha,\breve{\Omega}\beta)\left[\frac{1+d^o(\breve{K}\alpha,\breve{H}\alpha)}{1+d^o(\breve{H}\alpha,\breve{K}\alpha)}\right],$$

$$d^o(\breve{H}\alpha,\breve{K}\alpha)\left[\frac{1+d^o(\breve{K}\alpha,\breve{H}\alpha)}{1+d^o(\breve{K}\alpha,\breve{\Omega}\beta)}\right],$$

$$\frac{{d^o}^2(\breve{K}\alpha,\breve{\Omega}\beta)}{1+d^o(\breve{\Omega}\beta,\breve{K}\alpha)}, \frac{{d^o}^2(\breve{H}\alpha,\breve{K}\alpha)}{1+d^o(\breve{\Omega}\beta,\breve{K}\alpha)},$$

$$d^o(\breve{K}\alpha,\breve{\Omega}\beta)\left[\frac{1+d^o(\breve{K}\alpha,\breve{K}\alpha)+d^o(\breve{K}\alpha,\breve{\Omega}\beta)}{1+d^o(\breve{K}\alpha,\breve{H}\alpha)+d^o(\breve{\Omega}\beta,\breve{K}\alpha)}\right],$$

$$d^o(\breve{H}\alpha,\breve{K}\alpha)\left[\frac{1+d^o(\breve{K}\alpha,\breve{K}\alpha)+d^o(\breve{\Omega}\beta,\breve{K}\alpha)}{1+d^o(\breve{K}\alpha,\breve{H}\alpha)+d^o(\breve{\Omega}\beta,\breve{K}\alpha)}\right]\}$$

$$= \max\{0, d^o(\breve{K}\alpha,\breve{\Omega}\beta), 0, \frac{1}{2}\left(0 + d^o(\breve{K}\alpha,\breve{\Omega}\beta)\right), d^o(\breve{K}\alpha,\breve{\Omega}\beta), 0,$$

$$\frac{d^2(\breve{K}\alpha,\breve{\Omega}\beta)}{1+d^o(\breve{K}\alpha,\breve{\Omega}\beta)}, 0, d^o(\breve{K}\alpha,\breve{\Omega}\beta)\}$$

$$= d^o(\breve{\Omega}\beta,\breve{K}\alpha).$$

From (2.24), we have

$$\psi\big(d^o(\breve{\Omega}\beta,\breve{K}\alpha)\big) \le \psi\big(d^o(\breve{\Omega}\beta,\breve{K}\alpha)\big) - \phi\big(d^o(\breve{\Omega}\beta,\breve{K}\alpha)\big)$$
$$< \psi\big(d^o(\breve{\Omega}\beta,\breve{K}\alpha)\big),$$

which is a contradiction. Therefore, $\breve{\Omega}\beta = \breve{K}\alpha = \breve{Z}\beta$.

Now, since $(\breve{Z},\breve{\Omega})$ is weakly compatible, it follows that $\breve{Z}\breve{\Omega}\beta = \breve{\Omega}\breve{Z}\beta = \breve{\Omega}\breve{\Omega}\beta = \breve{Z}\breve{Z}\beta$.

Now, we claim that $\breve{K}\alpha$ is common fixed point of $\breve{Z},\breve{H},\breve{\Omega}$ and $\breve{K}$.

Let, if possible, $\breve{K}\breve{K}\alpha \neq \breve{K}\alpha$.

From (2.2), we have

$$\psi\left(d^0(\breve{K}\alpha,\breve{K}\breve{K}\alpha)\right) = \psi\left(d^0(\breve{\Omega}\beta,\breve{K}\breve{K}\alpha)\right) \le \psi\left(\delta(\beta,\breve{K}\alpha)\right) - \phi\left(\delta(\beta,\breve{K}\alpha)\right), \tag{2.25}$$

where

$$\begin{aligned}\delta(\beta,\breve{K}\alpha) = \max\Big\{& d^o(\breve{Z}\beta,\breve{H}\breve{K}\alpha), d^o(\breve{Z}\beta,\breve{\Omega}\beta), d^o(\breve{H}\breve{K}\alpha,\breve{K}\breve{K}\alpha),\\
&\tfrac{1}{2}\left[d^o(\breve{Z}\beta,\breve{K}\breve{K}\alpha) + d^o(\breve{\Omega}\beta,\breve{H}\breve{K}\alpha)\right],\\
&d^o(\breve{Z}\beta,\breve{\Omega}\beta)\left[\frac{1+d^o(\breve{Z}\beta,\breve{H}\breve{K}\alpha)}{1+d^o(\breve{H}\breve{K}\alpha,\breve{K}\breve{K}\alpha)}\right],\\
&d^o(\breve{H}\breve{K}\alpha,\breve{K}\breve{K}\alpha)\left[\frac{1+d^o(\breve{Z}\beta,\breve{H}\breve{K}\alpha)}{1+d^o(\breve{Z}\beta,\breve{\Omega}\beta)}\right],\\
&\frac{d^{o2}(\breve{Z}\beta,\breve{\Omega}\beta)}{1+d^o(\breve{\Omega}\beta,\breve{K}\breve{K}\alpha)}, \frac{d^{o2}(\breve{H}\breve{K}\alpha,\breve{K}\breve{K}\alpha)}{1+d^o(\breve{\Omega}\beta,\breve{K}\breve{K}\alpha)},\\
&d^o(\breve{Z}\beta,\breve{\Omega}\beta)\left[\frac{1+d^o(\breve{Z}\beta,\breve{K}\breve{K}\alpha)+d^o(\breve{\Omega}\beta,\breve{H}\breve{K}\alpha)}{1+d^o(\breve{Z}\beta,\breve{H}\breve{K}\alpha)+d^o(\breve{\Omega}\beta,\breve{K}\breve{K}\alpha)}\right],\\
&d^o(\breve{H}\breve{K}\alpha,\breve{K}\breve{K}\alpha)\left[\frac{1+d^o(\breve{Z}\beta,\breve{K}\breve{K}\alpha)+d^o(\breve{\Omega}\beta,\breve{H}\breve{K}\alpha)}{1+d^o(\breve{Z}\beta,\breve{H}\breve{K}\alpha)+d^o(\breve{\Omega}\beta,\breve{K}\breve{K}\alpha)}\right]\Big\}\end{aligned}$$

$$\begin{aligned}= \max\Big\{& d^o(\breve{K}\alpha,\breve{K}\breve{K}\alpha), d^o(\breve{K}\alpha,\breve{K}\alpha), d^o(\breve{H}\breve{K}\alpha,\breve{K}\breve{K}\alpha),\\
&\tfrac{1}{2}\left[d^o(\breve{K}\alpha,\breve{K}\breve{K}\alpha) + d^o(\breve{K}\alpha,\breve{K}\breve{K}\alpha)\right],\\
&d^o(\breve{K}\alpha,\breve{K}\alpha)\left[\frac{1+d^o(\breve{K}\alpha,\breve{K}\breve{K}\alpha)}{1+d^o(\breve{K}\breve{K}\alpha,\breve{K}\breve{K}\alpha\alpha)}\right],\\
&d^o(\breve{K}\breve{K}\alpha,\breve{K}\breve{K}\alpha)\left[\frac{1+d^o(\breve{K}\alpha,\breve{K}\breve{K}\alpha)}{1+d^o(\breve{K}\alpha,\breve{K}\alpha)}\right],\\
&\frac{d^{o2}(\breve{K}\alpha,\breve{K}\alpha)}{1+d^o(\breve{K}\alpha,\breve{K}\breve{K}\alpha)}, \frac{d^{o2}(\breve{K}\breve{K}\alpha,\breve{K}\breve{K}\alpha)}{1+d^o(\breve{K}\alpha,\breve{K}\alpha)},\end{aligned}$$

$$d^o(\breve{K}\alpha, \breve{K}\alpha)\left[\frac{1 + d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha) + d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha)}{1 + d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha) + d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha)}\right],$$

$$d^o(\breve{K}\breve{K}\alpha, \breve{K}\breve{K}\alpha)\left[\frac{1 + d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha) + d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha)}{1 + d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha) + d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha)}\right]\Bigg\}$$

$$= max\{d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha), 0, 0, d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha), 0, 0, 0, 0, 0, 0\}$$

$$= d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha).$$

Thus, from (2.25), we have

$$\psi\left(d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha)\right) \le \psi\left(d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha)\right) - \phi\left(d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha)\right)$$

$$< \psi\left(d^o(\breve{K}\alpha, \breve{K}\breve{K}\alpha)\right),$$

which is a contradiction. Therefore, $\breve{K}\alpha = \breve{K}\breve{K}\alpha = \breve{H}\breve{K}\alpha$.

