

Common fixed point theorems for four self-maps satisfying generalized (ψ, ϕ) - weak contraction in metric space

Abstract: In this manuscript, we shall prove a common fixed point theorem for four weakly compatible self-maps \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} on a metric space (M, d^o) satisfying the following generalized (ψ, ϕ) - weak contraction:

$$\psi \left(d^o(\check{\Omega}\sigma, \check{K}\omega) \right) \leq \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)), \text{ where}$$

$$\delta(\sigma, \omega) = \max\{d^o(\check{Z}\sigma, \check{H}\omega), d^o(\check{Z}\sigma, \check{\Omega}\sigma), d^o(\check{H}\omega, \check{K}\omega), \frac{1}{2}[d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)],$$

$$d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{H}\omega, \check{K}\omega)} \right], d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{\Omega}\sigma)} \right], \frac{d^{o^2}(\check{Z}\sigma, \check{\Omega}\sigma)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)},$$

$$\frac{d^{o^2}(\check{H}\omega, \check{K}\omega)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right],$$

$$d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right]\}.$$

Also, we have proved common fixed point theorems for the above mentioned weakly compatible self-maps along with E.A. property and (CLR) property. An illustrative example is also provided to support our results.

Keywords: fixed point, weakly compatible maps, E.A. property, (CLR) property, generalized (ψ, ϕ) - weak contraction.

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1. Introduction

Definition 1.1 A coincidence point of a pair of self-maps $P, Q : M \rightarrow M$ is a point $u \in M$ for which $Pu = Qu$.

A common fixed point of a pair of self – maps $P, Q : M \rightarrow M$ is a point $u \in M$ for which $Pu = Qu = u$.

In 1996, Jungck [2] introduced the concept of weakly compatible maps to study common fixed point theorems:

Definition 1.2 Let (M, d^o) be a metric space. A pair of self – maps $P, Q : M \rightarrow M$ is weakly compatible if they commute at their coincidence points, that is, if there exists $u \in M$ such that $PQu = QPu$, where u is coincidence point of P and Q .

In 2002, Aamri and EI Moutawakil [1] introduced the notion of E.A. property as follows:

Definition 1.3 Let (M, d^o) be a metric space. Two self-maps P and Q on M are said to satisfy the E.A. property, if there exists a sequence $\{u_n\}$ in M such that,

$$\lim_{n \rightarrow \infty} Pu_n = \lim_{n \rightarrow \infty} Qu_n = t, \text{ for some } t \in M.$$

In 2011, Sintunavarat *et.al* [5] introduced the notion of (CLR) property as follows:

Definition 1.4 Let (M, d^o) be a metric space. Two self-maps P and Q on M are said to satisfy the (CLR_P) property, if there exists a sequence $\{u_n\}$ in M such that,

$$\lim_{n \rightarrow \infty} Pu_n = \lim_{n \rightarrow \infty} Qu_n = P(t), \text{ for some } t \in M.$$

2. Main Results

Theorem 2.1. Let (M, d^o) be a metric space and let $\check{Z}, \check{H}, \check{\Omega}$ and \check{K} be self-maps on M satisfying the followings:

$$(2.1) \check{\Omega}M \subseteq \check{H}M, \check{K}M \subseteq \check{Z}M,$$

for all $\sigma, \omega \in M$, there exists right continuous functions $\psi, \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, with $\psi(0) = 0 = \phi(0)$ and $\psi(a) < a$ for $a > 0$ such that:

$$(2.2) \psi(d^o(\check{\Omega}\sigma, \check{K}\omega)) \leq \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)), \text{ where}$$

$$\delta(\sigma, \omega) = \max\{d^o(\check{Z}\sigma, \check{H}\omega), d^o(\check{Z}\sigma, \check{\Omega}\sigma), d^o(\check{H}\omega, \check{K}\omega), \frac{1}{2}[d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)],$$

$$d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{H}\omega, \check{K}\omega)} \right], d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{\Omega}\sigma)} \right], \frac{d^{o^2}(\check{Z}\sigma, \check{\Omega}\sigma)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)},$$

$$\frac{d^{o^2}(\check{H}\omega, \check{K}\omega)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right],$$

$$d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right] \}.$$

If one of $\check{Z}M, \check{H}M, \check{\Omega}M$ or $\check{K}M$ is complete subspace of M , then the pair $(\check{Z}, \check{\Omega})$ or (\check{H}, \check{K}) have a coincidence point. Moreover, if the pair $(\check{Z}, \check{\Omega})$ or (\check{H}, \check{K}) is weakly compatible, then $\check{Z}, \check{H}, \check{\Omega}$ and \check{K} have a unique common fixed point.

Proof: Let $\sigma_0 \in M$ be an arbitrary point of M . From (2.2), we can construct a sequence $\{\omega_n\}$ in M as follows:

$$\omega_{2n+1} = \check{\Omega}\sigma_{2n} = \check{H}\sigma_{2n+1}, \quad \omega_{2n+2} = \check{K}\sigma_{2n+1} = \check{Z}\sigma_{2n+2}, \tag{2.3}$$

for all $n = 0, 1, 2, \dots$

Now, we define $d^o_n = d^o(\omega_n, \omega_{n+1})$. If $d^o_{2n} = 0$ for some n , then $d^o(\omega_{2n}, \omega_{2n+1}) = 0$.

Then $\omega_{2n} = \omega_{2n+1}$, that is, $\check{K}\sigma_{2n-1} = \check{Z}\sigma_{2n} = \check{\Omega}\sigma_{2n} = \check{H}\sigma_{2n+1}$.

We can note that \check{Z} and $\check{\Omega}$ have a coincidence point. Similarly, if $d^o_{2n+1} = 0$, then \check{H} and \check{K} have a coincidence point. Assume that $d^o_n \neq 0$ for each n .

On putting, $\sigma = \sigma_{2n}$ and $\omega = \sigma_{2n+1}$ in (2.2), we get

$$\psi\left(d^o(\check{\Omega}\sigma_{2n}, \check{K}\sigma_{2n+1})\right) \leq \psi(\delta(\sigma_{2n}, \sigma_{2n+1})) - \phi(\delta(\sigma_{2n}, \sigma_{2n+1})), \tag{2.4}$$

where

$$\begin{aligned} \delta(\sigma_{2n}, \sigma_{2n+1}) &= \max\{d^o(\check{Z}\sigma_{2n}, \check{H}\sigma_{2n+1}), d^o(\check{Z}\sigma_{2n}, \check{\Omega}\sigma_{2n}), d^o(\check{H}\sigma_{2n+1}, \check{K}\sigma_{2n+1}), \\ &\quad \frac{1}{2}[d^o(\check{Z}\sigma_{2n}, \check{K}\sigma_{2n+1}) + d^o(\check{\Omega}\sigma_{2n}, \check{H}\sigma_{2n+1})], \\ &\quad d^o(\check{Z}\sigma_{2n}, \check{\Omega}\sigma_{2n}) \left[\frac{1 + d^o(\check{Z}\sigma_{2n}, \check{H}\sigma_{2n+1})}{1 + d^o(\check{H}\sigma_{2n+1}, \check{K}\sigma_{2n+1})} \right], \\ &\quad d^o(\check{H}\sigma_{2n+1}, \check{K}\sigma_{2n+1}) \left[\frac{1 + d^o(\check{Z}\sigma_{2n}, \check{H}\sigma_{2n+1})}{1 + d^o(\check{Z}\sigma_{2n}, \check{\Omega}\sigma_{2n})} \right], \\ &\quad \frac{d^{o^2}(\check{Z}\sigma_{2n}, \check{\Omega}\sigma_{2n})}{1 + d^o(\check{\Omega}\sigma_{2n}, \check{K}\sigma_{2n+1})}, \frac{d^{o^2}(\check{H}\sigma_{2n+1}, \check{K}\sigma_{2n+1})}{1 + d^o(\check{\Omega}\sigma_{2n}, \check{K}\sigma_{2n+1})}, \\ &\quad d^o(\check{Z}\sigma_{2n}, \check{\Omega}\sigma_{2n}) \left[\frac{1 + d^o(\check{Z}\sigma_{2n}, \check{K}\sigma_{2n+1}) + d^o(\check{\Omega}\sigma_{2n}, \check{H}\sigma_{2n+1})}{1 + d^o(\check{Z}\sigma_{2n}, \check{H}\sigma_{2n+1}) + d^o(\check{\Omega}\sigma_{2n}, \check{K}\sigma_{2n+1})} \right], \\ &\quad d^o(\check{H}\sigma_{2n+1}, \check{K}\sigma_{2n+1}) \left[\frac{1 + d^o(\check{Z}\sigma_{2n}, \check{K}\sigma_{2n+1}) + d^o(\check{\Omega}\sigma_{2n}, \check{H}\sigma_{2n+1})}{1 + d^o(\check{Z}\sigma_{2n}, \check{H}\sigma_{2n+1}) + d^o(\check{\Omega}\sigma_{2n}, \check{K}\sigma_{2n+1})} \right]\}, \\ &= \max\{d^o(\omega_{2n}, \omega_{2n+1}), d^o(\omega_{2n}, \omega_{2n+1}), d^o(\omega_{2n+1}, \omega_{2n+2}), \\ &\quad \frac{1}{2}[d^o(\omega_{2n}, \omega_{2n+2}) + d^o(\omega_{2n+1}, \omega_{2n+1})], \end{aligned}$$

$$\begin{aligned}
 & d^o(\omega_{2n}, \omega_{2n+1}) \left[\frac{1 + d^o(\omega_{2n}, \omega_{2n+1})}{1 + d^o(\omega_{2n+1}, \omega_{2n+2})} \right], d^o(\omega_{2n+1}, \omega_{2n+2}) \left[\frac{1 + d^o(\omega_{2n}, \omega_{2n+1})}{1 + d^o(\omega_{2n}, \omega_{2n+1})} \right], \\
 & \frac{d^{o^2}(\omega_{2n}, \omega_{2n+1})}{1 + d^o(\omega_{2n+1}, \omega_{2n+2})}, \frac{d^{o^2}(\omega_{2n+1}, \omega_{2n+2})}{1 + d^o(\omega_{2n+1}, \omega_{2n+2})}, \\
 & d^o(\omega_{2n}, \omega_{2n+1}) \left[\frac{1 + d^o(\omega_{2n}, \omega_{2n+2}) + d^o(\omega_{2n+1}, \omega_{2n+1})}{1 + d^o(\omega_{2n}, \omega_{2n+1}) + d^o(\omega_{2n+1}, \omega_{2n+2})} \right] \\
 & d^o(\omega_{2n+1}, \omega_{2n+2}) \left[\frac{1 + d^o(\omega_{2n}, \omega_{2n+2}) + d^o(\omega_{2n+1}, \omega_{2n+1})}{1 + d^o(\omega_{2n}, \omega_{2n+1}) + d^o(\omega_{2n+1}, \omega_{2n+2})} \right] \Big\}. \\
 & = \max \left\{ d^o_{2n}, d^o_{2n}, d^o_{2n+1}, \frac{1}{2} [d^o_{2n} + d^o_{2n+1} + 0], \right. \\
 & \quad d^o_{2n} \frac{1+d^o_{2n}}{1+d^o_{2n+1}}, d^o_{2n+1}, \frac{d^{o^2}_{2n}}{1+d^o_{2n+1}}, \frac{d^{o^2}_{2n+1}}{1+d^o_{2n+1}}, \\
 & \quad \left. d^o_{2n} \frac{1 + d^o_{2n} + d^o_{2n+1}}{1 + d^o_{2n} + d^o_{2n+1}}, d^o_{2n+1} \frac{1 + d^o_{2n} + d^o_{2n+1}}{1 + d^o_{2n} + d^o_{2n+1}} \right\}. \\
 & = \max \{ d^o_{2n}, d^o_{2n+1} \}. \tag{2.5}
 \end{aligned}$$

Now, from (2.4), we have

$$\psi(d^o(\omega_{2n+1}, \omega_{2n+2})) \leq \psi(\max\{d^o_{2n}, d^o_{2n+1}\}) - \phi(\max\{d^o_{2n}, d^o_{2n+1}\}), \tag{2.6}$$

Now, if $d^o_{2n+1} \geq d^o_{2n}$ for some n , then from (2.6), we get

$$\begin{aligned}
 \psi(d^o_{2n+1}) & \leq \psi(d^o_{2n+1}) - \phi(d^o_{2n+1}) \\
 & < \psi(d^o_{2n+1}), \tag{2.7}
 \end{aligned}$$

which is a contradiction. Thus, $d^o_{2n} > d^o_{2n+1}$ for all n , and so, from (2.6), we have

$$\psi(d^o_{2n+1}) \leq \psi(d^o_{2n}) - \phi(d^o_{2n}) \text{ for all } n \in N \tag{2.8}$$

