Original Research Article
Spatially Closed FRW Model leads to Einstein's Universe with Cosmological Constant

Abstract: Einstein revisited his equations and altered by introducing something known as a universal constant Einstein's 1917 concept has been with us ever since in various variants and incarnations, including the broader concept of 'Dark Energy. Einstein's field equation has been studying under the assumption of a power law time variation of the expansion factor. The Hubble's parameter and distance modulus in our descended model are in good concordance with recent data of astrophysical observations under appropriate conditions. Theoretically, the cosmological constant Λ and the density parameter ρ are determined using dynamical tests like as density and velocity profiles around clusters and virialization. This paper, we discuss and briefly other classes of models that have a close relationship with the Freidmann models, and these are models derived from modified Einstein equations containing the cosmological constant. Thus the cosmological constant is not any old value but rather simply the inverse of the scale factor squared, where the scale factor is a fixed value in static closed universe. Ultimately we derived to describe the final radius of a virialized cluster, in which a repulsive cosmological constant lambda (Λ) gives a smaller value. Based on the results, two scenarios for the universe are proposed, one with a huge proportion of nonbaryonic matter and a zero cosmological constant, and the other with all matter being baryonic. The cosmological constant is added to save inflation and build up a static universe model.

Keywords: Friedmann Model, Cosmological Constant, Einstein's Universe, Ricci tensor, Energy momentum tensor, Hubble's Parameter.

1. Introduction

In Einstein's equation an additional term that allowed him to avoid predictions and once again bask in the comfort of a static universe. However, 12 years later, through detail measurements of distance galaxies, the American astronomer Edwin Powel Hubble observationally established that the universe is expanding. In a now famous story in the annals of science, Einstein then returned to the original form of his equations, citing his temporary modification of them as the biggest blunder of his life. His initial unwillingness to accept the conclusion not with standing. Einstein's theory predicated the expansion of the universe. In fact in the early 1920's, before Hubble's measurements the Russian meteorologist Alexander Friedmann had used Einstein's original equation to show, in some details, that all galaxies would be carried along on the substrate of stretching spatial fabric, thereby spatially moving away from all others. Hubble's observations and numerous subsequent ones have thoroughly verified this astonishing conclusion of general relativity. By offering the explanation for the expansion of the universe, Einstein [2, 3, 12] achieved one of the greatest intellect feats of all time. If the fabric of space is stretching, thereby increasing the distance between galaxies that are carried along on the cosmic flow, we can imagine running the evolution backward in time to learn about the origin of the universe. In reverse, the fabric of space shrinks bringing all galaxies closer and closer to each other. Like the contents of a pressure cooker, as the shrinking universe compress the galaxies together the temperature dramatically increase, stars disintegrate and a hot plasma of matter elementary constituents in formed. As the fabric continues to shrink, the temperature rise unabated as does the density of the primordial plasma. As we imagine running the clock backward from the age of the presently observed universe about 15 billion years, the universe as we know, it as crushed to an ever smaller size. And the clock is turned back to ever earlier times; the whole of the cosmos is compressed to the size of an orange, a lemon, a pea, a grain of sand and to yet tinnier size still. Extrapolating all the way back to the beginning of the universe would appear to have begun as a point an image we will critically reexamine in which all matter and energy is squeezed together to unimaginable density and temperature. It is believed that a cosmic fireball the BIG BANG erupted from this volatile mixture spewing forth the seeds from which the universe as we know it evolved.

2. Preliminaries

The standard Friedmann Robertson Walker (FRW) [1, 7, 8] line element will be assumed. For a closed universe with the denoted expansion parameter by R , which is not the function of time‘t’,

$$ds^2 = c^2dt^2 - R^2(t)\left[1+\frac{1}{4kr^2}\right]^{-1}\delta_{ij}dx^i dx^j \qquad (1)$$

with $i, j = 1,2,3$ and $r^2 = \delta_{ij}x^i x^j$. There will eventually be a focus on a static solution.