Hence $\breve{K}\alpha$ is the common fixed point of $\breve{H}$ and $\breve{K}$.

Similarly, we can prove that $\breve{\Omega}\beta$ is common fixed point of $\breve{\Omega}$ and $\breve{Z}$. Since $\breve{K}\alpha = \breve{\Omega}\beta$, $\breve{K}\alpha$ is the common fixed point of $\breve{Z}, \breve{H}, \breve{\Omega}$ and $\breve{K}$. If we assume $\breve{\Omega}M$ is complete subspace of M, the proof is similar.

Similarly, we can prove the theorem for cases when $\breve{Z}M$ or $\breve{H}M$ is a complete subspace of M. Since $\breve{K}M \subseteq \breve{Z}M$ and $\breve{\Omega}M \subseteq \breve{H}M$.

Now, we shall prove the uniqueness of common fixed point. If possible, let σ and ω be two common fixed points of $\breve{Z}, \breve{H}, \breve{\Omega}$ and $\breve{K}$ such that $\sigma \neq \omega$.

From (2.2), we have

$$\psi\left(d^o(\breve{\Omega}\sigma, \breve{K}\omega)\right) \le \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)), \tag{2.26}$$

where

$$\delta(\sigma, \omega) = max\{d^o(\breve{Z}\sigma, \breve{H}\omega), d^o(\breve{Z}\sigma, \breve{\Omega}\sigma), d^o(\breve{H}\omega, \breve{K}\omega),$$

$$\tfrac{1}{2}[d^o(\breve{Z}\sigma, \breve{K}\omega) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)],$$

$$d^o(\breve{Z}\sigma, \breve{\Omega}\sigma)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{H}\omega)}{1 + d^o(\breve{H}\omega, \breve{K}\omega)}\right],$$

$$d^o(\breve{H}\omega, \breve{K}\omega)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{H}\omega)}{1 + d^o(\breve{Z}\sigma, \breve{\Omega}\sigma)}\right],$$

$$\frac{{d^o}^2(\breve{Z}\sigma, \breve{\Omega}\sigma)}{1 + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}, \frac{{d^o}^2(\breve{H}\omega, \breve{K}\omega)}{1 + d^o(\breve{\Omega}\sigma, \breve{K}\omega)},$$

$$d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{K}\omega)+d^o(\breve{\Omega}\sigma,\breve{H}\omega)}{1+d^o(\breve{Z}\sigma,\breve{H}\omega)+d^o(\breve{\Omega}\sigma,\breve{K}\omega)}\right],$$

$$d^o(\breve{H}\omega,\breve{K}\omega)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{K}\omega)+d^o(\breve{\Omega}\sigma,\breve{H}\omega)}{1+d^o(\breve{Z}\sigma,\breve{H}\omega)+d^o(\breve{\Omega}\sigma,\breve{K}\omega)}\right]\Bigg\}$$

$$= max\{d^o(\sigma,\omega), d^o(\sigma,\sigma), d^o(\omega,\omega), \tfrac{1}{2}[d^o(\sigma,\omega)+d^o(\sigma,\omega)],$$

$$d^o(\sigma,\sigma)\left[\frac{1+d^o(\sigma,\omega)}{1+d^o(\omega,\omega)}\right], d^o(\omega,\omega)\left[\frac{1+d^o(\sigma,\omega)}{1+d^o(\sigma,\sigma)}\right], \frac{{d^o}^2(\sigma,\sigma)}{1+d^o(\sigma,\omega)},$$

$$\frac{{d^o}^2(\omega,\omega)}{1+d^o(\sigma,\omega)}, d^o(\sigma,\sigma)\left[\frac{1+d^o(\sigma,\omega)+d^o(\sigma,\omega)}{1+d^o(\sigma,\omega)+d^o(\sigma,\omega)}\right],$$

$$d^o(\omega,\omega)\left[\frac{1+d^o(\sigma,\omega)+d^o(\sigma,\omega)}{1+d^o(\sigma,\omega)+d^o(\sigma,\omega)}\right]\Bigg\}$$

$$= max\{d^o(\sigma,\omega), 0, 0, d^o(\sigma,\omega), 0, 0, 0, 0, 0, 0\}$$

From (2.26), we have

$$\psi(d^o(\sigma,\omega)) \le \psi(d^o(\sigma,\omega)) - \phi(d^o(\sigma,\omega))$$
$$< \psi(d^o(\sigma,\omega)),$$

which is a contradiction.

Therefore, $\sigma = \omega$ and this follows the uniqueness and completes the proof of the theorem.

Theorem 2.3. Let (M, d^0) be a metric space. let Z̆, H̆, Ω̆ and $\breve{K}$ be self-maps on M satisfying (2.2) and (2.21) and the followings:

(2.27) $\breve{\Omega}M \subseteq \breve{H}M$ and the pair (Z̆, Ω̆) satisfied $(CLR_{\breve{Z}})$ or

$\breve{K}M \subseteq \breve{Z}M$ and the pair (H̆, $\breve{K}$) satisfied $(CLR_{\breve{H}})$.

Then Z̆, H̆, Ω̆ and $\breve{K}$ have unique common fixed point.

Proof: Without loss of generality, assume that $\breve{\Omega}M \subseteq \breve{H}M$ and the pair (Z̆, Ω̆) satisfied $(CLR_{\breve{Z}})$ property. Then, there exists a sequence $\{\sigma_n\}$ in M such that

$$\lim_{n\to\infty} \breve{Z}\sigma_n = \lim_{n\to\infty} \breve{\Omega}u_n = \breve{Z}\mu \text{ for some } \mu \text{ in } M.$$

Since $\breve{\Omega}M \subseteq \breve{H}M$, there exists a sequence $\{\omega_n\}$ in M such that $\breve{\Omega}\sigma_n = \breve{H}\omega_n$.

Hence

$$\lim_{n\to\infty} \breve{H}\omega_n = \breve{Z}\mu.$$

Now, we shall show that $\lim_{n\to\infty} \breve{K}\omega_n = \breve{Z}\mu$. Let if possible, $\lim_{n\to\infty} \breve{K}\omega_n = \pi \ne \breve{Z}\mu$.

From (2.2), we have

$$\psi\left(d^0\left(\breve{\Omega}\sigma_n, \breve{K}\omega_n\right)\right) \le \psi\left(\delta(\sigma_n, \omega_n)\right) - \phi\left(\delta(\sigma_n, \omega_n)\right), \tag{2.28}$$

where

$$\begin{aligned}
\delta(\sigma_n, \omega_n) = max\Big\{ & d^o\left(\breve{Z}\sigma_n, \breve{H}\omega_n\right), d^o\left(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n\right), d^o\left(\breve{H}\omega_n, \breve{K}\omega_n\right), \\
& \frac{1}{2}\left[d^o\left(\breve{Z}\sigma_n, \breve{K}\omega_n\right) + d^o(\breve{\Omega}\sigma_n, \breve{H}\omega_n)\right], \\
& d^o\left(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n\right)\left[\frac{1 + d^o\left(\breve{Z}\sigma_n, \breve{H}\omega_n\right)}{1 + d^o\left(\breve{H}\omega_n, \breve{K}\omega_n\right)}\right], \\
& d^o\left(\breve{H}\omega_n, \breve{K}\omega_n\right)\left[\frac{1 + d^o\left(\breve{Z}\sigma_n, \breve{H}\omega_n\right)}{1 + d^o\left(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n\right)}\right], \\
& \frac{d^{o^2}\left(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n\right)}{1 + d^o(\breve{\Omega}\sigma_n, \breve{K}\omega_n)}, \frac{d^{o^2}\left(\breve{H}\omega_n, \breve{K}\omega_n\right)}{1 + d^o\left(\breve{\Omega}\sigma_n, \breve{K}\omega_n\right)}, \\
& d^o\left(\breve{Z}\sigma_n, \breve{\Omega}\sigma_n\right)\left[\frac{1 + d^o\left(\breve{Z}\sigma_n, \breve{K}\omega_n\right) + d^o(\breve{\Omega}\sigma_n, \breve{H}\omega_n)}{1 + d^o\left(\breve{Z}\sigma_n, \breve{H}\omega_n\right) + d^o(\breve{\Omega}\sigma_n, \breve{K}\omega_n)}\right], \\
& d^o\left(\breve{H}\omega_n, \breve{K}\omega_n\right)\left[\frac{1 + d^o\left(\breve{Z}\sigma_n, \breve{K}\omega_n\right) + d^o(\breve{\Omega}\sigma_n, \breve{H}\omega_n)}{1 + d^o\left(\breve{Z}\sigma_n, \breve{H}\omega_n\right) + d^o(\breve{\Omega}\sigma_{2n}, \breve{K}\omega_n)}\right]\Big\}
\end{aligned}$$