Similarly,

$$\begin{aligned}
 \psi(d^o_{2n}) & \leq \psi(d^o_{2n-1}) - \phi(d^o_{2n-1}) \\
 \psi(d^o_{2n-1}) & \leq \psi(d^o_{2n-2}) - \phi(d^o_{2n-2})
 \end{aligned}$$

In general, we have for all $n = 1, 2, 3 \dots$

$$\begin{aligned}
 \psi(d^o_n) & \leq \psi(d^o_{n-1}) - \phi(d^o_{n-1}) \\
 & < \psi(d^o_{n-1}) \tag{2.9}
 \end{aligned}$$

Hence sequence $\{ \psi(d^o_n) \}$ is monotonically decreasing and bounded below. Thus, there exists $s \geq 0$, such that

$$\lim_{n \rightarrow \infty} \psi(d^o_n) = s \tag{2.10}$$

From (2.9), we deduce that

$$0 \leq \phi(d^o_{n-1}) \leq \psi(d^o_{n-1}) - \psi(d^o_n)$$

Taking limit as $n \rightarrow \infty$ and using (2.10), we get

$$\lim_{n \rightarrow \infty} \phi(d^o_{n-1}) = 0,$$

this implies that

$$\lim_{n \rightarrow \infty} \phi(d^o_{n-1}) = \lim_{n \rightarrow \infty} \phi(d^o(\omega_{n-1}, \omega_n)) = 0 \tag{2.11}$$

$$\text{or } \lim_{n \rightarrow \infty} d^o_n = \lim_{n \rightarrow \infty} d^o(\omega_n, \omega_{n+1}) = 0 \tag{2.12}$$

Now, we claim that $\{\omega_n\}$ is a Cauchy sequence. For this, it is sufficient to show that $\{\omega_{2n}\}$ is a Cauchy sequence. Let, if possible, $\{\omega_{2n}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$, such that for each even integer $2a$ there exists even integers $2m(a) > 2n(a) > 2a$ such that

$$d^o(\omega_{2n(a)}, \omega_{2m(a)}) \geq \epsilon \tag{2.13}$$

for every even integer $2a$, suppose that $2m(a)$ be the least positive integer exceeding $2n(a)$ satisfying (2.13), such that

$$d^o(\omega_{2n(a)}, \omega_{2m(a)-2}) < \epsilon. \tag{2.14}$$

From (2.13), we get

$$\begin{aligned} \epsilon &\leq d^o(\omega_{2n(a)}, \omega_{2m(a)}); \\ &\leq d^o(\omega_{2n(a)}, \omega_{2m(a)-2}) + d^o(\omega_{2m(a)-2}, \omega_{2m(a)-1}) + d^o(\omega_{2m(a)-1}, \omega_{2m(a)}). \end{aligned}$$

Using (2.12) and (2.14) in the above inequality, we get

$$\lim_{n \rightarrow \infty} d^o(\omega_{2n(a)}, \omega_{2m(a)}) = \epsilon \tag{2.15}$$

Also by the triangular inequality,

$$\begin{aligned} |d^o(\omega_{2n(a)}, \omega_{2m(a)-1}) - d^o(\omega_{2n(a)}, \omega_{2m(a)})| &\leq d^o(\omega_{2m(a)-1}, \omega_{2m(a)}) = d^o_{2m(a)-1}, \\ |d^o(\omega_{2n(a)+1}, \omega_{2m(a)-1}) - d^o(\omega_{2n(a)}, \omega_{2m(a)})| &\leq d^o_{2m(a)-1} + d^o_{2m(a)} \end{aligned} \tag{2.16}$$

Using (2.12), we have

$$\lim_{n \rightarrow \infty} d^o(\omega_{2n(a)}, \omega_{2m(a)-1}) = \lim_{n \rightarrow \infty} d^o(\omega_{2n(a)+1}, \omega_{2m(a)-1}) = \epsilon \tag{2.17}$$

Now, from (2.2), we have

$$\psi(d^o(\check{\Omega}\sigma_{2n(a)}, \check{\mathbb{K}}\sigma_{2m(a)-1})) \leq \psi(\delta(\sigma_{2n(a)}, \sigma_{2m(a)-1})) - \phi(\delta(\sigma_{2n(a)}, \sigma_{2m(a)-1})), \tag{2.18}$$

where

$$\begin{aligned}
 & \delta(\sigma_{2n(a)}, \sigma_{2m(a)-1}) \\
 &= \max\{d^o(\check{Z}\sigma_{2n(a)}, \check{H}\sigma_{2m(a)-1}), d^o(\check{Z}\sigma_{2n(a)}, \check{\Omega}\sigma_{2n(a)}), d^o(\check{H}\sigma_{2m(a)-1}, \check{K}\sigma_{2m(a)-1}), \\
 & \quad \frac{1}{2}[d^o(\check{Z}\sigma_{2n(a)}, \check{K}\sigma_{2m(a)-1}) + d^o(\check{\Omega}\sigma_{2n(a)}, \check{H}\sigma_{2m(a)-1})], \\
 & \quad d^o(\check{Z}\sigma_{2n(a)}, \check{\Omega}\sigma_{2n(a)}) \left[\frac{1 + d^o(\check{Z}\sigma_{2n(a)}, \check{H}\sigma_{2m(a)-1})}{1 + d^o(\check{H}\sigma_{2m(a)-1}, \check{K}\sigma_{2m(a)-1})} \right], \\
 & \quad d^o(\check{H}\sigma_{2m(a)-1}, \check{K}\sigma_{2m(a)-1}) \left[\frac{1 + d^o(\check{Z}\sigma_{2n(a)}, \check{H}\sigma_{2m(a)-1})}{1 + d^o(\check{Z}\sigma_{2n(a)}, \check{\Omega}\sigma_{2n(a)})} \right], \\
 & \quad \frac{d^{o^2}(\check{Z}\sigma_{2n(a)}, \check{\Omega}\sigma_{2n(a)})}{1 + d^o(\check{\Omega}\sigma_{2n(a)}, \check{K}\sigma_{2m(a)-1})}, \frac{d^{o^2}(\check{H}\sigma_{2m(a)-1}, \check{K}\sigma_{2m(a)-1})}{1 + d^o(\check{\Omega}\sigma_{2n(a)}, \check{K}\sigma_{2m(a)-1})}, \\
 & \quad d^o(\check{Z}\sigma_{2n(a)}, \check{\Omega}\sigma_{2n(a)}) \left[\frac{1 + d^o(\check{Z}\sigma_{2n(a)}, \check{K}\sigma_{2m(a)-1}) + d^o(\check{\Omega}\sigma_{2n(a)}, \check{H}\sigma_{2m(a)-1})}{1 + d^o(\check{Z}\sigma_{2n(a)}, \check{H}\sigma_{2m(a)-1}) + d^o(\check{\Omega}\sigma_{2n(a)}, \check{K}\sigma_{2m(a)-1})} \right], \\
 & \quad d^o(\check{H}\sigma_{2m(a)-1}, \check{K}\sigma_{2m(a)-1}) \left[\frac{1 + d^o(\check{Z}\sigma_{2n(a)}, \check{K}\sigma_{2m(a)-1}) + d^o(\check{\Omega}\sigma_{2n(a)}, \check{H}\sigma_{2m(a)-1})}{1 + d^o(\check{Z}\sigma_{2n(a)}, \check{H}\sigma_{2m(a)-1}) + d^o(\check{\Omega}\sigma_{2n(a)}, \check{K}\sigma_{2m(a)-1})} \right]\}, \\
 &= \max\{d^o(\omega_{2n(a)}, \omega_{2m(a)-1}), d^o(\omega_{2n(a)}, \omega_{2n(a)+1}), d^o(\omega_{2m(a)-1}, \omega_{2m(a)}), \\
 & \quad \frac{1}{2}[d^o(\omega_{2n(a)}, \omega_{2m(a)}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)-1})], \\
 & \quad d^o(\omega_{2n(a)}, \omega_{2n(a)+1}) \left[\frac{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)-1})}{1 + d^o(\omega_{2m(a)-1}, \omega_{2m(a)})} \right], \\
 & \quad d^o(\omega_{2m(a)-1}, \omega_{2m(a)}) \left[\frac{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)-1})}{1 + d^o(\omega_{2n(a)}, \omega_{2n(a)+1})} \right], \\
 & \quad \frac{d^{o^2}(\omega_{2n(a)}, \omega_{2n(a)+1})}{1 + d^o(\omega_{2n(a)+1}, \omega_{2m(a)})}, \frac{d^{o^2}(\omega_{2m(a)-1}, \omega_{2m(a)})}{1 + d^o(\omega_{2n(a)+1}, \omega_{2m(a)})}, \\
 & \quad d^o(\omega_{2n(a)}, \omega_{2n(a)+1}) \left[\frac{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)-1})}{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)-1}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)})} \right], \\
 & \quad d^o(\omega_{2m(a)-1}, \omega_{2m(a)}) \left[\frac{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)-1})}{1 + d^o(\omega_{2n(a)}, \omega_{2m(a)-1}) + d^o(\omega_{2n(a)+1}, \omega_{2m(a)})} \right]\}.
 \end{aligned}$$

Assuming $a \rightarrow \infty$ and using (2.14), (2.15) and (2.17), we have

$$\begin{aligned}
 &= \max\left\{\epsilon, 0, 0, \frac{1}{2}(\epsilon + \epsilon), 0, 0, 0, 0, 0, 0\right\}. \\
 &= \epsilon.
 \end{aligned}$$

Hence, $\delta(\sigma_{2n(a)}, \sigma_{2m(a)-1}) = \epsilon$.

Now, by (2.18), we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction, since $\epsilon > 0$.

Thus, $\{\omega_{2n}\}$ is a Cauchy sequence. So, $\{\omega_n\}$ is a Cauchy sequence.

Now, suppose that $\check{Z}M$ is complete. Since $\{\omega_{2n}\}$ is contained in $\check{Z}M$ and has limit in $\check{Z}M$ say μ , that is, $\lim_{n \rightarrow \infty} \omega_{2n} = \mu$. let $\nu \in \check{Z}^{-1}(\mu)$ then $\check{Z}\nu = \mu$.

Now, we shall prove that $\check{\Omega}\nu = \mu$.

Let, if possible, $\check{\Omega}\nu \neq \mu$ that is, $d^o(\check{\Omega}\nu, \mu) = b > 0$.

Now, on putting, $\sigma = \nu, \omega = \sigma_{2n-1}$ in (2.2), we have

$$\psi\left(d^o(\check{\Omega}\nu, \check{K}\sigma_{2n-1})\right) \leq \psi(\delta(\nu, \sigma_{2n-1})) - \phi(\delta(\nu, \sigma_{2n-1})), \quad (2.19)$$

where

$$\begin{aligned} \delta(\nu, \sigma_{2n-1}) = & \max\{d^o(\check{Z}\nu, \check{H}\sigma_{2n-1}), d^o(\check{Z}\nu, \check{\Omega}\nu), d^o(\check{H}\sigma_{2n-1}, \check{K}\sigma_{2n-1}), \\ & \frac{1}{2}[d^o(\check{Z}\nu, \check{K}\sigma_{2n-1}) + d^o(\check{\Omega}\nu, \check{H}\sigma_{2n-1})], \\ & d^o(\check{Z}\nu, \check{\Omega}\nu) \left[\frac{1 + d^o(\check{Z}\nu, \check{H}\sigma_{2n-1})}{1 + d^o(\check{H}\sigma_{2n-1}, \check{K}\sigma_{2n-1})} \right], \\ & d^o(\check{H}\sigma_{2n-1}, \check{K}\sigma_{2n-1}) \left[\frac{1 + d^o(\check{Z}\nu, \check{H}\sigma_{2n-1})}{1 + d^o(\check{Z}\nu, \check{\Omega}\nu)} \right], \frac{d^{o^2}(\check{Z}\nu, \check{\Omega}\nu)}{1 + d^o(\check{\Omega}\nu, \check{K}\sigma_{2n-1})}, \\ & \frac{d^{o^2}(\check{H}\sigma_{2n-1}, \check{K}\sigma_{2n-1})}{1 + d^o(\check{\Omega}\nu, \check{K}\sigma_{2n-1})}, d^o(\check{Z}\nu, \check{\Omega}\nu) \left[\frac{1 + d^o(\check{Z}\nu, \check{K}\sigma_{2n-1}) + d^o(\check{\Omega}\nu, \check{H}\sigma_{2n-1})}{1 + d^o(\check{Z}\nu, \check{H}\sigma_{2n-1}) + d^o(\check{\Omega}\nu, \check{K}\sigma_{2n-1})} \right], \\ & d^o(\check{H}\sigma_{2n-1}, \check{K}\sigma_{2n-1}) \left[\frac{1 + d^o(\check{Z}\nu, \check{K}\sigma_{2n-1}) + d^o(\check{\Omega}\nu, \check{H}\sigma_{2n-1})}{1 + d^o(\check{Z}\nu, \check{H}\sigma_{2n-1}) + d^o(\check{\Omega}\nu, \check{K}\sigma_{2n-1})} \right] \}. \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \delta(\nu, \sigma_{2n-1}) = \max\{d^o(\mu, \mu), d^o(\mu, b), d^o(\mu, \mu), \frac{1}{2}[d^o(\mu, \mu) + d^o(b, \mu)],$$

$$\begin{aligned} & d^o(\mu, b) \left[\frac{1 + d^o(\mu, \mu)}{1 + d^o(\mu, \mu)} \right], d^o(\mu, \mu) \left[\frac{1 + d^o(\mu, \mu)}{1 + d^o(\mu, b)} \right], \\ & \frac{d^{o^2}(\mu, b)}{1 + d^o(b, \mu)}, \frac{d^{o^2}(\mu, \mu)}{1 + d^o(b, \mu)} \end{aligned}$$

$$\begin{aligned}
 & d^o(\mu, b) \left[\frac{1 + d^o(\mu, \mu) + d^o(b, \mu)}{1 + d^o(\mu, \mu) + d^o(b, \mu)} \right], \\
 & d^o(\mu, \mu) \left[\frac{1 + d^o(\mu, \mu) + d^o(b, \mu)}{1 + d^o(\mu, \mu) + d^o(b, \mu)} \right] \Bigg\}. \\
 & = \max \left\{ 0, d^o(\mu, b), 0, \frac{1}{2}(0 + d^o(\mu, b)), d^o(\mu, b), 0, \frac{d^{o^2}(\mu, b)}{1 + d^o(b, \mu)}, 0, d^o(\mu, b), 0 \right\}. \\
 & = d^o(\mu, b).
 \end{aligned}$$