It will be easier to work with the field equations for the time-dependent case of an incoherent dust source plus a positive cosmological term and then examine the static solution as a function of the total mass of the dust. The general equations for this case are as follows: $R_\nu^\mu - \frac{1}{2}\delta_\nu^\mu R + \frac{\Lambda}{c^2}g_{\mu\nu} = -kT_\nu^\mu(dust) - \Lambda\delta_\nu^\mu$ (2)

with $\Lambda \rangle 0$ and in the co-moving frame one has $T_\nu^\mu(dust) = \rho diag\ (1,0,0,0)$ (3)

Where is the dust's mass density? Because the universe is assumed to be closed, we have set, $x^1 = x, \quad x^2 = y, \quad x^3 = z$

so that $r^2 = x^2 + y^2 + z^2,$ with $x = r'\sin\theta\cos\varphi, y = r'\sin\theta\sin\varphi, z = r'\cos\theta$, then the equation (1) reduces to the following

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\varphi^2)}{(1+\frac{kr'^2}{4})^2}\right] \qquad (4)$$

The transformation $r = r' \big/ (1+\frac{kr'^2}{4})$ yields the standard form of the Robertson- Walker metric as follows

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 d\varphi^2)\right] \qquad (5)$$

In equation (5), the Constant can take the values -1, 0, or +1, resulting in three different geometrical configurations for an open, flat, or closed universe is the time-dependent scale factor determines the length scale and grows as the universe expands.

In his cosmological problem, Einstein assumed homogeneity and isotropy. He also assumed that space time is static. This enabled him to select a time coordinate such that the space time line element could be described by

$$ds^2 = c^2dt^2 - \alpha_{\mu\nu}dx^{\mu}dx^{\nu} \qquad (\mu,\nu = 1,2,3) \quad (6)$$

where $\alpha_{\mu\nu}$ are the functions of space coordinates only. The homogeneity constraint implies that the coefficient can only be a constant, which we have normalized to. Similarly, the isotropy condition [4, 5] states that there should be no form terms in the line element. This is easily demonstrated in the following way. If we have the period in the line element, then spatial displacements would contribute in opposite directions over a short time interval, and such directional variation would be observable and inconsistent with isotropy. Einstein believed that the universe has so much matter as to close the space. This assumption led him to a specific form for $\alpha_{\mu\nu}$.

We can now see how to build the homogeneous and isotropic three-dimensional closed space that Einstein desired for his universe model. It is S_3, the three surface of a four dimensional hyper surface of radius R. The equation of such a 3- surface is given by,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$$

To use co-ordinates intrinsic to the surface, we define

$$x_1 = R\sin\chi\cos\theta, x_2 = R\sin\chi\sin\theta\cos\varphi, x_3 = R\sin\chi\sin\theta\sin\varphi, x_4 = R\cos\chi$$

As a result, the spatial line element on the surface is given by

$$d\sigma^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2$$

$$= R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin\theta d\varphi^2)] \qquad (7)$$

$$0 \le \chi \le \pi \quad, 0 \le \theta \le \pi \quad, 0 \le \varphi \le 2\pi$$

$$V = \int_{\chi=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}(Rd\chi)(R\sin\chi d\theta)(R\sin\chi\sin\theta d\varphi)$$

$$= 2\pi^2R^3$$

$$d\sigma^2 = R^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 d\varphi^2)\right] \qquad (8)$$

The radius of the universe is the name given to this constant. As a result, the Einstein universe's line element is provided by [2, 3, 12].

$$ds^2 = c^2dt^2 - R^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2 d\varphi^2)\right] \quad (9)$$

3. Friedmann equation and the Einstein equations.

The Robertson-Walker metric [14, 15, 16] is

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 d\varphi^2)\right] \quad (9a)$$