$$\begin{aligned}
= max\Big\{ & d^o\left(\breve{Z}\mu, \breve{Z}\mu\right), d^o\left(\breve{Z}\mu, \breve{Z}\mu\right), d^o\left(\breve{Z}\mu, \pi\right), \\
& \tfrac{1}{2}\left[d^o\left(\breve{Z}\mu, \pi\right) + d^o(\breve{Z}\mu, \breve{Z}\mu)\right], \\
& d^o\left(\breve{Z}\mu, \breve{Z}\mu\right)\left[\frac{1 + d^o\left(\breve{Z}\mu, \breve{Z}\mu\right)}{1 + d^o\left(\breve{Z}\mu, \pi\right)}\right], d^o\left(\breve{Z}\mu, \pi\right)\left[\frac{1 + d^o\left(\breve{Z}\mu, \breve{Z}\mu\right)}{1 + d^o\left(\breve{Z}\mu, \breve{Z}\mu\right)}\right], \\
& \frac{d^{o^2}\left(\breve{Z}\mu, \breve{Z}\mu\right)}{1 + d^o(\breve{Z}\mu, \pi)}, \frac{d^{o^2}\left(\breve{Z}\mu, \pi\right)}{1 + d^o(\breve{Z}\mu, \pi)}, \\
& d^o\left(\breve{Z}\mu, \breve{Z}\mu\right)\left[\frac{1 + d^o\left(\breve{Z}\mu, \pi\right) + d^o(\breve{Z}\mu, \breve{Z}\mu)}{1 + d^o(\breve{Z}\mu, \breve{Z}\mu) + d^o(\breve{Z}\mu, \pi)}\right], \\
& d^o\left(\breve{Z}\mu, \pi\right)\left[\frac{1 + d^o\left(\breve{Z}\mu, \pi\right) + d^o(\breve{Z}\mu, \breve{Z}\mu)}{1 + d^o(\breve{Z}\mu, \breve{Z}\mu) + d^o(\breve{Z}\mu, \pi)}\right]\Big\}
\end{aligned}$$

$$= max\left\{0, 0, d^o\left(\breve{Z}\mu, \pi\right), \tfrac{1}{2}d^o\left(\breve{Z}\mu, \pi\right), 0, d^o\left(\breve{Z}\mu, \pi\right), 0, \tfrac{d^{o^2}(\breve{Z}\mu, \pi)}{1 + d^o(\breve{Z}\mu, \pi)}, 0, d^o\left(\breve{Z}\mu, \pi\right)\right\}$$

$$= d^o\left(\breve{Z}\mu, \pi\right)$$

From (2.28), we have

$$\psi\left(d^o(\breve{Z}\mu,\pi)\right) \le \psi\left(d^o(\breve{Z}\mu,\pi)\right) - \phi\left(d^o(\breve{Z}\mu,\pi)\right)$$
$$< \psi\left(d^o(\breve{Z}\mu,\pi)\right),$$

which is a contradiction.

Therefore, $\breve{Z}\mu = \pi$, that is, $\lim_{n\to\infty} \breve{K}\omega_n = \pi \neq \breve{Z}\mu$..

Subsequently, we have $\lim_{n\to\infty} \breve{K}\omega_n = \lim_{n\to\infty} \breve{\Omega}\mu_n = \lim_{n\to\infty} \breve{Z}\mu_n = \lim_{n\to\infty} \breve{H}\omega_n = \breve{Z}\mu = \pi.$

Now, we shall show that $\breve{\Omega}\mu = \pi$. let, if possible, $\breve{\Omega}\mu \neq \pi$.

From (2.2), we have

$$\psi\left(d^0(\breve{\Omega}\mu,\breve{K}\omega_n)\right) \le \psi(\delta(\mu,\omega_n)) - \phi(\delta(\mu,\omega_n)),$$

$$\lim_{n\to\infty} \psi\left(d^0(\breve{\Omega}\mu,\breve{K}\omega_n)\right) \le \lim_{n\to\infty}\psi(\delta(\mu,\omega_n)) - \lim_{n\to\infty}\phi(\delta(\mu,\omega_n)), \tag{2.29}$$

where

$$\begin{aligned}\delta(\mu,\omega_n) = max\Big\{ & d^o(\breve{Z}\mu,\breve{H}\omega_n), d^o(\breve{Z}\mu,\breve{\Omega}\mu), d^o(\breve{H}\omega_n,\breve{K}\omega_n), \\ & \frac{1}{2}\left[d^o(\breve{Z}\mu,\breve{K}\omega_n) + d^o(\breve{\Omega}\mu,\breve{H}\omega_n)\right], \\ & d^o(\breve{Z}\mu,\breve{\Omega}\mu)\left[\frac{1+d^o(\breve{Z}\mu,\breve{H}\omega_n)}{1+d^o(\breve{H}\omega_n,\breve{K}\omega_n)}\right], \\ & d^o(\breve{Z}\mu,\breve{\Omega}\mu)\left[\frac{1+d^o(\breve{Z}\mu,\breve{H}\omega_n)}{1+d^o(\breve{H}\omega_n,\breve{K}\omega_n)}\right], \\ & d^o(\breve{H}\omega_n,\breve{K}\omega_n)\left[\frac{1+d^o(\breve{Z}\mu,\breve{H}\omega_n)}{1+d^o(\breve{Z}\mu,\breve{\Omega}\mu)}\right], \\ & \frac{d^{o2}(\breve{Z}\mu,\breve{\Omega}\mu)}{1+d^o(\breve{\Omega}\mu,\breve{K}\omega_n)}, \frac{d^{o2}(\breve{H}\omega_n,\breve{K}\omega_n)}{1+d^o(\breve{\Omega}\mu,\breve{K}\omega_n)}, \\ & d^o(\breve{Z}\mu,\breve{\Omega}\mu)\left[\frac{1+d^o(\breve{Z}\mu,\breve{K}\omega_n)+d^o(\breve{\Omega}\mu,\breve{H}\omega_n)}{1+d^o(\breve{Z}\mu,\breve{H}\omega_n)+d^o(\breve{\Omega}\mu,\breve{K}\omega_n)}\right], \\ & d^o(\breve{H}\omega_n,\breve{K}\omega_n)\left[\frac{1+d^o(\breve{Z}\mu,\breve{K}\omega_n)+d^o(\breve{\Omega}\mu,\breve{H}\omega_n)}{1+d^o(\breve{Z}\mu,\breve{H}\omega_n)+d^o(\breve{\Omega}\mu,\breve{K}\omega_n)}\right]\Big\}\end{aligned}$$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty}\delta(\mu,\omega_n) = max\{d^o(\pi,\pi), d^o(\pi,\breve{\Omega}\mu), d^o(\pi,\pi), \tfrac{1}{2}\left[d^o(\pi,\pi) + d^o(\breve{\Omega}\mu,\pi)\right],$$
$$d^o(\pi,\breve{\Omega}\mu)\left[\frac{1+d^o(\pi,\pi)}{1+d^o(\pi,\pi)}\right], d^o(\pi,\pi)\left[\frac{1+d^o(\pi,\pi)}{1+d^o(\pi,\breve{\Omega}\mu)}\right],$$