Thus, from (2.19), we have

$$\begin{aligned}
 \psi(d^o(\mu, b)) & \leq \psi(d^o(\mu, b)) - \phi(d^o(\mu, b)), \\
 & < \psi(d^o(\mu, b)),
 \end{aligned}$$

which is a contradiction, since $\phi(d^o(\mu, b)) > 0$.

Thus, $\mu = b$. Hence $\check{Z}v = \check{\Omega}v = \mu$, which implies that v is coincidence point of the pair $(\check{Z}, \check{\Omega})$. Since $\check{\Omega}M \subseteq \check{H}M$, $\check{\Omega}v = \mu$ implies that, $\mu \in \check{H}M$.

Let $\theta \in \check{H}^{-1}\mu$. Then $\check{H}\theta = \mu$. By using the same arguments as above, we can easily verify that $\check{K}\theta = \mu = \check{H}\theta$, that is, θ is the coincidence point of the pair (\check{H}, \check{K}) .

Similarly, we can prove the result if $\check{H}M$ is complete subspace of M instead of $\check{Z}M$. Now, if $\check{K}M$ is complete then by (2.1), $\mu \in \check{K}M \subseteq \check{Z}M$.

In the same manner, if $\check{\Omega}M$ is complete then $\mu \in \check{\Omega}M \subseteq \check{H}M$. Now, since the pair $(\check{Z}, \check{\Omega})$ and (\check{H}, \check{K}) are weakly compatible, so

$$\begin{aligned}
 \mu & = \check{\Omega}v = \check{Z}v = \check{K}\theta = \check{H}\theta \\
 \check{Z}\mu & = \check{Z}\check{\Omega}v = \check{\Omega}\check{Z}v = \check{\Omega}\mu \\
 \check{H}\mu & = \check{H}\check{K}\theta = \check{K}\check{H}\theta = \check{K}\mu
 \end{aligned} \tag{2.20}$$

Now, we shall prove that $\check{K}\mu = \mu$.

Let, if possible, $\check{K}\mu \neq \mu$.

From (2.2), we have

$$\psi(d^o(\mu, \check{K}\mu)) = \psi(d^o(\check{\Omega}v, \check{K}\mu)) \leq \psi(\delta(v, \mu)) - \phi(\delta(v, \mu)),$$

where

$$\begin{aligned}
 \delta(v, \mu) & = \max\{d^o(\check{Z}v, \check{H}\mu), d^o(\check{Z}v, \check{\Omega}v), d^o(\check{H}\mu, \check{K}\mu), \\
 & \frac{1}{2}[d^o(\check{Z}v, \check{K}\mu) + d^o(\check{\Omega}v, \check{H}\mu)],
 \end{aligned}$$

$$\begin{aligned}
 & d^o(\check{Z}v, \check{\Omega}v) \left[\frac{1 + d^o(\check{Z}v, \check{H}\mu)}{1 + d^o(\check{H}\mu, \check{K}\mu)} \right], \\
 & d^o(\check{H}\mu, \check{K}\mu) \left[\frac{1 + d^o(\check{Z}v, \check{H}\mu)}{1 + d^o(\check{Z}v, \check{\Omega}v)} \right], \\
 & \frac{d^{o^2}(\check{Z}v, \check{\Omega}v)}{1 + d^o(\check{\Omega}v, \check{K}\mu)}, \quad \frac{d^{o^2}(\check{H}\mu, \check{K}\mu)}{1 + d^o(\check{\Omega}v, \check{K}\mu)}, \\
 & d^o(\check{Z}v, \check{\Omega}v) \left[\frac{1 + d^o(\check{Z}v, \check{K}\mu) + d^o(\check{\Omega}v, \check{H}\mu)}{1 + d^o(\check{Z}v, \check{H}\mu) + d^o(\check{\Omega}v, \check{K}\mu)} \right], \\
 & d^o(\check{H}\mu, \check{K}\mu) \left[\frac{1 + d^o(\check{Z}v, \check{K}\mu) + d^o(\check{\Omega}v, \check{H}\mu)}{1 + d^o(\check{Z}v, \check{H}\mu) + d^o(\check{\Omega}v, \check{K}\mu)} \right] \}.
 \end{aligned}$$

Using (2.20), we have

$$\begin{aligned}
 & = \max\{d^o(\mu, \check{K}\mu), d^o(\mu, \mu), d^o(\check{K}\mu, \check{K}\mu), \\
 & \quad \frac{1}{2}[d^o(\mu, \check{K}\mu) + d^o(\mu, \check{K}\mu)], \\
 & \quad d^o(\mu, \mu) \left[\frac{1 + d^o(\mu, \check{K}\mu)}{1 + d^o(\check{K}\mu, \check{K}\mu)} \right], d^o(\check{K}\mu, \check{K}\mu) \left[\frac{1 + d^o(\mu, \check{K}\mu)}{1 + d^o(\mu, \mu)} \right], \\
 & \quad \frac{d^{o^2}(\mu, \mu)}{1 + d^o(\mu, \check{K}\mu)}, \frac{d^{o^2}(\check{K}\mu, \check{K}\mu)}{1 + d^o(\mu, \check{K}\mu)}, \\
 & \quad d^o(\mu, \mu) \left[\frac{1 + d^o(\mu, \check{K}\mu) + d^o(\mu, \check{K}\mu)}{1 + d^o(\mu, \check{K}\mu) + d^o(\mu, \check{K}\mu)} \right], \\
 & \quad d^o(\check{K}\mu, \check{K}\mu) \left[\frac{1 + d^o(\mu, \check{K}\mu) + d^o(\mu, \check{K}\mu)}{1 + d^o(\mu, \check{K}\mu) + d^o(\mu, \check{K}\mu)} \right] \}. \\
 & = \max\{d^o(\mu, \check{K}\mu), 0, 0, d^o(\mu, \check{K}\mu), 0, 0, 0, 0, 0, 0, 0\}. \\
 & = d^o(\mu, \check{K}\mu).
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 \psi(d^o(\mu, \check{K}\mu)) & \leq \psi(d^o(\mu, \check{K}\mu)) - \phi(d^o(\mu, \check{K}\mu)) \\
 & < \psi(d^o(\mu, \check{K}\mu)),
 \end{aligned}$$

which is a contradiction. So, $\mu = \check{K}\mu$. Similarly, $\check{\Omega}\mu = \mu$.

Thus, we get $\check{K}\mu = \check{H}\mu = \check{Z}\mu = \check{\Omega}\mu = \mu$.

Hence μ is the common fixed point of \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} .

For the uniqueness, let κ be another common fixed point of \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} . Now, we claim that $\kappa = \mu$. Let, if possible $\kappa \neq \mu$.

From (2.2), we have

$$\begin{aligned} \psi(d^o(\mu, \kappa)) &= \psi(d^o(\check{\Omega}\mu, \check{K}\kappa)) \\ &\leq \psi(\delta(\mu, \kappa)) - \phi(\delta(\mu, \kappa)) \\ &= \psi(d^o(p, \kappa)) - \phi(d^o(p, \kappa)), \\ &< \psi(d^o(p, \kappa)), \end{aligned}$$

which is a contradiction. Thus, $\kappa = \mu$ and hence the uniqueness follows. This completes the proof of the theorem.

Theorem 2.2. Let (M, d^o) be a metric space and let $\check{Z}, \check{H}, \check{\Omega}$ and \check{K} be self-maps on M satisfying (2.1), (2.2) and the followings:

(2.21) The pairs $(\check{Z}, \check{\Omega})$ and (\check{H}, \check{K}) are weakly compatible.

(2.22) The pair $(\check{Z}, \check{\Omega})$ or (\check{H}, \check{K}) satisfy the E.A. property.

If one of $\check{Z}M, \check{H}M, \check{\Omega}M$ or $\check{K}M$ is complete subspace of M , then $\check{Z}, \check{H}, \check{\Omega}$ and \check{K} have a unique common fixed point.

Proof: Suppose that the pair $(\check{Z}, \check{\Omega})$ satisfies the E.A. property. Then, there exists a sequence $\{\sigma_n\}$ in M , such that $\lim_{n \rightarrow \infty} \check{Z}\sigma_n = \lim_{n \rightarrow \infty} \check{\Omega}\sigma_n = \mu$ for some μ in M .

Suppose that $\check{H}M$ is a complete subspace of M . Then $\mu = \check{H}\alpha$ for some $\alpha \in M$.

Now, we shall show that $\check{K}\alpha = \check{H}\alpha$. Let, if possible $\check{K}\alpha \neq \check{H}\alpha$.

From (2.2), we have

$$\psi(d^o(\check{\Omega}\sigma_n, \check{K}\alpha)) \leq \psi(\delta(\sigma_n, \alpha)) - \phi(\delta(\sigma_n, \alpha))$$

Taking limit as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \psi(d^o(\check{\Omega}\sigma_n, \check{K}\alpha)) \leq \lim_{n \rightarrow \infty} \psi(\delta(\sigma_n, \alpha)) - \lim_{n \rightarrow \infty} \phi(\delta(\sigma_n, \alpha)), \quad (2.23)$$

where

$$\begin{aligned} \delta(\sigma_n, \alpha) &= \max\{d^o(\check{Z}\sigma_n, \check{H}\alpha), d^o(\check{Z}\sigma_n, \check{\Omega}\sigma_n), d^o(\check{H}\alpha, \check{K}\alpha), \\ &\quad \frac{1}{2}[d^o(\check{Z}\sigma_n, \check{K}\alpha) + d^o(\check{\Omega}\sigma_n, \check{H}\alpha)], \\ &\quad d^o(\check{Z}\sigma_n, \check{\Omega}\sigma_n) \left[\frac{1 + d^o(\check{Z}\sigma_n, \check{H}\alpha)}{1 + d^o(\check{H}\alpha, \check{K}\alpha)} \right]\}, \end{aligned}$$