The non- zero components of the Robertson-Walker metric is as follows

$$g_{00}=c^2, g_{11}=-\frac{R^2}{1-kr^2}, g_{22}=-r^2R^2, g_{33}=-R^2r^2\sin^2\theta$$

with $g^{ii}=(g_{ii})^{-1}$ for $i=1,2,3$

We know that

$$\Gamma^s_{pq}=\frac{1}{2}g^{rs}\left(\frac{\partial g_{qr}}{\partial x^p}+\frac{\partial g_{pr}}{\partial x^q}-\frac{\partial g_{pq}}{\partial x^r}\right)$$

Now calculate the non-zero Christoffel symbols are

$$\Gamma_{rr}{}^{0}=\frac{R\dot{R}}{1-kr^2},$$

$$\Gamma^0_{\theta\theta}=r^2R\dot{R}, \quad \Gamma^0_{\varphi\varphi}=r^2R\dot{R}\sin^2\theta$$

$$\Gamma^r_{0r}=\Gamma^\theta_{0\theta}=\Gamma^\varphi_{0\varphi}=\frac{\dot{R}}{R}, \Gamma^r_{\theta\theta}=-r\left(1-kr^2\right)$$

$$\Gamma^r_{rr}=\frac{r}{1-kr^2}, \Gamma^\theta_{r\theta}=\Gamma^\varphi_{r\theta}=\frac{1}{r}, \Gamma^r_{\varphi\varphi}=-r\left(1-kr^2\right)\sin^2\theta$$

$$\Gamma^\theta_{\varphi\varphi}=-\sin\theta\cos\theta, \Gamma^\varphi_{\theta\varphi}=\cot\theta \qquad (10)$$

Now we get the non- zero components [6, 9] of the Ricci tensor $R_{\mu\nu}$

$$R_{00}=\frac{3\ddot{R}}{c^2R}, R_{ii}=\frac{(R\ddot{R}+2\dot{R}^2+2kc^2)}{c^2R}, R=\frac{6(R\ddot{R}+\dot{R}^2+kc^2)}{c^2R^2} \qquad (11)$$

In any other Lorentz frames we then find the energy momentum tensor [19, 20] as follows

$$T_{\mu\nu}=(p+c^2\rho)u_\mu u_\nu-pg_{\mu\nu} \qquad (12)$$

$$T_{00}=\rho c^2, T_{ii}=pg_{ii} \qquad (13)$$

The Einstein field equation with cosmological constant is as follows,

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{\Lambda g_{\mu\nu}}{c^2}=-\frac{8\pi G_N}{c^4}T_{\mu\nu} \qquad (14)$$

Newtonian gravitational constant, $G_N=6.673\times10^{-11}m^3kg^{-1}\sec^{-2}$.The 00 and 11 components of equation (14) gives the equations

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_N \rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3} \qquad (15)$$

$$\left(\frac{\ddot{R}}{R}\right) = -\frac{4\pi G_N}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3} \qquad (15a)$$

Matter, which exerts no pressure (p=0) at present, determines the universe's dynamics [21, 22, 23] as we will see below, Λ is very small (probably $\Lambda = 0$) .Then

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_N \rho}{3} - \frac{kc^2}{R^2} \qquad (16)$$

$$\left(\frac{\ddot{R}}{R}\right) = -\frac{4\pi G_N \rho}{3} \qquad (17)$$

H being the Hubble's constant. Since the total amount of matter in a given commoving volume is (presumably) fixed, ρR^3 is a constant, and is dependent on time t.

$$\dot{R}^2 = \frac{8\pi G_N}{3}\left(\rho R^3\right)\frac{1}{R} - kc^2 \qquad (18)$$