$$\frac{{d^o}^2(\pi,\breve{\Omega}\mu)}{1+d^o(\breve{\Omega}\mu,\pi)},\frac{{d^o}^2(\pi,\pi)}{1+d^o(\breve{\Omega}\mu,\pi)},$$

$$d^o(\pi,\pi)\left[\frac{1+d^o(\pi,\pi)+d^o(\breve{\Omega}\mu,\pi)}{1+d^o(\pi,\pi)+d^o(\breve{\Omega}\mu,\pi)}\right],$$

$$d^o(\pi,\pi)\left[\frac{1+d^o(\pi,\pi)+d^o(\breve{\Omega}\mu,\pi)}{1+d^o(\pi,\pi)+d^o(\breve{\Omega}\mu,\pi)}\right]\Bigg\}$$

$$= max\left\{0, d^o(\pi,\breve{\Omega}\mu), 0, \ \tfrac{1}{2}d^o(\pi,\breve{\Omega}\mu), d^o(\pi,\breve{\Omega}\mu), 0, \ \tfrac{{d^o}^2(\pi,\breve{\Omega}\mu)}{1+d^o(\breve{\Omega}\mu,\pi)}, 0, 0, 0\right\}$$

$$= d^o(\pi,\breve{\Omega}\mu).$$

Thus, from (2.29), we get

$$\psi\left(d^o(\pi,\breve{\Omega}\mu)\right) \le \psi\left(d^o(\pi,\breve{\Omega}\mu)\right) - \ \phi\left(d^o(\pi,\breve{\Omega}\mu)\right).$$

$$< \psi\left(d^o(\pi,\breve{\Omega}\mu)\right),$$

which is a contradiction. Therefore, $\breve{\Omega}\mu \ = \ \pi = \breve{Z}\mu$. Since the pair $(\breve{Z},\breve{\Omega})$ is weakly compatible, it follows that $\breve{Z}\pi = \breve{\Omega}\pi$. Also, since $\breve{\Omega}M \subseteq \breve{H}M$, there exists some θ in M, such that, $\breve{\Omega}\mu = \breve{H}\theta$, that is, $\breve{H}\theta = \pi$.

Now, we show that $\breve{K}\theta = \pi$. Let if possible $\breve{K}\theta \neq \pi$.

From (2.2), we have

$$\psi\left(d^0(\breve{\Omega}\mu_n, \breve{K}\theta)\right) \le \psi(\delta(\mu_n,\theta)) - \ \phi(\delta(\mu_n,\theta)),$$

$$\lim_{n\to\infty}\psi\left(d^0(\breve{\Omega}\mu_n, \breve{K}\theta)\right) \le \lim_{n\to\infty}\psi(\delta(\mu_n,\theta)) - \lim_{n\to\infty}\phi(\delta(\mu_n,\theta)), \qquad (2.30)$$

where

$$\delta(\mu_n,\theta) = max\{d^o(\breve{Z}\mu_n,\breve{H}\theta), d^o(\breve{Z}\mu_n,\breve{\Omega}\mu_n), d^o(\breve{H}\theta,\breve{K}\theta),$$

$$\frac{1}{2}\left[d^o(\breve{Z}\mu_n,\breve{K}\theta) + d^o(\breve{\Omega}\mu_n,\breve{H}\theta)\right],$$

$$d^o(\breve{Z}\mu_n,\breve{\Omega}\mu_n)\left[\frac{1+d^o(\breve{Z}\mu_n,\breve{H}\theta)}{1+d^o(\breve{H}\theta,\breve{K}\theta)}\right],$$

$$d^o(\breve{Z}\mu_n,\breve{\Omega}\mu_n)\left[\frac{1+d^o(\breve{Z}\mu_n,\breve{H}\theta)}{1+d^o(\breve{H}\theta,\breve{K}\theta)}\right],$$

$$d^o(\breve{H}\theta,\breve{K}\theta)\left[\frac{1+d^o(\breve{Z}\mu_n,\breve{H}\theta)}{1+d^o(\breve{Z}\mu_n,\breve{\Omega}\mu_n)}\right],$$

$$\frac{d^{o^2}(\breve{Z}\mu_n, \breve{\Omega}\mu_n)}{1 + d^o(\breve{\Omega}\mu_n, \breve{K}\theta)}, \frac{d^{o^2}(\breve{H}\theta, \breve{K}\theta)}{1 + d^o(\breve{\Omega}\mu, \breve{K}\theta)},$$

$$d^o(\breve{Z}\mu_n, \breve{\Omega}\mu_n)\left[\frac{1 + d^o(\breve{Z}\mu_n, \breve{K}\theta) + d^o(\breve{\Omega}\mu_n, \breve{H}\theta)}{1 + d^o(\breve{Z}\mu_n, \breve{H}\theta) + d^o(\breve{\Omega}\mu_n, \breve{K}\theta)}\right],$$

$$d^o(\breve{H}\theta, \breve{K}\theta)\left[\frac{1 + d^o(\breve{Z}\mu_n, \breve{K}\theta) + d^o(\breve{\Omega}\mu_n, \breve{H}\theta)}{1 + d^o(\breve{Z}\mu_n, \breve{H}\theta) + d^o(\breve{\Omega}\mu_n, \breve{K}\theta)}\right]\Bigg\}$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \delta(\mu_n, \theta) = max\{d^o(\pi,\pi), d^o(\pi,\pi), d^o(\pi, \breve{K}\theta), \tfrac{1}{2}\left[d^o(\pi, \breve{K}\theta) + d^o(\pi,\pi)\right],$$

$$d^o(\pi, \breve{K}\theta)\left[\frac{1 + d^o(\pi,\pi)}{1 + d^o(\pi, \breve{K}\theta)}\right], d^o(\pi, \breve{K}\theta)\left[\frac{1 + d^o(\pi,\pi)}{1 + d^o(\pi,\pi)}\right],$$

$$\frac{d^{o^2}(\pi,\pi)}{1 + d^o(\pi, \breve{K}\theta)}, \frac{d^{o^2}(\pi, \breve{K}\theta)}{1 + d^o(\pi, \breve{K}\theta)},$$

$$d^o(\pi,\pi)\left[\frac{1 + d^o(\pi, \breve{K}\theta) + d^o(\pi,\pi)}{1 + d^o(\pi,\pi) + d^o(\pi, \breve{K}\theta)}\right],$$

$$d^o(\pi, \breve{K}\theta)\left[\frac{1 + d^o(\pi, \breve{K}\theta) + d^o(\pi,\pi)}{1 + d^o(\pi,\pi) + d^o(\pi, \breve{K}\theta)}\right]\Bigg\}$$

$$= max\left\{0,\ 0, d^o(\pi, \breve{K}\theta),\ \tfrac{1}{2} d^o(\pi, \breve{K}\theta),\ \tfrac{d^o(\pi, \breve{K}\theta)}{1 + d^o(\pi, \breve{K}\theta)}, d^o(\pi, \breve{K}\theta), 0,\right.$$

$$\left.\tfrac{d^{o^2}(\pi, \breve{K}\theta)}{1 + d^o(\pi, \breve{K}\theta)}, 0, d^o(\pi, \breve{K}\theta)\right\}$$

$$= d^o(\pi, \breve{K}\theta).$$

Thus, from (2.30), we get

$$\psi(d^o(\pi, \breve{K}\theta)) \le \psi(d^o(\pi, \breve{K}\theta)) - \phi(d^o(\pi, \breve{K}\theta)).$$

$$< \psi(d^o(\pi, \breve{K}\theta)),$$

which is a contradiction. Therefore, $\breve{K}\theta = \pi = \breve{H}\theta$. Since the pair $(\breve{H}, \breve{K})$ is weakly compatible, it follows that $\breve{K}\pi = \breve{H}\pi$. Now, we claim that $\breve{\Omega}\pi = \breve{K}\pi$. Let, if possible, $\breve{\Omega}\pi \ne \breve{K}\pi$.