$$\begin{aligned}
 & d^o(\check{H}\alpha, \check{K}\alpha) \left[\frac{1 + d^o(\check{Z}\sigma_n, \check{H}\alpha)}{1 + d^o(\check{Z}\sigma_n, \check{\Omega}\sigma_n)} \right], \\
 & \frac{d^{o^2}(\check{Z}\sigma_n, \check{\Omega}\sigma_n)}{1 + d^o(\check{\Omega}\sigma_n, \check{K}\alpha)}, \frac{d^{o^2}(\check{H}\alpha, \check{K}\alpha)}{1 + d^o(\check{\Omega}\sigma_n, \check{K}\alpha)}, \\
 & d^o(\check{Z}\sigma_n, \check{\Omega}\sigma_n) \left[\frac{1 + d^o(\check{Z}\sigma_n, \check{K}\alpha) + d^o(\check{\Omega}\sigma_n, \check{H}\alpha)}{1 + d^o(\check{Z}\sigma_n, \check{H}\alpha) + d^o(\check{\Omega}\sigma_n, \check{K}\alpha)} \right], \\
 & d^o(\check{H}\alpha, \check{K}\alpha) \left[\frac{1 + d^o(\check{Z}\sigma_n, \check{K}\alpha) + d^o(\check{\Omega}\sigma_n, \check{H}\alpha)}{1 + d^o(\check{Z}\sigma_n, \check{H}\alpha) + d^o(\check{\Omega}\sigma_n, \check{K}\alpha)} \right] \}. \\
 = & \max \left\{ d^o(\mu, \mu), d^o(\mu, \mu), d^o(\mu, \check{K}\alpha), \frac{1}{2} [d^o(\mu, \check{K}\alpha) + d^o(\mu, \mu)], \right. \\
 & d^o(\mu, \mu) \left[\frac{1 + d^o(\mu, \mu)}{1 + d^o(\mu, \check{K}\alpha)} \right], d^o(\mu, \check{K}\alpha) \left[\frac{1 + d^o(\mu, \mu)}{1 + d^o(\mu, \mu)} \right], \\
 & \frac{d^{o^2}(\mu, \mu)}{1 + d^o(\mu, \check{K}\alpha)}, \frac{d^{o^2}(\mu, \check{K}\alpha)}{1 + d^o(\mu, \check{K}\alpha)}, \\
 & d^o(\mu, \mu) \left[\frac{1 + d^o(\mu, \check{K}\alpha) + d^o(\mu, \mu)}{1 + d^o(\mu, \mu) + d^o(\mu, \check{K}\alpha)} \right], \\
 & \left. d^o(\mu, \check{K}\alpha) \left[\frac{1 + d^o(\mu, \check{K}\alpha) + d^o(\mu, \mu)}{1 + d^o(\mu, \mu) + d^o(\mu, \check{K}\alpha)} \right] \right\}. \\
 = & \max \left\{ 0, 0, d^o(\mu, \check{K}\alpha), \frac{1}{2} d^o(\mu, \check{K}\alpha), 0, d^o(\mu, \check{K}\alpha), 0, d^o(\mu, \check{K}\alpha), 0, d^o(\mu, \check{K}\alpha) \right\}. \\
 = & \max \{ d^o(\mu, \check{K}\alpha) \}.
 \end{aligned}$$

Thus, from (2.23), we have

$$\begin{aligned}
 \psi \left(d^o(\mu, \check{K}\alpha) \right) & \leq \psi \left(d^o(\mu, \check{K}\alpha) \right) - \phi \left(d^o(\mu, \check{K}\alpha) \right) \\
 & < \psi \left(d^o(\mu, \check{K}\alpha) \right),
 \end{aligned}$$

which is possible only when $\check{K}\alpha = \mu = \check{H}\alpha$. Since \check{H} and \check{K} are weakly compatible, therefore, $\check{K}\check{H}\alpha = \check{H}\check{K}\alpha$, implies that, $\check{K}\check{K}\alpha = \check{K}\check{H}\alpha = \check{H}\check{K}\alpha = \check{H}\check{H}\alpha$. Since $\check{K}M \subseteq \check{Z}M$, there exists $\beta \in M$, such that, $\check{K}\alpha = \check{Z}\beta$.

Now, we claim that $\check{Z}\beta = \check{\Omega}\beta$. Let, if possible, $\check{Z}\beta \neq \check{\Omega}\beta$.

From (2.2), we have

$$\psi \left(d^o(\check{\Omega}\beta, \check{K}\alpha) \right) \leq \psi(\delta(\beta, \alpha)) - \phi(\delta(\beta, \alpha)), \tag{2.24}$$

where

$$\begin{aligned}
 \delta(\beta, \alpha) &= \max\{d^\circ(\check{Z}\beta, \check{H}\alpha), d^\circ(\check{Z}\beta, \check{\Omega}\beta), d^\circ(\check{H}\alpha, \check{K}\alpha), \\
 &\quad \frac{1}{2}[d^\circ(\check{Z}\beta, \check{K}\alpha) + d^\circ(\check{\Omega}\beta, \check{H}\alpha)], \\
 &\quad d^\circ(\check{Z}\beta, \check{\Omega}\beta) \left[\frac{1 + d^\circ(\check{Z}\beta, \check{H}\alpha)}{1 + d^\circ(\check{H}\alpha, \check{K}\alpha)} \right], \\
 &\quad d^\circ(\check{H}\alpha, \check{K}\alpha) \left[\frac{1 + d^\circ(\check{Z}\beta, \check{H}\alpha)}{1 + d^\circ(\check{Z}\beta, \check{\Omega}\beta)} \right], \\
 &\quad \frac{d^{\circ 2}(\check{Z}\beta, \check{\Omega}\beta)}{1 + d^\circ(\check{\Omega}\beta, \check{K}\alpha)}, \frac{d^{\circ 2}(\check{H}\alpha, \check{K}\alpha)}{1 + d^\circ(\check{\Omega}\beta, \check{K}\alpha)}, \\
 &\quad d^\circ(\check{Z}\beta, \check{\Omega}\beta) \left[\frac{1 + d^\circ(\check{Z}\beta, \check{K}\alpha) + d^\circ(\check{\Omega}\beta, \check{H}\alpha)}{1 + d^\circ(\check{Z}\beta, \check{H}\alpha) + d^\circ(\check{\Omega}\beta, \check{K}\alpha)} \right], \\
 &\quad d^\circ(\check{H}\alpha, \check{K}\alpha) \left[\frac{1 + d^\circ(\check{Z}\beta, \check{K}\alpha) + d^\circ(\check{\Omega}\beta, \check{H}\alpha)}{1 + d^\circ(\check{Z}\beta, \check{H}\alpha) + d^\circ(\check{\Omega}\beta, \check{K}\alpha)} \right]\}, \\
 &= \max\{d^\circ(\check{K}\alpha, \check{H}\alpha), d^\circ(\check{K}\alpha, \check{\Omega}\beta), d^\circ(\check{H}\alpha, \check{K}\alpha), \\
 &\quad \frac{1}{2}[d^\circ(\check{K}\alpha, \check{K}\alpha) + d^\circ(\check{\Omega}\beta, \check{K}\alpha)], \\
 &\quad d^\circ(\check{K}\alpha, \check{\Omega}\beta) \left[\frac{1 + d^\circ(\check{K}\alpha, \check{H}\alpha)}{1 + d^\circ(\check{H}\alpha, \check{K}\alpha)} \right], \\
 &\quad d^\circ(\check{H}\alpha, \check{K}\alpha) \left[\frac{1 + d^\circ(\check{K}\alpha, \check{H}\alpha)}{1 + d^\circ(\check{K}\alpha, \check{\Omega}\beta)} \right], \\
 &\quad \frac{d^{\circ 2}(\check{K}\alpha, \check{\Omega}\beta)}{1 + d^\circ(\check{\Omega}\beta, \check{K}\alpha)}, \frac{d^{\circ 2}(\check{H}\alpha, \check{K}\alpha)}{1 + d^\circ(\check{\Omega}\beta, \check{K}\alpha)}, \\
 &\quad d^\circ(\check{K}\alpha, \check{\Omega}\beta) \left[\frac{1 + d^\circ(\check{K}\alpha, \check{K}\alpha) + d^\circ(\check{K}\alpha, \check{\Omega}\beta)}{1 + d^\circ(\check{K}\alpha, \check{H}\alpha) + d^\circ(\check{\Omega}\beta, \check{K}\alpha)} \right], \\
 &\quad d^\circ(\check{H}\alpha, \check{K}\alpha) \left[\frac{1 + d^\circ(\check{K}\alpha, \check{K}\alpha) + d^\circ(\check{\Omega}\beta, \check{K}\alpha)}{1 + d^\circ(\check{K}\alpha, \check{H}\alpha) + d^\circ(\check{\Omega}\beta, \check{K}\alpha)} \right]\}, \\
 &= \max\{0, d^\circ(\check{K}\alpha, \check{\Omega}\beta), 0, \frac{1}{2}(0 + d^\circ(\check{K}\alpha, \check{\Omega}\beta)), d^\circ(\check{K}\alpha, \check{\Omega}\beta), 0, \\
 &\quad \frac{d^2(\check{K}\alpha, \check{\Omega}\beta)}{1 + d^\circ(\check{K}\alpha, \check{\Omega}\beta)}, 0, d^\circ(\check{K}\alpha, \check{\Omega}\beta)\}, \\
 &= d^\circ(\check{\Omega}\beta, \check{K}\alpha).
 \end{aligned}$$

From (2.24), we have

$$\begin{aligned} \psi(d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\alpha)) &\leq \psi(d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\alpha)) - \phi(d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\alpha)) \\ &< \psi(d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\alpha)), \end{aligned}$$

which is a contradiction. Therefore, $\tilde{\Omega}\beta = \check{\kappa}\alpha = \check{z}\beta$.

Now, since $(\check{z}, \tilde{\Omega})$ is weakly compatible. This implies that $\check{z}\tilde{\Omega}\beta = \tilde{\Omega}\check{z}\beta = \tilde{\Omega}\tilde{\Omega}\beta = \check{z}\check{z}\beta$.

Now, we claim that $\check{\kappa}\alpha$ is common fixed point of $\check{z}, \check{h}, \tilde{\Omega}$ and $\check{\kappa}$.

Let, if possible, $\check{\kappa}\check{\kappa}\alpha \neq \check{\kappa}\alpha$.

From (2.2), we have

$$\psi(d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)) = \psi(d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\check{\kappa}\alpha)) \leq \psi(\delta(\beta, \check{\kappa}\alpha)) - \phi(\delta(\beta, \check{\kappa}\alpha)), \quad (2.25)$$

where

$$\begin{aligned} \delta(\beta, \check{\kappa}\alpha) &= \max\{d^{\circ}(\check{z}\beta, \check{h}\check{\kappa}\alpha), d^{\circ}(\check{z}\beta, \tilde{\Omega}\beta), d^{\circ}(\check{h}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha), \\ &\quad \frac{1}{2}[d^{\circ}(\check{z}\beta, \check{\kappa}\check{\kappa}\alpha) + d^{\circ}(\tilde{\Omega}\beta, \check{h}\check{\kappa}\alpha)], \\ &\quad d^{\circ}(\check{z}\beta, \tilde{\Omega}\beta) \left[\frac{1 + d^{\circ}(\check{z}\beta, \check{h}\check{\kappa}\alpha)}{1 + d^{\circ}(\check{h}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)} \right], \\ &\quad d^{\circ}(\check{h}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha) \left[\frac{1 + d^{\circ}(\check{z}\beta, \check{h}\check{\kappa}\alpha)}{1 + d^{\circ}(\check{z}\beta, \tilde{\Omega}\beta)} \right], \\ &\quad \frac{d^{\circ 2}(\check{z}\beta, \tilde{\Omega}\beta)}{1 + d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\check{\kappa}\alpha)}, \frac{d^{\circ 2}(\check{h}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)}{1 + d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\check{\kappa}\alpha)}, \\ &\quad d^{\circ}(\check{z}\beta, \tilde{\Omega}\beta) \left[\frac{1 + d^{\circ}(\check{z}\beta, \check{\kappa}\check{\kappa}\alpha) + d^{\circ}(\tilde{\Omega}\beta, \check{h}\check{\kappa}\alpha)}{1 + d^{\circ}(\check{z}\beta, \check{h}\check{\kappa}\alpha) + d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\check{\kappa}\alpha)} \right], \\ &\quad d^{\circ}(\check{h}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha) \left[\frac{1 + d^{\circ}(\check{z}\beta, \check{\kappa}\check{\kappa}\alpha) + d^{\circ}(\tilde{\Omega}\beta, \check{h}\check{\kappa}\alpha)}{1 + d^{\circ}(\check{z}\beta, \check{h}\check{\kappa}\alpha) + d^{\circ}(\tilde{\Omega}\beta, \check{\kappa}\check{\kappa}\alpha)} \right]\}, \\ &= \max\{d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha), d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\alpha), d^{\circ}(\check{h}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha), \\ &\quad \frac{1}{2}[d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha) + d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)], \\ &\quad d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\alpha) \left[\frac{1 + d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)}{1 + d^{\circ}(\check{\kappa}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)} \right], \\ &\quad d^{\circ}(\check{\kappa}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha) \left[\frac{1 + d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)}{1 + d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\alpha)} \right], \\ &\quad \frac{d^{\circ 2}(\check{\kappa}\alpha, \check{\kappa}\alpha)}{1 + d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)}, \frac{d^{\circ 2}(\check{\kappa}\check{\kappa}\alpha, \check{\kappa}\check{\kappa}\alpha)}{1 + d^{\circ}(\check{\kappa}\alpha, \check{\kappa}\alpha)}\}, \end{aligned}$$

$$\begin{aligned}
 & d^o(\check{K}\alpha, \check{K}\alpha) \left[\frac{1 + d^o(\check{K}\alpha, \check{K}\check{K}\alpha) + d^o(\check{K}\alpha, \check{K}\check{K}\alpha)}{1 + d^o(\check{K}\alpha, \check{K}\check{K}\alpha) + d^o(\check{K}\alpha, \check{K}\check{K}\alpha)} \right], \\
 & d^o(\check{K}\check{K}\alpha, \check{K}\check{K}\alpha) \left[\frac{1 + d^o(\check{K}\alpha, \check{K}\check{K}\alpha) + d^o(\check{K}\alpha, \check{K}\check{K}\alpha)}{1 + d^o(\check{K}\alpha, \check{K}\check{K}\alpha) + d^o(\check{K}\alpha, \check{K}\check{K}\alpha)} \right] \Big\}. \\
 & = \max\{d^o(\check{K}\alpha, \check{K}\check{K}\alpha), 0, 0, d^o(\check{K}\alpha, \check{K}\check{K}\alpha), 0, 0, 0, 0, 0, 0\}. \\
 & = d^o(\check{K}\alpha, \check{K}\check{K}\alpha).
 \end{aligned}$$

Thus, from (2.25), we have

$$\begin{aligned}
 \psi(d^o(\check{K}\alpha, \check{K}\check{K}\alpha)) & \leq \psi(d^o(\check{K}\alpha, \check{K}\check{K}\alpha)) - \phi(d^o(\check{K}\alpha, \check{K}\check{K}\alpha)) \\
 & < \psi(d^o(\check{K}\alpha, \check{K}\check{K}\alpha)),
 \end{aligned}$$

which is a contradiction. Therefore, $\check{K}\alpha = \check{K}\check{K}\alpha = \check{H}\check{K}\alpha$.