From (2.2), we have

$$\psi\left(d^o(\breve{\Omega}\pi, \breve{K}\pi)\right) \le \psi(\delta(\pi,\pi)) - \phi(\delta(\pi,\pi)), \tag{2.31}$$

where

$$\delta(\pi,\pi) = max\{d^o(\breve{Z}\pi, \breve{H}\pi), d^o(\breve{Z}\pi, \breve{\Omega}\pi), d^o(\breve{H}\pi, \breve{K}\pi),$$

$$\frac{1}{2}\left[d^o\left(\breve{Z}\pi,\breve{K}\pi\right)+d^o(\breve{\Omega}\pi,\breve{H}\pi)\right],$$

$$d^o\left(\breve{Z}\pi,\breve{\Omega}\pi\right)\left[\frac{1+d^o\left(\breve{Z}\pi,\breve{H}\pi\right)}{1+d^o\left(\breve{H}\pi,\breve{K}\pi\right)}\right],$$

$$d^o\left(\breve{H}\pi,\breve{K}\pi\right)\left[\frac{1+d^o\left(\breve{Z}\pi,\breve{H}\pi\right)}{1+d^o\left(\breve{Z}\pi,\breve{\Omega}\pi\right)}\right],$$

$$\frac{{d^o}^2\left(\breve{Z}\pi,\breve{\Omega}\pi\right)}{1+d^o(\breve{\Omega}\pi,\breve{K}\pi)},\ \frac{{d^o}^2\left(\breve{H}\pi,\breve{K}\pi\right)}{1+d^o(\breve{\Omega}\pi,\breve{K}\pi)},$$

$$d^o\left(\breve{Z}\pi,\breve{\Omega}\pi\right)\left[\frac{1+d^o\left(\breve{Z}\pi,\breve{K}\pi\right)+d^o(\breve{\Omega}\pi,\breve{H}\pi)}{1+d^o\left(\breve{Z}\pi,\breve{H}\pi\right)+d^o(\breve{\Omega}\pi,\breve{K}\pi)}\right],$$

$$\left.d^o\left(\breve{H}\pi,\breve{K}\pi\right)\left[\frac{1+d^o\left(\breve{Z}\pi,\breve{K}\pi\right)+d^o(\breve{\Omega}\pi,\breve{H}\pi)}{1+d^o\left(\breve{Z}\pi,\breve{H}\pi\right)+d^o(\breve{\Omega}\pi,\breve{K}\pi)}\right]\right\}$$

$$= max\left\{d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right),d^o\left(\breve{\Omega}\pi,\breve{\Omega}\pi\right),d^o\left(\breve{K}\pi,\breve{K}\pi\right),\right.$$

$$\frac{1}{2}\left[d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)+d^o(\breve{\Omega}\pi,\breve{K}\pi)\right],$$

$$d^o\left(\breve{\Omega}\pi,\breve{\Omega}\pi\right)\left[\frac{1+d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)}{1+d^o\left(\breve{K}\pi,\breve{K}\pi\right)}\right],$$

$$d^o\left(\breve{K}\pi,\breve{K}\pi\right)\left[\frac{1+d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)}{1+d^o\left(\breve{\Omega}\pi,\breve{\Omega}\pi\right)}\right],$$

$$\frac{{d^o}^2\left(\breve{\Omega}\pi,\breve{\Omega}\pi\right)}{1+d^o(\breve{\Omega}\pi,\breve{K}\pi)},\frac{{d^o}^2\left(\breve{K}\pi,\breve{K}\pi\right)}{1+d^o(\breve{\Omega}\pi,\breve{K}\pi)},$$

$$d^o\left(\breve{\Omega}\pi,\breve{\Omega}\pi\right)\left[\frac{1+d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)+d^o(\breve{\Omega}\pi,\breve{K}\pi)}{1+d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)+d^o(\breve{\Omega}\pi,\breve{K}\pi)}\right],$$

$$\left.d^o\left(\breve{K}\pi,\breve{K}\pi\right)\left[\frac{1+d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)+d^o(\breve{\Omega}\pi,\breve{K}\pi)}{1+d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)+d^o(\breve{\Omega}\pi,\breve{K}\pi)}\right]\right\}$$

$$= max\left\{d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right),0,0,d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right),0,0,0,0,0,0\right\}.$$

From (2.31), we have

$$\psi\left(d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)\right)\le\psi\left(d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)\right)-\ \phi\left(d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)\right)$$

$$<\psi\left(d^o\left(\breve{\Omega}\pi,\breve{K}\pi\right)\right),$$

which is a contradiction. Thus, $\breve{\Omega}\pi=\ \breve{K}\pi$, that is, $\breve{Z}\pi\ =\ \breve{\Omega}\pi=\ \breve{K}\pi\ =\ \breve{H}\pi$.

Now, we shall show that $\pi\ =\ \breve{K}\pi$. Let, if possible, $\pi\neq\breve{K}\pi$.

From (2.2), we have

$$\psi\left(d^o\left(\breve{\Omega}\mu,\breve{K}\pi\right)\right) \leq \psi\left(\delta(\mu,\pi)\right) - \phi\left(\delta(\mu,\pi)\right), \tag{2.32}$$

where

$$\begin{aligned}\delta(\mu,\pi) = max\Big\{ & d^o\left(\breve{Z}\mu,\breve{H}\pi\right), d^o\left(\breve{Z}\mu,\breve{\Omega}\mu\right), d^o\left(\breve{H}\pi,\breve{K}\pi\right),\\
& \frac{1}{2}\left[d^o\left(\breve{Z}\mu,\breve{K}\pi\right) + d^o(\breve{\Omega}\mu,\breve{H}\pi)\right],\\
& d^o\left(\breve{Z}\mu,\breve{\Omega}\mu\right)\left[\frac{1+d^o\left(\breve{Z}\mu,\breve{H}\pi\right)}{1+d^o\left(\breve{H}\pi,\breve{K}\pi\right)}\right],\\
& d^o\left(\breve{Z}\mu,\breve{\Omega}\mu\right)\left[\frac{1+d^o\left(\breve{Z}\mu,\breve{H}\pi\right)}{1+d^o\left(\breve{H}\pi,\breve{K}\pi\right)}\right],\\
& d^o\left(\breve{H}\pi,\breve{K}\pi\right)\left[\frac{1+d^o\left(\breve{Z}\mu,\breve{H}\pi\right)}{1+d^o\left(\breve{Z}\mu,\breve{\Omega}\mu\right)}\right],\\
& \frac{{d^o}^2\left(\breve{Z}\mu,\breve{\Omega}\mu\right)}{1+d^o(\breve{\Omega}\mu,\breve{K}\pi)}, \frac{{d^o}^2\left(\breve{H}\omega_n,\breve{K}\pi\right)}{1+d^o\left(\breve{\Omega}\mu,\breve{K}\pi\right)},\\
& d^o\left(\breve{Z}\mu,\breve{\Omega}\mu\right)\left[\frac{1+d^o\left(\breve{Z}\mu,\breve{K}\pi\right)+d^o(\breve{\Omega}\mu,\breve{H}\pi)}{1+d^o\left(\breve{Z}\mu,\breve{H}\pi\right)+d^o(\breve{\Omega}\mu,\breve{K}\pi)}\right],\\
& d^o\left(\breve{H}\pi,\breve{K}\pi\right)\left[\frac{1+d^o\left(\breve{Z}\mu,\breve{K}\pi\right)+d^o(\breve{\Omega}\mu,\breve{H}\pi)}{1+d^o\left(\breve{Z}\mu,\breve{H}\pi\right)+d^o(\breve{\Omega}\mu,\breve{K}\pi)}\right]\Big\}\\
= {} & d^o\left(\pi,\breve{K}\pi\right).\end{aligned}$$

From (2.32), we have

$$\begin{aligned}\psi\left(d^o\left(\pi,\breve{K}\pi\right)\right) &\leq \psi\left(d^o\left(\pi,\breve{K}\pi\right)\right) - \phi\left(d^o\left(\pi,\breve{K}\pi\right)\right)\\ &< \psi\left(d^o\left(\pi,\breve{K}\pi\right)\right),\end{aligned}$$

which is a contradiction. Therefore, $\pi = \breve{K}\pi = \breve{H}\pi = \breve{Z}\pi = \breve{\Omega}\pi$.

Hence π is the common fixed point of $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$.

Now, we shall prove the uniqueness of common fixed point. If possible, let σ and ω be two common fixed points of $\breve{Z}, \breve{H}, \breve{\Omega}$ and $\breve{K}$ such that $\sigma \neq \omega$.