Hence $\check{K}\alpha$ is the common fixed point of \check{H} and \check{K} .

Similarly, we can prove that $\check{\Omega}\beta$ is common fixed point of $\check{\Omega}$ and \check{Z} . Since $\check{K}\alpha = \check{\Omega}\beta$, $\check{K}\alpha$ is the common fixed point of \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} . If we assume $\check{\Omega}M$ is complete subspace of M , the proof is similar.

Similarly we can prove the theorem for cases when $\check{Z}M$ or $\check{H}M$ is a complete subspace of M . Since $\check{K}M \subseteq \check{Z}M$ and $\check{\Omega}M \subseteq \check{H}M$.

Now, we shall prove the uniqueness of common fixed point. If possible, let σ and ω be two common fixed points of \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} such that $\sigma \neq \omega$.

From (2.2), we have

$$\psi(d^o(\check{\Omega}\sigma, \check{K}\omega)) \leq \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)), \tag{2.26}$$

where

$$\begin{aligned}
 \delta(\sigma, \omega) & = \max\{d^o(\check{Z}\sigma, \check{H}\omega), d^o(\check{Z}\sigma, \check{\Omega}\sigma), d^o(\check{H}\omega, \check{K}\omega), \\
 & \quad \frac{1}{2}[d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)], \\
 & \quad d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{H}\omega, \check{K}\omega)} \right], \\
 & \quad d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{\Omega}\sigma)} \right], \\
 & \quad \frac{d^{o^2}(\check{Z}\sigma, \check{\Omega}\sigma)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, \frac{d^{o^2}(\check{H}\omega, \check{K}\omega)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)} \Big\}
 \end{aligned}$$

$$\begin{aligned}
 & d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right], \\
 & d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right] \Big\}. \\
 = & \max\{d^o(\sigma, \omega), d^o(\sigma, \sigma), d^o(\omega, \omega), \frac{1}{2}[d^o(\sigma, \omega) + d^o(\sigma, \omega)], \\
 & d^o(\sigma, \sigma) \left[\frac{1 + d^o(\sigma, \omega)}{1 + d^o(\omega, \omega)} \right], d^o(\omega, \omega) \left[\frac{1 + d^o(\sigma, \omega)}{1 + d^o(\sigma, \sigma)} \right], \frac{d^{o^2}(\sigma, \sigma)}{1 + d^o(\sigma, \omega)}, \\
 & \frac{d^{o^2}(\omega, \omega)}{1 + d^o(\sigma, \omega)}, d^o(\sigma, \sigma) \left[\frac{1 + d^o(\sigma, \omega) + d^o(\sigma, \omega)}{1 + d^o(\sigma, \omega) + d^o(\sigma, \omega)} \right], \\
 & d^o(\omega, \omega) \left[\frac{1 + d^o(\sigma, \omega) + d^o(\sigma, \omega)}{1 + d^o(\sigma, \omega) + d^o(\sigma, \omega)} \right] \Big\}. \\
 = & \max\{d^o(\sigma, \omega), 0, 0, d^o(\sigma, \omega), 0, 0, 0, 0, 0, 0\}.
 \end{aligned}$$

From (2.26), we have

$$\begin{aligned}
 \psi(d^o(\sigma, \omega)) & \leq \psi(d^o(\sigma, \omega)) - \phi(d^o(\sigma, \omega)) \\
 & < \psi(d^o(\sigma, \omega)),
 \end{aligned}$$

which is a contradiction.

Therefore, $\sigma = \omega$ and this follows the uniqueness and completes the proof of the theorem.

Theorem 2.3. Let (M, d^o) be a metric space. let \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} be self-maps on M satisfying (2.2) and (2.21) and the followings:

(2.27) $\check{\Omega}M \subseteq \check{H}M$ and the pair $(\check{Z}, \check{\Omega})$ satisfied $(CLR_{\check{Z}})$ or

$\check{K}M \subseteq \check{Z}M$ and the pair (\check{H}, \check{K}) satisfied $(CLR_{\check{H}})$.

Then \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} have unique common fixed point.

Proof: Without loss of generality, assume that $\check{\Omega}M \subseteq \check{H}M$ and the pair $(\check{Z}, \check{\Omega})$ satisfied $(CLR_{\check{Z}})$ property. Then, there exists a sequence $\{\sigma_n\}$ in M such that

$$\lim_{n \rightarrow \infty} \check{Z}\sigma_n = \lim_{n \rightarrow \infty} \check{\Omega}\sigma_n = \check{Z}\mu \text{ for some } \mu \text{ in } M.$$

Since $\check{\Omega}M \subseteq \check{H}M$, there exists a sequence $\{\omega_n\}$ in M such that $\check{\Omega}\sigma_n = \check{H}\omega_n$.

Hence

$$\lim_{n \rightarrow \infty} \check{H}\omega_n = \check{Z}\mu.$$

Now, we shall show that $\lim_{n \rightarrow \infty} \check{K}\omega_n = \check{Z}\mu$. Let if possible, $\lim_{n \rightarrow \infty} \check{K}\omega_n = \pi \neq \check{Z}\mu$.

From (2.2), we have

$$\psi \left(d^o(\check{\Omega}\sigma_n, \check{K}\omega_n) \right) \leq \psi(\delta(\sigma_n, \omega_n)) - \phi(\delta(\sigma_n, \omega_n)), \quad (2.28)$$

where

$$\delta(\sigma_n, \omega_n) = \max\{d^o(\check{Z}\sigma_n, \check{H}\omega_n), d^o(\check{Z}\sigma_n, \check{\Omega}\sigma_n), d^o(\check{H}\omega_n, \check{K}\omega_n),$$

$$\frac{1}{2} [d^o(\check{Z}\sigma_n, \check{K}\omega_n) + d^o(\check{\Omega}\sigma_n, \check{H}\omega_n)],$$

$$d^o(\check{Z}\sigma_n, \check{\Omega}\sigma_n) \left[\frac{1 + d^o(\check{Z}\sigma_n, \check{H}\omega_n)}{1 + d^o(\check{H}\omega_n, \check{K}\omega_n)} \right],$$

$$d^o(\check{H}\omega_n, \check{K}\omega_n) \left[\frac{1 + d^o(\check{Z}\sigma_n, \check{H}\omega_n)}{1 + d^o(\check{Z}\sigma_n, \check{\Omega}\sigma_n)} \right],$$

$$\frac{d^{o^2}(\check{Z}\sigma_n, \check{\Omega}\sigma_n)}{1 + d^o(\check{\Omega}\sigma_n, \check{K}\omega_n)}, \frac{d^{o^2}(\check{H}\omega_n, \check{K}\omega_n)}{1 + d^o(\check{\Omega}\sigma_n, \check{K}\omega_n)},$$

$$d^o(\check{Z}\sigma_n, \check{\Omega}\sigma_n) \left[\frac{1 + d^o(\check{Z}\sigma_n, \check{K}\omega_n) + d^o(\check{\Omega}\sigma_n, \check{H}\omega_n)}{1 + d^o(\check{Z}\sigma_n, \check{H}\omega_n) + d^o(\check{\Omega}\sigma_n, \check{K}\omega_n)} \right],$$

$$d^o(\check{H}\omega_n, \check{K}\omega_n) \left[\frac{1 + d^o(\check{Z}\sigma_n, \check{K}\omega_n) + d^o(\check{\Omega}\sigma_n, \check{H}\omega_n)}{1 + d^o(\check{Z}\sigma_n, \check{H}\omega_n) + d^o(\check{\Omega}\sigma_n, \check{K}\omega_n)} \right] \}.$$

$$= \max\{d^o(\check{Z}\mu, \check{Z}\mu), d^o(\check{Z}\mu, \check{Z}\mu), d^o(\check{Z}\mu, \pi),$$

$$\frac{1}{2} [d^o(\check{Z}\mu, \pi) + d^o(\check{Z}\mu, \check{Z}\mu)],$$

$$d^o(\check{Z}\mu, \check{Z}\mu) \left[\frac{1 + d^o(\check{Z}\mu, \check{Z}\mu)}{1 + d^o(\check{Z}\mu, \pi)} \right], d^o(\check{Z}\mu, \pi) \left[\frac{1 + d^o(\check{Z}\mu, \check{Z}\mu)}{1 + d^o(\check{Z}\mu, \check{Z}\mu)} \right],$$

$$\frac{d^{o^2}(\check{Z}\mu, \check{Z}\mu)}{1 + d^o(\check{Z}\mu, \pi)}, \frac{d^{o^2}(\check{Z}\mu, \pi)}{1 + d^o(\check{Z}\mu, \pi)},$$

$$d^o(\check{Z}\mu, \check{Z}\mu) \left[\frac{1 + d^o(\check{Z}\mu, \pi) + d^o(\check{Z}\mu, \check{Z}\mu)}{1 + d^o(\check{Z}\mu, \check{Z}\mu) + d^o(\check{Z}\mu, \pi)} \right],$$

$$d^o(\check{Z}\mu, \pi) \left[\frac{1 + d^o(\check{Z}\mu, \pi) + d^o(\check{Z}\mu, \check{Z}\mu)}{1 + d^o(\check{Z}\mu, \check{Z}\mu) + d^o(\check{Z}\mu, \pi)} \right] \}.$$

$$= \max \left\{ 0, 0, d^o(\check{Z}\mu, \pi), \frac{1}{2} d^o(\check{Z}\mu, \pi), 0, d^o(\check{Z}\mu, \pi), 0, \frac{d^{o^2}(\check{Z}\mu, \pi)}{1 + d^o(\check{Z}\mu, \pi)}, 0, d^o(\check{Z}\mu, \pi) \right\}.$$

$$= d^o(\check{Z}\mu, \pi)$$

From (2.28), we have

$$\begin{aligned}\psi\left(d^{\circ}(\check{Z}\mu, \pi)\right) &\leq \psi\left(d^{\circ}(\check{Z}\mu, \pi)\right) - \phi\left(d^{\circ}(\check{Z}\mu, \pi)\right) \\ &< \psi\left(d^{\circ}(\check{Z}\mu, \pi)\right),\end{aligned}$$

which is a contradiction.

Therefore, $\check{Z}\mu = \pi$, that is, $\lim_{n \rightarrow \infty} \check{K}\omega_n = \pi \neq \check{Z}\mu..$

Subsequently, we have $\lim_{n \rightarrow \infty} \check{K}\omega_n = \lim_{n \rightarrow \infty} \check{\Omega}\mu_n = \lim_{n \rightarrow \infty} \check{Z}\mu_n = \lim_{n \rightarrow \infty} \check{H}\omega_n = \check{Z}\mu = \pi$.

Now, we shall show that $\check{\Omega}\mu = \pi$. let, if possible, $\check{\Omega}\mu \neq \pi$.

From (2.2), we have

$$\begin{aligned}\psi\left(d^{\circ}(\check{\Omega}\mu, \check{K}\omega_n)\right) &\leq \psi(\delta(\mu, \omega_n)) - \phi(\delta(\mu, \omega_n)), \\ \lim_{n \rightarrow \infty} \psi\left(d^{\circ}(\check{\Omega}\mu, \check{K}\omega_n)\right) &\leq \lim_{n \rightarrow \infty} \psi(\delta(\mu, \omega_n)) - \lim_{n \rightarrow \infty} \phi(\delta(\mu, \omega_n)),\end{aligned}\quad (2.29)$$

where

$$\delta(\mu, \omega_n) = \max\{d^{\circ}(\check{Z}\mu, \check{H}\omega_n), d^{\circ}(\check{Z}\mu, \check{\Omega}\mu), d^{\circ}(\check{H}\omega_n, \check{K}\omega_n),$$

$$\begin{aligned}&\frac{1}{2}[d^{\circ}(\check{Z}\mu, \check{K}\omega_n) + d^{\circ}(\check{\Omega}\mu, \check{H}\omega_n)], \\ &d^{\circ}(\check{Z}\mu, \check{\Omega}\mu) \left[\frac{1 + d^{\circ}(\check{Z}\mu, \check{H}\omega_n)}{1 + d^{\circ}(\check{H}\omega_n, \check{K}\omega_n)} \right], \\ &d^{\circ}(\check{Z}\mu, \check{\Omega}\mu) \left[\frac{1 + d^{\circ}(\check{Z}\mu, \check{H}\omega_n)}{1 + d^{\circ}(\check{H}\omega_n, \check{K}\omega_n)} \right], \\ &d^{\circ}(\check{H}\omega_n, \check{K}\omega_n) \left[\frac{1 + d^{\circ}(\check{Z}\mu, \check{H}\omega_n)}{1 + d^{\circ}(\check{Z}\mu, \check{\Omega}\mu)} \right], \\ &\frac{d^{\circ 2}(\check{Z}\mu, \check{\Omega}\mu)}{1 + d^{\circ}(\check{\Omega}\mu, \check{K}\omega_n)}, \frac{d^{\circ 2}(\check{H}\omega_n, \check{K}\omega_n)}{1 + d^{\circ}(\check{\Omega}\mu, \check{K}\omega_n)}, \\ &d^{\circ}(\check{Z}\mu, \check{\Omega}\mu) \left[\frac{1 + d^{\circ}(\check{Z}\mu, \check{K}\omega_n) + d^{\circ}(\check{\Omega}\mu, \check{H}\omega_n)}{1 + d^{\circ}(\check{Z}\mu, \check{H}\omega_n) + d^{\circ}(\check{\Omega}\mu, \check{K}\omega_n)} \right], \\ &d^{\circ}(\check{H}\omega_n, \check{K}\omega_n) \left[\frac{1 + d^{\circ}(\check{Z}\mu, \check{K}\omega_n) + d^{\circ}(\check{\Omega}\mu, \check{H}\omega_n)}{1 + d^{\circ}(\check{Z}\mu, \check{H}\omega_n) + d^{\circ}(\check{\Omega}\mu, \check{K}\omega_n)} \right]\}.\end{aligned}$$

Taking limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \delta(\mu, \omega_n) = \max\{d^{\circ}(\pi, \pi), d^{\circ}(\pi, \check{\Omega}\mu), d^{\circ}(\pi, \pi), \frac{1}{2}[d^{\circ}(\pi, \pi) + d^{\circ}(\check{\Omega}\mu, \pi)],$$

$$d^{\circ}(\pi, \check{\Omega}\mu) \left[\frac{1 + d^{\circ}(\pi, \pi)}{1 + d^{\circ}(\pi, \pi)} \right], d^{\circ}(\pi, \pi) \left[\frac{1 + d^{\circ}(\pi, \pi)}{1 + d^{\circ}(\pi, \check{\Omega}\mu)} \right],$$

$$\begin{aligned}
 & \frac{d^{o^2}(\pi, \check{\Omega}\mu)}{1 + d^o(\check{\Omega}\mu, \pi)}, \frac{d^{o^2}(\pi, \pi)}{1 + d^o(\check{\Omega}\mu, \pi)}, \\
 & d^o(\pi, \pi) \left[\frac{1 + d^o(\pi, \pi) + d^o(\check{\Omega}\mu, \pi)}{1 + d^o(\pi, \pi) + d^o(\check{\Omega}\mu, \pi)} \right], \\
 & d^o(\pi, \pi) \left[\frac{1 + d^o(\pi, \pi) + d^o(\check{\Omega}\mu, \pi)}{1 + d^o(\pi, \pi) + d^o(\check{\Omega}\mu, \pi)} \right] \Big\}. \\
 & = \max \left\{ 0, d^o(\pi, \check{\Omega}\mu), 0, \frac{1}{2} d^o(\pi, \check{\Omega}\mu), d^o(\pi, \check{\Omega}\mu), 0, \frac{d^{o^2}(\pi, \check{\Omega}\mu)}{1 + d^o(\check{\Omega}\mu, \pi)}, 0, 0, 0 \right\}. \\
 & = d^o(\pi, \check{\Omega}\mu).
 \end{aligned}$$

Thus, from (2.29), we get

$$\begin{aligned}
 \psi \left(d^o(\pi, \check{\Omega}\mu) \right) & \leq \psi \left(d^o(\pi, \check{\Omega}\mu) \right) - \phi \left(d^o(\pi, \check{\Omega}\mu) \right). \\
 & < \psi \left(d^o(\pi, \check{\Omega}\mu) \right),
 \end{aligned}$$

which is a contradiction. Therefore, $\check{\Omega}\mu = \pi = \check{Z}\mu$. Since the pair $(\check{Z}, \check{\Omega})$ is weakly compatible, it follows that $\check{Z}\pi = \check{\Omega}\pi$. Also, since $\check{\Omega}M \subseteq \check{H}M$, there exists some θ in M , such that, $\check{\Omega}\mu = \check{H}\theta$, that is, $\check{H}\theta = \pi$.

Now, we show that $\check{K}\theta = \pi$. Let if possible $\check{K}\theta \neq \pi$.

From (2.2), we have

$$\begin{aligned}
 \psi \left(d^o(\check{\Omega}\mu_n, \check{K}\theta) \right) & \leq \psi(\delta(\mu_n, \theta)) - \phi(\delta(\mu_n, \theta)), \\
 \lim_{n \rightarrow \infty} \psi \left(d^o(\check{\Omega}\mu_n, \check{K}\theta) \right) & \leq \lim_{n \rightarrow \infty} \psi(\delta(\mu_n, \theta)) - \lim_{n \rightarrow \infty} \phi(\delta(\mu_n, \theta)), \quad (2.30)
 \end{aligned}$$

where

$$\begin{aligned}
 \delta(\mu_n, \theta) & = \max \{ d^o(\check{Z}\mu_n, \check{H}\theta), d^o(\check{Z}\mu_n, \check{\Omega}\mu_n), d^o(\check{H}\theta, \check{K}\theta), \\
 & \frac{1}{2} [d^o(\check{Z}\mu_n, \check{K}\theta) + d^o(\check{\Omega}\mu_n, \check{H}\theta)], \\
 & d^o(\check{Z}\mu_n, \check{\Omega}\mu_n) \left[\frac{1 + d^o(\check{Z}\mu_n, \check{H}\theta)}{1 + d^o(\check{H}\theta, \check{K}\theta)} \right], \\
 & d^o(\check{Z}\mu_n, \check{\Omega}\mu_n) \left[\frac{1 + d^o(\check{Z}\mu_n, \check{H}\theta)}{1 + d^o(\check{H}\theta, \check{K}\theta)} \right], \\
 & d^o(\check{H}\theta, \check{K}\theta) \left[\frac{1 + d^o(\check{Z}\mu_n, \check{H}\theta)}{1 + d^o(\check{Z}\mu_n, \check{\Omega}\mu_n)} \right],
 \end{aligned}$$

$$\begin{aligned} & \frac{d^{o^2}(\check{Z}\mu_n, \check{\Omega}\mu_n)}{1 + d^o(\check{\Omega}\mu_n, \check{K}\theta)}, \frac{d^{o^2}(\check{H}\theta, \check{K}\theta)}{1 + d^o(\check{\Omega}\mu, \check{K}\theta)}, \\ & d^o(\check{Z}\mu_n, \check{\Omega}\mu_n) \left[\frac{1 + d^o(\check{Z}\mu_n, \check{K}\theta) + d^o(\check{\Omega}\mu_n, \check{H}\theta)}{1 + d^o(\check{Z}\mu_n, \check{H}\theta) + d^o(\check{\Omega}\mu_n, \check{K}\theta)} \right], \\ & d^o(\check{H}\theta, \check{K}\theta) \left[\frac{1 + d^o(\check{Z}\mu_n, \check{K}\theta) + d^o(\check{\Omega}\mu_n, \check{H}\theta)}{1 + d^o(\check{Z}\mu_n, \check{H}\theta) + d^o(\check{\Omega}\mu_n, \check{K}\theta)} \right]. \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \delta(\mu_n, \theta) &= \max\{d^o(\pi, \pi), d^o(\pi, \pi), d^o(\pi, \check{K}\theta), \frac{1}{2}[d^o(\pi, \check{K}\theta) + d^o(\pi, \pi)], \\ & d^o(\pi, \check{K}\theta) \left[\frac{1 + d^o(\pi, \pi)}{1 + d^o(\pi, \check{K}\theta)} \right], d^o(\pi, \check{K}\theta) \left[\frac{1 + d^o(\pi, \pi)}{1 + d^o(\pi, \pi)} \right], \\ & \frac{d^{o^2}(\pi, \pi)}{1 + d^o(\pi, \check{K}\theta)}, \frac{d^{o^2}(\pi, \check{K}\theta)}{1 + d^o(\pi, \check{K}\theta)}, \\ & d^o(\pi, \pi) \left[\frac{1 + d^o(\pi, \check{K}\theta) + d^o(\pi, \pi)}{1 + d^o(\pi, \pi) + d^o(\pi, \check{K}\theta)} \right], \\ & d^o(\pi, \check{K}\theta) \left[\frac{1 + d^o(\pi, \check{K}\theta) + d^o(\pi, \pi)}{1 + d^o(\pi, \pi) + d^o(\pi, \check{K}\theta)} \right]\}. \\ &= \max \left\{ 0, 0, d^o(\pi, \check{K}\theta), \frac{1}{2} d^o(\pi, \check{K}\theta), \frac{d^o(\pi, \check{K}\theta)}{1 + d^o(\pi, \check{K}\theta)}, d^o(\pi, \check{K}\theta), 0, \right. \\ & \left. \frac{d^{o^2}(\pi, \check{K}\theta)}{1 + d^o(\pi, \check{K}\theta)}, 0, d^o(\pi, \check{K}\theta) \right\}. \\ &= d^o(\pi, \check{K}\theta). \end{aligned}$$

Thus, from (2.30), we get

$$\begin{aligned} \psi(d^o(\pi, \check{K}\theta)) &\leq \psi(d^o(\pi, \check{K}\theta)) - \phi(d^o(\pi, \check{K}\theta)). \\ &< \psi(d^o(\pi, \check{K}\theta)), \end{aligned}$$

which is a contradiction. Therefore, $\check{K}\theta = \pi = \check{H}\theta$. Since the pair (\check{H}, \check{K}) is weakly compatible, it follows that $\check{K}\pi = \check{H}\pi$. Now, we claim that $\check{\Omega}\pi = \check{K}\pi$. Let, if possible, $\check{\Omega}\pi \neq \check{K}\pi$.