From (2.2), we have

$$\psi\left(d^o\left(\breve{\Omega}\sigma,\breve{K}\omega\right)\right) \leq \psi\left(\delta(\sigma,\omega)\right) - \phi\left(\delta(\sigma,\omega)\right), \tag{2.33}$$

where

$$\delta(\sigma,\omega) = max\{d^o(\breve{Z}\sigma,\breve{H}\omega), d^o(\breve{Z}\sigma,\breve{\Omega}\sigma), d^o(\breve{H}\omega,\breve{K}\omega),$$

$$\frac{1}{2}\left[d^o(\breve{Z}\sigma,\breve{K}\omega) + d^o(\breve{\Omega}\sigma,\breve{H}\omega)\right],$$

$$d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)\left[\frac{1 + d^o(\breve{Z}\sigma,\breve{H}\omega)}{1 + d^o(\breve{H}\omega,\breve{K}\omega)}\right],$$

$$d^o(\breve{H}\omega,\breve{K}\omega)\left[\frac{1 + d^o(\breve{Z}\sigma,\breve{H}\omega)}{1 + d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)}\right],$$

$$\frac{{d^o}^2(\breve{Z}\sigma,\breve{\Omega}\sigma)}{1 + d^o(\breve{\Omega}\sigma,\breve{K}\omega)}, \frac{{d^o}^2(\breve{H}\omega,\breve{K}\omega)}{1 + d^o(\breve{\Omega}\sigma,\breve{K}\omega)},$$

$$d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)\left[\frac{1 + d^o(\breve{Z}\sigma,\breve{K}\omega) + d^o(\breve{\Omega}\sigma,\breve{H}\omega)}{1 + d^o(\breve{Z}\sigma,\breve{H}\omega) + d^o(\breve{\Omega}\sigma,\breve{K}\omega)}\right],$$

$$d^o(\breve{H}\omega,\breve{K}\omega)\left[\frac{1 + d^o(\breve{Z}\sigma,\breve{K}\omega) + d^o(\breve{\Omega}\sigma,\breve{H}\omega)}{1 + d^o(\breve{Z}\sigma,\breve{H}\omega) + d^o(\breve{\Omega}\sigma,\breve{K}\omega)}\right]\Bigg\}$$

$$= max\{d^o(\sigma,\omega), d^o(\sigma,\sigma), d^o(\omega,\omega),$$

$$\frac{1}{2}[d^o(\sigma,\omega) + d^o(\sigma,\omega)],$$

$$d^o(\sigma,\sigma)\left[\frac{1 + d^o(\sigma,\omega)}{1 + d^o(\omega,\omega)}\right],$$

$$d^o(\omega,\omega)\left[\frac{1 + d^o(\sigma,\omega)}{1 + d^o(\sigma,\sigma)}\right],$$

$$\frac{{d^o}^2(\sigma,\sigma)}{1 + d^o(\sigma,\omega)}, \frac{{d^o}^2(\omega,\omega)}{1 + d^o(\sigma,\omega)},$$

$$d^o(\sigma,\sigma)\left[\frac{1 + d^o(\sigma,\omega) + d^o(\sigma,\omega)}{1 + d^o(\sigma,\omega) + d^o(\sigma,\omega)}\right],$$

$$d^o(\omega,\omega)\left[\frac{1 + d^o(\sigma,\omega) + d^o(\sigma,\omega)}{1 + d^o(\sigma,\omega) + d^o(\sigma,\omega)}\right]\Bigg\}$$

$$= max\{d^o(\sigma,\omega), 0, 0, d^o(\sigma,\omega), 0, 0, 0, 0, 0, 0\}.$$

From (2.33), we have

$$\psi\big(d^o(\sigma,\omega)\big) \le \psi\big(d^o(\sigma,\omega)\big) - \phi\big(d^o(\sigma,\omega)\big)$$
$$< \psi\big(d^o(\sigma,\omega)\big),$$

which is a contradiction.

Therefore, $\sigma = \omega$ and this follows the uniqueness and completes the proof of the theorem.

Example 2.4. Let $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$ be self - mappings on M. $M = [0,1]$ be endowed with the Euclidean metric $d^o(\sigma,\omega) = |\sigma - \omega|$ for all σ,ω in M. Let $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$ are defined by

$$\breve{K}\sigma = \begin{cases} 0 & \sigma = 0 \\ \frac{\sigma}{4} & \sigma > 0, \end{cases}$$

$$\breve{\Omega}\sigma = \begin{cases} 0 & \sigma = 0 \\ \frac{\sigma}{8} & \sigma > 0, \end{cases}$$

$$\breve{H}\sigma = \begin{cases} 0 & \sigma = 0 \\ \frac{\sigma}{2} & \sigma > 0, \end{cases}$$

$$\breve{Z}\sigma = \begin{cases} 0 & \sigma = 0 \\ \sigma & \sigma > 0 \end{cases}$$

$$\psi(t) = \frac{t}{2}, \ \ \phi(t) = \frac{t}{4}, \text{ for all } t \text{ in } \mathbb{R}.$$

Let $\{\mu_n\}$ be a sequence in M such that $\mu_n = \frac{1}{n+1}$ for each n.

Clearly, $\breve{\Omega}M = \left[0,\frac{1}{8}\right] \subseteq \left[0,\frac{1}{2}\right] = \breve{H}M$ *and* $\breve{K}M \left[0,\frac{1}{4}\right] \subseteq [0,1] = \breve{Z}M$, implies that (2.1) satisfied.

Since $\breve{Z}\breve{\Omega}(0) = \breve{\Omega}\breve{Z}(0) = 0$, implies that the pair $(\breve{Z},\breve{\Omega})$ is weakly compatible and $\breve{H}\breve{K}(0) = \breve{K}\breve{H}(0) = 0$, implies that the pair $(\breve{H},\breve{K})$ is weakly compatible.

Now, we check condition (2.2) for the following cases:

Case 1. If $\sigma = 0$ and $\omega = 0$.

$$\psi\left(d^0(\breve{\Omega}\sigma,\breve{K}\omega)\right) = \psi(|\breve{\Omega}\sigma - \breve{K}\omega|) = \psi(0) = 0.$$

Also

$$\psi(\delta(\sigma,\omega)) = \psi(0) = 0,$$

$$\phi(\delta(\sigma,\omega)) = \phi(0) = 0.$$

Hence

$$\psi\left(d^0(\breve{\Omega}\sigma,\breve{K}\omega)\right) = \psi(\delta(\sigma,\omega)) - \phi(\delta(\sigma,\omega)).$$

Clearly, inequality (2.2) holds.

Case 2. If $\sigma = 0$, $\omega \neq 0$.

$$\psi\left(d^0(\breve{\Omega}\sigma,\breve{K}\omega)\right) = \psi(|\breve{\Omega}\sigma - \breve{K}\omega|) = \psi\left(\left|0 - \frac{\omega}{4}\right|\right) = \frac{\omega}{8}.$$