From (2.2), we have

$$\psi(d^o(\check{\Omega}\pi, \check{K}\pi)) \leq \psi(\delta(\pi, \pi)) - \phi(\delta(\pi, \pi)), \quad (2.31)$$

where

$$\delta(\pi, \pi) = \max\{d^o(\check{Z}\pi, \check{H}\pi), d^o(\check{Z}\pi, \check{\Omega}\pi), d^o(\check{H}\pi, \check{K}\pi),$$

$$\begin{aligned}
 & \frac{1}{2} [d^\circ(\check{Z}\pi, \check{K}\pi) + d^\circ(\check{\Omega}\pi, \check{H}\pi)], \\
 & d^\circ(\check{Z}\pi, \check{\Omega}\pi) \left[\frac{1 + d^\circ(\check{Z}\pi, \check{H}\pi)}{1 + d^\circ(\check{H}\pi, \check{K}\pi)} \right], \\
 & d^\circ(\check{H}\pi, \check{K}\pi) \left[\frac{1 + d^\circ(\check{Z}\pi, \check{H}\pi)}{1 + d^\circ(\check{Z}\pi, \check{\Omega}\pi)} \right], \\
 & \frac{d^{\circ 2}(\check{Z}\pi, \check{\Omega}\pi)}{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi)}, \quad \frac{d^{\circ 2}(\check{H}\pi, \check{K}\pi)}{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi)}, \\
 & d^\circ(\check{Z}\pi, \check{\Omega}\pi) \left[\frac{1 + d^\circ(\check{Z}\pi, \check{K}\pi) + d^\circ(\check{\Omega}\pi, \check{H}\pi)}{1 + d^\circ(\check{Z}\pi, \check{H}\pi) + d^\circ(\check{\Omega}\pi, \check{K}\pi)} \right], \\
 & d^\circ(\check{H}\pi, \check{K}\pi) \left[\frac{1 + d^\circ(\check{Z}\pi, \check{K}\pi) + d^\circ(\check{\Omega}\pi, \check{H}\pi)}{1 + d^\circ(\check{Z}\pi, \check{H}\pi) + d^\circ(\check{\Omega}\pi, \check{K}\pi)} \right] \Big\}. \\
 = & \max\{d^\circ(\check{\Omega}\pi, \check{K}\pi), d^\circ(\check{\Omega}\pi, \check{\Omega}\pi), d^\circ(\check{K}\pi, \check{K}\pi), \\
 & \frac{1}{2} [d^\circ(\check{\Omega}\pi, \check{K}\pi) + d^\circ(\check{\Omega}\pi, \check{K}\pi)], \\
 & d^\circ(\check{\Omega}\pi, \check{\Omega}\pi) \left[\frac{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi)}{1 + d^\circ(\check{K}\pi, \check{K}\pi)} \right], \\
 & d^\circ(\check{K}\pi, \check{K}\pi) \left[\frac{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi)}{1 + d^\circ(\check{\Omega}\pi, \check{\Omega}\pi)} \right], \\
 & \frac{d^{\circ 2}(\check{\Omega}\pi, \check{\Omega}\pi)}{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi)}, \quad \frac{d^{\circ 2}(\check{K}\pi, \check{K}\pi)}{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi)}, \\
 & d^\circ(\check{\Omega}\pi, \check{\Omega}\pi) \left[\frac{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi) + d^\circ(\check{\Omega}\pi, \check{K}\pi)}{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi) + d^\circ(\check{\Omega}\pi, \check{K}\pi)} \right], \\
 & d^\circ(\check{K}\pi, \check{K}\pi) \left[\frac{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi) + d^\circ(\check{\Omega}\pi, \check{K}\pi)}{1 + d^\circ(\check{\Omega}\pi, \check{K}\pi) + d^\circ(\check{\Omega}\pi, \check{K}\pi)} \right] \Big\}. \\
 = & \max\{d^\circ(\check{\Omega}\pi, \check{K}\pi), 0, 0, d^\circ(\check{\Omega}\pi, \check{K}\pi), 0, 0, 0, 0, 0, 0\}.
 \end{aligned}$$

From (2.31), we have

$$\begin{aligned}
 \psi(d^\circ(\check{\Omega}\pi, \check{K}\pi)) & \leq \psi(d^\circ(\check{\Omega}\pi, \check{K}\pi)) - \phi(d^\circ(\check{\Omega}\pi, \check{K}\pi)) \\
 & < \psi(d^\circ(\check{\Omega}\pi, \check{K}\pi)),
 \end{aligned}$$

which is a contradiction. Thus, $\check{\Omega}\pi = \check{K}\pi$, that is, $\check{Z}\pi = \check{\Omega}\pi = \check{K}\pi = \check{H}\pi$.

Now, we shall show that $\pi = \check{K}\pi$. Let, if possible, $\pi \neq \check{K}\pi$.

From (2.2), we have

$$\psi \left(d^o(\check{\Omega}\mu, \check{K}\pi) \right) \leq \psi(\delta(\mu, \pi)) - \phi(\delta(\mu, \pi)), \tag{2.32}$$

where

$$\delta(\mu, \pi) = \max\{d^o(\check{Z}\mu, \check{H}\pi), d^o(\check{Z}\mu, \check{\Omega}\mu), d^o(\check{H}\pi, \check{K}\pi),$$

$$\frac{1}{2} [d^o(\check{Z}\mu, \check{K}\pi) + d^o(\check{\Omega}\mu, \check{H}\pi)],$$

$$d^o(\check{Z}\mu, \check{\Omega}\mu) \left[\frac{1 + d^o(\check{Z}\mu, \check{H}\pi)}{1 + d^o(\check{H}\pi, \check{K}\pi)} \right],$$

$$d^o(\check{Z}\mu, \check{\Omega}\mu) \left[\frac{1 + d^o(\check{Z}\mu, \check{H}\pi)}{1 + d^o(\check{H}\pi, \check{K}\pi)} \right],$$

$$d^o(\check{H}\pi, \check{K}\pi) \left[\frac{1 + d^o(\check{Z}\mu, \check{H}\pi)}{1 + d^o(\check{Z}\mu, \check{\Omega}\mu)} \right],$$

$$\frac{d^{o^2}(\check{Z}\mu, \check{\Omega}\mu)}{1 + d^o(\check{\Omega}\mu, \check{K}\pi)}, \frac{d^{o^2}(\check{H}\omega_n, \check{K}\pi)}{1 + d^o(\check{\Omega}\mu, \check{K}\pi)},$$

$$d^o(\check{Z}\mu, \check{\Omega}\mu) \left[\frac{1 + d^o(\check{Z}\mu, \check{K}\pi) + d^o(\check{\Omega}\mu, \check{H}\pi)}{1 + d^o(\check{Z}\mu, \check{H}\pi) + d^o(\check{\Omega}\mu, \check{K}\pi)} \right],$$

$$d^o(\check{H}\pi, \check{K}\pi) \left[\frac{1 + d^o(\check{Z}\mu, \check{K}\pi) + d^o(\check{\Omega}\mu, \check{H}\pi)}{1 + d^o(\check{Z}\mu, \check{H}\pi) + d^o(\check{\Omega}\mu, \check{K}\pi)} \right] \Big\}.$$

$$= d^o(\pi, \check{K}\pi).$$

From (2.32), we have

$$\begin{aligned} \psi \left(d^o(\pi, \check{K}\pi) \right) &\leq \psi \left(d^o(\pi, \check{K}\pi) \right) - \phi \left(d^o(\pi, \check{K}\pi) \right) \\ &< \psi \left(d^o(\pi, \check{K}\pi) \right), \end{aligned}$$

which is a contradiction. Therefore, $\pi = \check{K}\pi = \check{H}\pi = \check{Z}\pi = \check{\Omega}\pi$.

Hence π is the common fixed point of \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} .

Now, we shall prove the uniqueness of common fixed point. If possible, let σ and ω be two common fixed points of \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} such that $\sigma \neq \omega$.

From (2.2), we have

$$\psi \left(d^o(\check{\Omega}\sigma, \check{K}\omega) \right) \leq \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)), \tag{2.33}$$

where

$$\begin{aligned}
 \delta(\sigma, \omega) &= \max\{d^o(\check{Z}\sigma, \check{H}\omega), d^o(\check{Z}\sigma, \check{\Omega}\sigma), d^o(\check{H}\omega, \check{K}\omega), \\
 &\quad \frac{1}{2}[d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)], \\
 &\quad d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{H}\omega, \check{K}\omega)} \right], \\
 &\quad d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{\Omega}\sigma)} \right], \\
 &\quad \frac{d^{o^2}(\check{Z}\sigma, \check{\Omega}\sigma)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, \frac{d^{o^2}(\check{H}\omega, \check{K}\omega)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, \\
 &\quad d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right], \\
 &\quad d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right]\}, \\
 &= \max\{d^o(\sigma, \omega), d^o(\sigma, \sigma), d^o(\omega, \omega), \\
 &\quad \frac{1}{2}[d^o(\sigma, \omega) + d^o(\sigma, \omega)], \\
 &\quad d^o(\sigma, \sigma) \left[\frac{1 + d^o(\sigma, \omega)}{1 + d^o(\omega, \omega)} \right], \\
 &\quad d^o(\omega, \omega) \left[\frac{1 + d^o(\sigma, \omega)}{1 + d^o(\sigma, \sigma)} \right], \\
 &\quad \frac{d^{o^2}(\sigma, \sigma)}{1 + d^o(\sigma, \omega)}, \frac{d^{o^2}(\omega, \omega)}{1 + d^o(\sigma, \omega)}, \\
 &\quad d^o(\sigma, \sigma) \left[\frac{1 + d^o(\sigma, \omega) + d^o(\sigma, \omega)}{1 + d^o(\sigma, \omega) + d^o(\sigma, \omega)} \right], \\
 &\quad d^o(\omega, \omega) \left[\frac{1 + d^o(\sigma, \omega) + d^o(\sigma, \omega)}{1 + d^o(\sigma, \omega) + d^o(\sigma, \omega)} \right]\}, \\
 &= \max\{d^o(\sigma, \omega), 0, 0, d^o(\sigma, \omega), 0, 0, 0, 0, 0, 0\}.
 \end{aligned}$$

From (2.33), we have

$$\begin{aligned}
 \psi(d^o(\sigma, \omega)) &\leq \psi(d^o(\sigma, \omega)) - \phi(d^o(\sigma, \omega)) \\
 &< \psi(d^o(\sigma, \omega)),
 \end{aligned}$$

which is a contradiction.

Therefore, $\sigma = \omega$ and this follows the uniqueness and completes the proof of the theorem.

Example 2.4. Let \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} be self - mappings on M . $M = [0, 1]$ be endowed with the Euclidean metric $d^o(\sigma, \omega) = |\sigma - \omega|$ for all σ, ω in M . Let \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} are defined by

$$\check{K}\sigma = \begin{cases} 0 & \sigma = 0 \\ \frac{\sigma}{4} & \sigma > 0, \end{cases}$$

$$\check{\Omega}\sigma = \begin{cases} 0 & \sigma = 0 \\ \frac{\sigma}{8} & \sigma > 0, \end{cases}$$

$$\check{H}\sigma = \begin{cases} 0 & \sigma = 0 \\ \frac{\sigma}{2} & \sigma > 0, \end{cases}$$

$$\check{Z}\sigma = \begin{cases} 0 & \sigma = 0 \\ \sigma & \sigma > 0 \end{cases}$$

$$\psi(t) = \frac{t}{2}, \quad \phi(t) = \frac{t}{4} \text{ for all } t \text{ in } \mathbb{R}.$$

Let $\{\mu_n\}$ be a sequence in M such that $\mu_n = \frac{1}{n+1}$ for each n .

Clearly, $\check{\Omega}M = [0, \frac{1}{8}] \subseteq [0, \frac{1}{2}] = \check{H}M$ and $\check{K}M [0, \frac{1}{4}] \subseteq [0, 1] = \check{Z}M$, implies that (2.1) satisfied.

Since $\check{Z}\check{\Omega}(0) = \check{\Omega}\check{Z}(0) = 0$, implies that the pair $(\check{Z}, \check{\Omega})$ is weakly compatible and

$\check{H}\check{K}(0) = \check{K}\check{H}(0) = 0$, implies that the pair (\check{H}, \check{K}) is weakly compatible.

Now, we check condition (2.2) for the following cases:

Case 1. If $\sigma = 0$ and $\omega = 0$.

$$\psi(d^o(\check{\Omega}\sigma, \check{K}\omega)) = \psi(|\check{\Omega}\sigma - \check{K}\omega|) = \psi(0) = 0.$$

Also

$$\psi(\delta(\sigma, \omega)) = \psi(0) = 0,$$

$$\phi(\delta(\sigma, \omega)) = \phi(0) = 0.$$

Hence

$$\psi(d^o(\check{\Omega}\sigma, \check{K}\omega)) = \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)).$$

Clearly, inequality (2.2) holds.

Case 2. If $\sigma = 0, \omega \neq 0$.

$$\psi(d^o(\check{\Omega}\sigma, \check{K}\omega)) = \psi(|\check{\Omega}\sigma - \check{K}\omega|) = \psi\left(\left|0 - \frac{\omega}{4}\right|\right) = \frac{\omega}{8}.$$

$$\begin{aligned}
 \delta(\sigma, \omega) &= \max\{d^o(\check{Z}\sigma, \check{H}\omega), d^o(\check{Z}\sigma, \check{\Omega}\sigma), d^o(\check{H}\omega, \check{K}\omega), \\
 &\quad \frac{1}{2}[d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)], \\
 &\quad d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{H}\omega, \check{K}\omega)} \right], \\
 &\quad d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{\Omega}\sigma)} \right], \\
 &\quad \frac{d^{o^2}(\check{Z}\sigma, \check{\Omega}\sigma)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, \frac{d^{o^2}(\check{H}\omega, \check{K}\omega)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, \\
 &\quad d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right], \\
 &\quad d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right]\}. \\
 &= \max\{|\check{Z}\sigma - \check{H}\omega|, |\check{Z}\sigma - \check{\Omega}\sigma|, \quad |\check{H}\omega - \check{K}\omega|, \\
 &\quad \frac{1}{2}[|\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|], \\
 &\quad |\check{Z}\sigma - \check{\Omega}\sigma| \left[\frac{1 + |\check{Z}\sigma - \check{H}\omega|}{1 + |\check{H}\omega - \check{K}\omega|} \right], \\
 &\quad |\check{H}\omega - \check{K}\omega| \left[\frac{1 + |\check{Z}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma, \check{\Omega}\sigma|} \right], \\
 &\quad \frac{|\check{Z}\sigma - \check{\Omega}\sigma|^2}{1 + |\check{\Omega}\sigma, -\check{K}\omega|}, \frac{|\check{H}\omega - \check{K}\omega|^2}{1 + |\check{\Omega}\sigma - \check{K}\omega|}, \\
 &\quad |\check{Z}\sigma - \check{\Omega}\sigma| \left[\frac{1 + |\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma - \check{H}\omega| + |\check{\Omega}\sigma - \check{K}\omega|} \right], \\
 &\quad |\check{H}\omega - \check{K}\omega| \left[\frac{1 + |\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma - \check{H}\omega| + |\check{\Omega}\sigma - \check{K}\omega|} \right]\}. \\
 &= \max\left\{ \frac{\omega}{2}, 0, \frac{\omega}{4}, \frac{3\omega}{8}, 0, \frac{\omega}{8}(2 + \omega), 0, \frac{\omega^2}{4(4 + \omega)}, 0, \frac{\omega}{4} \left[\frac{1 + \frac{\omega}{4} + \frac{\omega}{2}}{1 + \frac{\omega}{2} + \frac{\omega}{4}} \right] \right\}.
 \end{aligned}$$

One can easily check that

$$\delta(\sigma, \omega) = \frac{\omega}{2}.$$

Now,

$$\psi\left(d^0(\check{\Omega}\sigma, \check{K}\omega)\right) = \psi(|\check{\Omega}\sigma - \check{K}\omega|) = \psi\left(\frac{\omega}{4}\right) = \frac{\omega}{8}.$$

Also

$$\psi(\delta(\sigma, \omega)) = \psi\left(\frac{\omega}{2}\right) = \frac{\omega}{4},$$

$$\phi(\delta(\sigma, \omega)) = \phi\left(\frac{\omega}{2}\right) = \frac{\omega}{8}.$$

Hence

$$\psi\left(d^0(\check{\Omega}\sigma, \check{K}\omega)\right) = \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)).$$

Clearly inequality (2.2) holds.

Case 3. If $\sigma \neq 0, \omega = 0$.

$$\psi\left(d^0(\check{\Omega}\sigma, \check{K}\omega)\right) = \psi(|\check{\Omega}\sigma - \check{K}\omega|) = \psi\left(\left|\frac{\sigma}{8} - 0\right|\right) = \frac{\sigma}{16}.$$

$$\delta(\sigma, \omega) = \max\{d^0(\check{Z}\sigma, \check{H}\omega), d^0(\check{Z}\sigma, \check{\Omega}\sigma), d^0(\check{H}\omega, \check{K}\omega),$$

$$\frac{1}{2}[d^0(\check{Z}\sigma, \check{K}\omega) + d^0(\check{\Omega}\sigma, \check{H}\omega)],$$

$$d^0(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^0(\check{Z}\sigma, \check{H}\omega)}{1 + d^0(\check{H}\omega, \check{K}\omega)} \right],$$

$$d^0(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^0(\check{Z}\sigma, \check{H}\omega)}{1 + d^0(\check{Z}\sigma, \check{\Omega}\sigma)} \right],$$

$$\frac{d^{o^2}(\check{Z}\sigma, \check{\Omega}\sigma)}{1 + d^0(\check{\Omega}\sigma, \check{K}\omega)}, \frac{d^{o^2}(\check{H}\omega, \check{K}\omega)}{1 + d^0(\check{\Omega}\sigma, \check{K}\omega)},$$

$$d^0(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^0(\check{Z}\sigma, \check{K}\omega) + d^0(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^0(\check{Z}\sigma, \check{H}\omega) + d^0(\check{\Omega}\sigma, \check{K}\omega)} \right],$$

$$d^0(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^0(\check{Z}\sigma, \check{K}\omega) + d^0(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^0(\check{Z}\sigma, \check{H}\omega) + d^0(\check{\Omega}\sigma, \check{K}\omega)} \right] \}.$$

$$= \max\{|\check{Z}\sigma - \check{H}\omega|, |\check{Z}\sigma - \check{\Omega}\sigma|, |\check{H}\omega - \check{K}\omega|, \frac{1}{2}[|\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|],$$

$$|\check{Z}\sigma - \check{\Omega}\sigma| \left[\frac{1 + |\check{Z}\sigma - \check{H}\omega|}{1 + |\check{H}\omega - \check{K}\omega|} \right],$$

$$|\check{H}\omega - \check{K}\omega| \left[\frac{1 + |\check{Z}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma, \check{\Omega}\sigma|} \right],$$

$$\frac{|\check{Z}\sigma - \check{\Omega}\sigma|^2}{1 + |\check{\Omega}\sigma, -\check{K}\omega|}, \frac{|\check{H}\omega - \check{K}\omega|^2}{1 + |\check{\Omega}\sigma - \check{K}\omega|}.$$

$$\begin{aligned}
 & |\check{Z}\sigma - \check{\Omega}\sigma| \left[\frac{1 + |\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma - \check{H}\omega| + |\check{\Omega}\sigma - \check{K}\omega|} \right], \\
 & |\check{H}\omega - \check{K}\omega| \left[\frac{1 + |\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma - \check{H}\omega| + |\check{\Omega}\sigma - \check{K}\omega|} \right] \Big\}. \\
 & = \max \left\{ \sigma, \frac{7\sigma}{8}, 0, \frac{9\sigma}{16}, \frac{7\sigma}{8}(1 + \sigma), 0, \frac{49\sigma^2}{8(8 + \sigma)}, 0, \frac{7\sigma}{8}, 0 \right\}.
 \end{aligned}$$

One can easily check that

$$\delta(\sigma, \omega) = \begin{cases} \sigma, & 0 < \sigma \leq \frac{1}{7} \\ \frac{7\sigma}{8}(1 + \sigma), & \frac{1}{7} \leq \sigma < 1 \end{cases}$$

Now, if $0 < \sigma \leq \frac{1}{7}$,

$$\psi(d^0(\check{\Omega}\sigma, \check{K}\omega)) = \frac{\sigma}{16}.$$

Also

$$\psi(\delta(\sigma, \omega)) = \psi(\sigma) = \frac{\sigma}{2},$$

$$\phi(\delta(\sigma, \omega)) = \phi(\sigma) = \frac{\sigma}{4}.$$

Clearly

$$\psi(d^0(\check{\Omega}\sigma, \check{K}\omega)) < \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)).$$

Hence inequality (2.2) holds.

If $\frac{1}{7} \leq \sigma < 1$,

$$\psi(d^0(\check{\Omega}\sigma, \check{K}\omega)) = \frac{\sigma}{16}.$$

$$\psi(\delta(\sigma, \omega)) = \psi\left(\frac{7\sigma}{8}(1 + \sigma)\right) = \frac{7\sigma}{16}(1 + \sigma),$$

$$\phi(\delta(\sigma, \omega)) = \phi\left(\frac{7\sigma}{8}(1 + \sigma)\right) = \frac{7\sigma}{32}(1 + \sigma).$$

$$\psi(d^0(\check{\Omega}\sigma, \check{K}\omega)) < \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)).$$

Hence inequality (2.2) holds.

Case 4: If $\sigma \neq 0, \omega \neq 0$.

$$\psi(d^0(\check{\Omega}\sigma, \check{K}\omega)) = \psi(|\check{\Omega}\sigma - \check{K}\omega|) = \psi\left(\left|\frac{\sigma}{8} - \frac{\omega}{4}\right|\right) = \frac{\sigma - 2\omega}{16}.$$

$$\begin{aligned}
 \delta(\sigma, \omega) &= \max\{d^o(\check{Z}\sigma, \check{H}\omega), d^o(\check{Z}\sigma, \check{\Omega}\sigma), d^o(\check{H}\omega, \check{K}\omega), \\
 &\quad \frac{1}{2}[d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)], \\
 &\quad d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{H}\omega, \check{K}\omega)} \right], \\
 &\quad d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{\Omega}\sigma)} \right], \\
 &\quad \frac{d^{o^2}(\check{Z}\sigma, \check{\Omega}\sigma)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, \frac{d^{o^2}(\check{H}\omega, \check{K}\omega)}{1 + d^o(\check{\Omega}\sigma, \check{K}\omega)}, \\
 &\quad d^o(\check{Z}\sigma, \check{\Omega}\sigma) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right], \\
 &\quad d^o(\check{H}\omega, \check{K}\omega) \left[\frac{1 + d^o(\check{Z}\sigma, \check{K}\omega) + d^o(\check{\Omega}\sigma, \check{H}\omega)}{1 + d^o(\check{Z}\sigma, \check{H}\omega) + d^o(\check{\Omega}\sigma, \check{K}\omega)} \right]\}. \\
 &= \max\{|\check{Z}\sigma - \check{H}\omega|, |\check{Z}\sigma - \check{\Omega}\sigma|, |\check{H}\omega - \check{K}\omega|, \\
 &\quad \frac{1}{2}[|\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|], \\
 &\quad |\check{Z}\sigma - \check{\Omega}\sigma| \left[\frac{1 + |\check{Z}\sigma - \check{H}\omega|}{1 + |\check{H}\omega - \check{K}\omega|} \right], \\
 &\quad |\check{H}\omega - \check{K}\omega| \left[\frac{1 + |\check{Z}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma, \check{\Omega}\sigma|} \right], \\
 &\quad \frac{|\check{Z}\sigma - \check{\Omega}\sigma|^2}{1 + |\check{\Omega}\sigma - \check{K}\omega|}, \frac{|\check{H}\omega - \check{K}\omega|^2}{1 + |\check{\Omega}\sigma - \check{K}\omega|}, \\
 &\quad |\check{Z}\sigma - \check{\Omega}\sigma| \left[\frac{1 + |\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma - \check{H}\omega| + |\check{\Omega}\sigma - \check{K}\omega|} \right], \\
 &\quad |\check{H}\omega - \check{K}\omega| \left[\frac{1 + |\check{Z}\sigma - \check{K}\omega| + |\check{\Omega}\sigma - \check{H}\omega|}{1 + |\check{Z}\sigma - \check{H}\omega| + |\check{\Omega}\sigma - \check{K}\omega|} \right]\}. \\
 &= \max \left\{ \sigma - \frac{\omega}{2}, \frac{\sigma}{2}, \frac{\omega}{4}, \frac{1}{2} \left[\frac{9\sigma - 6\omega}{8} \right], \frac{7\sigma}{8} \left(\frac{1 + \sigma - \frac{\omega}{2}}{1 + \frac{\omega}{2} - \frac{\omega}{4}} \right), \right. \\
 &\quad \left. \frac{\omega}{4} \left(\frac{1 + \sigma - \frac{\omega}{2}}{1 + \sigma - \frac{\sigma}{8}} \right), \frac{49\sigma^2}{8(8 + \sigma - 2\omega)}, \frac{\omega^2}{2(8 + \sigma - 2\omega)}, \frac{7\sigma}{8}, \frac{\omega}{4} \right\}.
 \end{aligned}$$

one can easily check that

$$\psi \left(d^0(\check{\Omega}\sigma, \check{K}\omega) \right) \leq \psi(\delta(\sigma, \omega)) - \phi(\delta(\sigma, \omega)).$$

for all the values of $\sigma, \omega \in [0, 1]$.

Hence inequality (2.2) holds for all the cases.

Also, $\lim_{n \rightarrow \infty} \check{Z}\mu_n = \lim_{n \rightarrow \infty} \check{\Omega}\mu_n = 0$, where $0 \in M$, implies that the pair $(\check{Z}, \check{\Omega})$ satisfies the E.A property.

Similarly, we can easily check that the pair (\check{H}, \check{K}) satisfies the E.A property.

Now, $\lim_{n \rightarrow \infty} \check{Z}\mu_n = \lim_{n \rightarrow \infty} \check{\Omega}\mu_n = 0 = \check{Z}(0)$, where $0 \in M$, implies that the pair $(\check{Z}, \check{\Omega})$ satisfies the $(CLR_{\check{Z}})$ property.

Similarly, we can easily check that the pair (\check{H}, \check{K}) satisfies the $(CLR_{\check{H}})$ property.

Therefore, all the conditions of the Theorems 2.1, 2.2 and 2.3 are satisfied.

Hence \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} have unique common fixed point. Clearly 1 is the unique common fixed point of \check{Z} , \check{H} , $\check{\Omega}$ and \check{K} .

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