$$\delta(\sigma,\omega) = max\{d^o(\breve{Z}\sigma,\breve{H}\omega), d^o(\breve{Z}\sigma,\breve{\Omega}\sigma), d^o(\breve{H}\omega,\breve{K}\omega),$$
$$\frac{1}{2}[d^o(\breve{Z}\sigma,\breve{K}\omega)+d^o(\breve{\Omega}\sigma,\breve{H}\omega)],$$
$$d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{H}\omega)}{1+d^o(\breve{H}\omega,\breve{K}\omega)}\right],$$
$$d^o(\breve{H}\omega,\breve{K}\omega)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{H}\omega)}{1+d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)}\right],$$
$$\frac{d^{o2}(\breve{Z}\sigma,\breve{\Omega}\sigma)}{1+d^o(\breve{\Omega}\sigma,\breve{K}\omega)},\frac{d^{o2}(\breve{H}\omega,\breve{K}\omega)}{1+d^o(\breve{\Omega}\sigma,\breve{K}\omega)},$$
$$d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{K}\omega)+d^o(\breve{\Omega}\sigma,\breve{H}\omega)}{1+d^o(\breve{Z}\sigma,\breve{H}\omega)+d^o(\breve{\Omega}\sigma,\breve{K}\omega)}\right],$$
$$d^o(\breve{H}\omega,\breve{K}\omega)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{K}\omega)+d^o(\breve{\Omega}\sigma,\breve{H}\omega)}{1+d^o(\breve{Z}\sigma,\breve{H}\omega)+d^o(\breve{\Omega}\sigma,\breve{K}\omega)}\right]\}$$
$$= max\{|\breve{Z}\sigma-\breve{H}\omega|,|\breve{Z}\sigma-\breve{\Omega}\sigma|,|\breve{H}\omega-\breve{K}\omega|,$$
$$\frac{1}{2}[|\breve{Z}\sigma-\breve{K}\omega|+|\breve{\Omega}\sigma-\breve{H}\omega|],$$
$$|\breve{Z}\sigma-\breve{\Omega}\sigma|\left[\frac{1+|\breve{Z}\sigma-\breve{H}\omega|}{1+|\breve{H}\omega-\breve{K}\omega|}\right],$$
$$|\breve{H}\omega-\breve{K}\omega|\left[\frac{1+|\breve{Z}\sigma-\breve{H}\omega|}{1+|\breve{Z}\sigma,\breve{\Omega}\sigma|}\right],$$
$$\frac{|\breve{Z}\sigma-\breve{\Omega}\sigma|^2}{1+|\breve{\Omega}\sigma,-\breve{K}\omega|},\frac{|\breve{H}\omega-\breve{K}\omega|^2}{1+|\breve{\Omega}\sigma-\breve{K}\omega|},$$
$$|\breve{Z}\sigma-\breve{\Omega}\sigma|\left[\frac{1+|\breve{Z}\sigma-\breve{K}\omega|+|\breve{\Omega}\sigma-\breve{H}\omega|}{1+|\breve{Z}\sigma-\breve{H}\omega|+|\breve{\Omega}\sigma-\breve{K}\omega|}\right],$$
$$|\breve{H}\omega-\breve{K}\omega|\left[\frac{1+|\breve{Z}\sigma-\breve{K}\omega|+|\breve{\Omega}\sigma-\breve{H}\omega|}{1+|\breve{Z}\sigma-\breve{H}\omega|+|\breve{\Omega}\sigma-\breve{K}\omega|}\right]\}$$
$$= \max\left\{\frac{\omega}{2},0,\frac{\omega}{4},\frac{3\omega}{8},0,\frac{\omega}{8}(2+\omega),0\,\frac{\omega^2}{4(4+\omega)},0,\frac{\omega}{4}\left[\frac{1+\frac{\omega}{4}+\frac{\omega}{2}}{1+\frac{\omega}{2}+\frac{\omega}{4}}\right]\right\}.$$

One can easily check that

$$\delta(\sigma,\omega)=\frac{\omega}{2}.$$

Now,

$$\psi\left(d^0(\breve{\Omega}\sigma,\breve{K}\omega)\right)=\psi(|\breve{\Omega}\sigma-\ \breve{K}\omega|)=\psi\left(\frac{\omega}{4}\right)=\frac{\omega}{8}.$$

Also

$$\psi(\delta(\sigma,\omega))=\psi\left(\frac{\omega}{2}\right)=\frac{\omega}{4},$$

$$\phi(\delta(\sigma,\omega))=\phi\left(\frac{\omega}{2}\right)=\frac{\omega}{8}.$$

Hence

$$\psi\left(d^0(\breve{\Omega}\sigma,\breve{K}\omega)\right)=\psi(\delta(\sigma,\omega))-\phi(\delta(\sigma,\omega)).$$

Clearly inequality (2.2) holds.

Case 3. If $\sigma\neq 0$, $\omega=0$.

$$\psi\left(d^0(\breve{\Omega}\sigma,\breve{K}\omega)\right)=\psi(|\breve{\Omega}\sigma-\ \breve{K}\omega|)=\psi\left(\left|\frac{\sigma}{8}-0\right|\right)=\frac{\sigma}{16}.$$

$$\begin{aligned}\delta(\sigma,\omega)=\ & max\{d^o(\breve{Z}\sigma,\breve{H}\omega),d^o(\breve{Z}\sigma,\breve{\Omega}\sigma\ ),d^o(\breve{H}\omega,\breve{K}\omega),\\
&\tfrac{1}{2}\left[d^o(\breve{Z}\sigma,\breve{K}\omega)+d^o(\breve{\Omega}\sigma,\breve{H}\omega)\right],\\
&d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{H}\omega)}{1+d^o(\ \breve{H}\omega,\breve{K}\omega)}\right],\\
&d^o(\ \breve{H}\omega,\breve{K}\omega)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{H}\omega)}{1+d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)}\right],\\
&\frac{d^{o^2}(\breve{Z}\sigma,\breve{\Omega}\sigma\ )}{1+d^o(\breve{\Omega}\sigma,\breve{K}\omega)},\frac{d^{o^2}(\ \breve{H}\omega,\breve{K}\omega)}{1+d^o(\breve{\Omega}\sigma,\breve{K}\omega)},\\
&d^o(\breve{Z}\sigma,\breve{\Omega}\sigma)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{K}\omega)+d^o(\breve{\Omega}\sigma,\breve{H}\omega)}{1+d^o(\breve{Z}\sigma,\breve{H}\omega)+d^o(\breve{\Omega}\sigma,\breve{K}\omega)}\right],\\
&d^o(\ \breve{H}\omega,\breve{K}\omega)\left[\frac{1+d^o(\breve{Z}\sigma,\breve{K}\omega)+d^o(\breve{\Omega}\sigma,\breve{H}\omega)}{1+d^o(\breve{Z}\sigma,\breve{H}\omega)+d^o(\breve{\Omega}\sigma,\breve{K}\omega)}\right]\Bigg\}\\
=\ & max\{|\breve{Z}\sigma-\ \breve{H}\omega|,|\breve{Z}\sigma-\breve{\Omega}\sigma|,\ |\breve{H}\omega-\breve{K}\omega|,\tfrac{1}{2}\left[|\breve{Z}\sigma-\ \breve{K}\omega|+|\breve{\Omega}\sigma-\breve{H}\omega|\right],\\
&|\breve{Z}\sigma-\breve{\Omega}\sigma|\left[\frac{1+|\breve{Z}\sigma-\breve{H}\omega|}{1+|\breve{H}\omega-\breve{K}\omega|}\right],\\
&|\breve{H}\omega-\breve{K}\omega|\left[\frac{1+|\breve{Z}\sigma-\breve{H}\omega|}{1+|\breve{Z}\sigma,\breve{\Omega}\sigma|}\right],\\
&\frac{|\breve{Z}\sigma-\ \breve{\Omega}\sigma\ |^2}{1+|\breve{\Omega}\sigma,-\breve{K}\omega|},\frac{|\breve{H}\omega-\breve{K}\omega|^2}{1+|\breve{\Omega}\sigma-\breve{K}\omega|},\end{aligned}$$

$$\left|\breve{Z}\sigma - \breve{\Omega}\sigma\right|\left[\frac{1+\left|\breve{Z}\sigma - \breve{K}\omega\right| + \left|\breve{\Omega}\sigma - \breve{H}\omega\right|}{1+\left|\breve{Z}\sigma - \breve{H}\omega\right| + \left|\breve{\Omega}\sigma - \breve{K}\omega\right|}\right],$$

$$\left|\breve{H}\omega - \breve{K}\omega\right|\left[\frac{1+\left|\breve{Z}\sigma - \breve{K}\omega\right| + \left|\breve{\Omega}\sigma - \breve{H}\omega\right|}{1+\left|\breve{Z}\sigma - \breve{H}\omega\right| + \left|\breve{\Omega}\sigma - \breve{K}\omega\right|}\right]\Bigg\}$$

$$= \max\left\{\sigma, \tfrac{7\sigma}{8}, 0, \tfrac{9\sigma}{16}, \tfrac{7\sigma}{8}(1+\sigma), 0\,\tfrac{49\sigma^2}{8(8+\sigma)}, 0, \tfrac{7\sigma}{8}, 0\right\}.$$

One can easily check that

$$\delta(\sigma,\omega) = \begin{cases} \sigma, & 0 < \sigma \leq \dfrac{1}{7} \\ \dfrac{7\sigma}{8}(1+\sigma), & \dfrac{1}{7} \leq \sigma < 1. \end{cases}$$

Now, if $0 < \sigma \leq \frac{1}{7}$,

$$\psi\left(d^0\left(\breve{\Omega}\sigma, \breve{K}\omega\right)\right) = \frac{\sigma}{16}.$$

Also

$$\psi\big(\delta(\sigma,\omega)\big) = \psi(\sigma) = \frac{\sigma}{2},$$

$$\phi\big(\delta(\sigma,\omega)\big) = \phi(\sigma) = \frac{\sigma}{4}.$$

Clearly

$$\psi\left(d^0\left(\breve{\Omega}\sigma, \breve{K}\omega\right)\right) < \psi\big(\delta(\sigma,\omega)\big) - \phi\big(\delta(\sigma,\omega)\big).$$

Hence inequality (2.2) holds.

If $\frac{1}{7} \leq \sigma < 1$,

$$\psi\left(d^0\left(\breve{\Omega}\sigma, \breve{K}\omega\right)\right) = \frac{\sigma}{16}.$$

$$\psi\big(\delta(\sigma,\omega)\big) = \psi\left(\frac{7\sigma}{8}(1+\sigma)\right) = \frac{7\sigma}{16}(1+\sigma),$$

$$\phi\big(\delta(\sigma,\omega)\big) = \phi\left(\frac{7\sigma}{8}(1+\sigma)\right) = \frac{7\sigma}{32}(1+\sigma).$$

$$\psi\left(d^0\left(\breve{\Omega}\sigma, \breve{K}\omega\right)\right) < \psi\big(\delta(\sigma,\omega)\big) - \phi\big(\delta(\sigma,\omega)\big).$$

Hence inequality (2.2) holds.

Case 4: If $\sigma \neq 0$, $\omega \neq 0$.

$$\psi\left(d^0\left(\breve{\Omega}\sigma, \breve{K}\omega\right)\right) = \psi\left(\left|\breve{\Omega}\sigma - \breve{K}\omega\right|\right) = \psi\left(\left|\frac{\sigma}{8} - \frac{\omega}{4}\right|\right) = \frac{\sigma - 2\omega}{16}.$$

$$\delta(\sigma,\omega) = max\{d^o(\breve{Z}\sigma, \breve{H}\omega), d^o(\breve{Z}\sigma, \breve{\Omega}\sigma), d^o(\breve{H}\omega, \breve{K}\omega),$$

$$\tfrac{1}{2}[d^o(\breve{Z}\sigma, \breve{K}\omega) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)],$$

$$d^o(\breve{Z}\sigma, \breve{\Omega}\sigma)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{H}\omega)}{1 + d^o(\breve{H}\omega, \breve{K}\omega)}\right],$$

$$d^o(\breve{H}\omega, \breve{K}\omega)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{H}\omega)}{1 + d^o(\breve{Z}\sigma, \breve{\Omega}\sigma)}\right],$$

$$\frac{{d^o}^2(\breve{Z}\sigma, \breve{\Omega}\sigma)}{1 + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}, \frac{{d^o}^2(\breve{H}\omega, \breve{K}\omega)}{1 + d^o(\breve{\Omega}\sigma, \breve{K}\omega)},$$

$$d^o(\breve{Z}\sigma, \breve{\Omega}\sigma)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{K}\omega) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)}{1 + d^o(\breve{Z}\sigma, \breve{H}\omega) + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}\right],$$

$$d^o(\breve{H}\omega, \breve{K}\omega)\left[\frac{1 + d^o(\breve{Z}\sigma, \breve{K}\omega) + d^o(\breve{\Omega}\sigma, \breve{H}\omega)}{1 + d^o(\breve{Z}\sigma, \breve{H}\omega) + d^o(\breve{\Omega}\sigma, \breve{K}\omega)}\right]\}$$

$$= max\{|\breve{Z}\sigma - \breve{H}\omega|, |\breve{Z}\sigma - \breve{\Omega}\sigma|, |\breve{H}\omega - \breve{K}\omega|,$$

$$\tfrac{1}{2}[|\breve{Z}\sigma - \breve{K}\omega| + |\breve{\Omega}\sigma - \breve{H}\omega|],$$

$$|\breve{Z}\sigma - \breve{\Omega}\sigma|\left[\frac{1 + |\breve{Z}\sigma - \breve{H}\omega|}{1 + |\breve{H}\omega - \breve{K}\omega|}\right],$$

$$|\breve{H}\omega - \breve{K}\omega|\left[\frac{1 + |\breve{Z}\sigma - \breve{H}\omega|}{1 + |\breve{Z}\sigma, \breve{\Omega}\sigma|}\right],$$

$$\frac{|\breve{Z}\sigma - \breve{\Omega}\sigma|^2}{1 + |\breve{\Omega}\sigma - \breve{K}\omega|}, \frac{|\breve{H}\omega - \breve{K}\omega|^2}{1 + |\breve{\Omega}\sigma - \breve{K}\omega|},$$

$$|\breve{Z}\sigma - \breve{\Omega}\sigma|\left[\frac{1 + |\breve{Z}\sigma - \breve{K}\omega| + |\breve{\Omega}\sigma - \breve{H}\omega|}{1 + |\breve{Z}\sigma - \breve{H}\omega| + |\breve{\Omega}\sigma - \breve{K}\omega|}\right],$$

$$|\breve{H}\omega - \breve{K}\omega|\left[\frac{1 + |\breve{Z}\sigma - \breve{K}\omega| + |\breve{\Omega}\sigma - \breve{H}\omega|}{1 + |\breve{Z}\sigma - \breve{H}\omega| + |\breve{\Omega}\sigma - \breve{K}\omega|}\right]\}$$

$$= max\left\{\sigma - \frac{\omega}{2}, \frac{\sigma}{2}, \frac{\omega}{4}, \frac{1}{2}\left[\frac{9\sigma - 6\omega}{8}\right], \frac{7\sigma}{8}\left(\frac{1 + \sigma - \frac{\omega}{2}}{1 + \frac{\omega}{2} - \frac{\omega}{4}}\right),\right.$$

$$\left.\frac{\omega}{4}\left(\frac{1 + \sigma - \frac{\omega}{2}}{1 + \sigma - \frac{\sigma}{8}}\right), \frac{49\sigma^2}{8(8 + \sigma - 2\omega)}, \frac{\omega^2}{2(8 + \sigma - 2\omega)}, \frac{7\sigma}{8}, \frac{\omega}{4}\right\}.$$

one can easily check that

$$\psi\left(d^0(\breve{\Omega}\sigma,\breve{K}\omega)\right) \leq \psi(\delta(\sigma,\omega)) - \phi(\delta(\sigma,\omega)).$$

for all the values of $\sigma, \omega \in [0,1]$.

Hence inequality (2.2) holds for all the cases.

Also, $\lim_{n\to\infty} \breve{Z}\mu_n = \lim_{n\to\infty} \breve{\Omega}\mu_n = 0$, where $0 \in M$, implies that the pair $(\breve{Z}, \breve{\Omega})$ satisfies the E.A. property.

Similarly, we can easily check that the pair $(\breve{H}, \breve{K})$ satisfies the E.A property.

Now, $\lim_{n\to\infty} \breve{Z}\mu_n = \lim_{n\to\infty} \breve{\Omega}\mu_n = 0 = \breve{Z}(0)$, where $0 \in M$, implies that the pair $(\breve{Z}, \breve{\Omega})$ satisfies the $(CLR_{\breve{Z}})$ property.

Similarly, we can easily check that the pair $(\breve{H}, \breve{K})$ satisfies the $(CLR_{\breve{H}})$ property.

Therefore, all the conditions of the Theorems 2.1, 2.2 and 2.3 are satisfied.

Hence $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$ have unique common fixed point. Clearly 1 is the unique common fixed point of $\breve{Z}$, $\breve{H}$, $\breve{\Omega}$ and $\breve{K}$.

3. Conclusions

In 2021, Kumar *et al.*[3] proved the fixed point results for (ψ, ϕ)- weak contraction in metric spaces. In this manuscript, we generalized the result of Kumar *et al.* [3]. We proved a common fixed point theorem for four weakly compatible self-maps on a metric space satisfying the generalized (ψ, ϕ)- weak contraction. Also, we have proved common fixed point theorems for the above mentioned weakly compatible self-maps along with E.A. property and (CLR) property. An illustrative example is also provided to support our results.

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