Differentiate w.r.to.'t '

$$2\dot{R}\ddot{R} = -\frac{8\pi G_N}{3}\left(\rho R^3\right)\frac{\dot{R}}{R^2} \qquad (19)$$

which is identical to $\left(\frac{\ddot{R}}{R}\right) = -\frac{4\pi G_N \rho}{3}$ As a result, these two equations are equivalent. They are easily understood from non-relativistic dynamics [4, 5].In Newtonian gravity, a particle of mass 'm' is gravitationally attracted by a sphere of matter of density ρ and radius R experience a force

$$\left(-\frac{4\pi R^3 \rho}{3R^2}\right) m G_N = m\ddot{R} \qquad (20)$$

which yields $\left(\frac{\ddot{R}}{R}\right) = -\frac{4\pi G_N \rho}{3}$, while energy conservation necessitates

$$\frac{1}{2} m\dot{R}^2 - \left(\frac{4\pi R^3 \rho}{3R}\right) m G_N = \text{Constant} = -\frac{1}{2} mc^2 k \text{ (say)} \qquad (21)$$

k gives the sign of the total energy and geometry of the curvature. However, general relativity is still required to justify the neglect of matter beyond the sphere and provide meaning of k. Since $\rho \sim R^{-3}$, if k= -1, the solution of

$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_N \rho}{3} - \frac{kc^2}{R^2}$, for large R is determined by the curvature and we find $R \sim ct$. So ultimately the universe will expand with the speed of light. If k=0 then $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_N \rho}{3}$.When k=1 then $\dot{R}$ vanishes for some finite $R_{\max} = \frac{4G_N M}{3\pi c^2}$, $M = 2\pi^2 R^3 \rho$ is the total mass of the universe.

4. New result for static closed universe.

From equation (21)

$$\frac{1}{2} m\dot{R}^2 - \left(\frac{4\pi R^3 \rho}{3R}\right) mG_N = -\frac{1}{2} mc^2 k$$

For $\dot{R} = 0$

$$\left(\frac{4\pi R^3 \rho}{3R}\right) mG_N = \frac{1}{2} mc^2 k$$

$$\Rightarrow R^2 = \frac{3c^2 k}{8\pi \rho G_N} \qquad ; \text{For } k = 1 \text{ and } 3c^2 \approx 1$$

$$\therefore R = \frac{1}{2\sqrt{2\pi G_N \rho}}$$

Equation (15a) yields $\Lambda = 4\pi G_N \rho$ for $\ddot{R} = 0$ and $p = 0$

Now the equation (15) for the closed ($k = 1, c = 1$) universe again be written as

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_N}{3}\left(\frac{M}{2\pi^2 R^3}\right) - \frac{1}{R^2} + \frac{\Lambda}{3} \qquad as\, M = 2\pi^2 R^3 \rho$$

$$\Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{4G_N M}{3\pi R^3} - \frac{1}{R^2} + \frac{\Lambda}{3}$$

$$\Rightarrow \frac{\dot{R}^2}{2} = \frac{2G_N M R^2}{3\pi R^3} - \frac{1}{2} + \frac{\Lambda R^2}{6}$$

$$\Rightarrow \frac{\dot{R}^2}{2} - \frac{2G_N M}{3\pi R} - \frac{\Lambda R^2}{6} = -\frac{1}{2} \qquad (22)$$

The equation (22) can be written as with the help of (21)

$$-\frac{2G_N M}{3\pi R}=-\frac{1}{2}+\frac{G_N M}{3\pi R} \quad ; \text{For } \dot{R}=0 \text{ and } \Lambda=4\pi G_N\rho$$

$$\Rightarrow \frac{3G_N M}{3\pi R}=\frac{1}{2}$$

$$\Rightarrow R=\frac{2G_N M}{\pi}$$

$$\Rightarrow R^2=\frac{1}{4\pi G_N\rho} \quad \text{where } M=2\pi^2R^3\rho \text{ and } \rho=\frac{1}{4\pi G_N R^2} \qquad (23)$$

$$\therefore R=\frac{1}{2\sqrt{\pi G_N\rho}} \quad \text{For static reason}$$

If $\dot{R}=0$ and $p=0$ then the equation (15) becomes $\Lambda=4\pi G_N\rho$

$$\text{Therefore, } R=\sqrt{\frac{1}{\Lambda}}=\frac{1}{2\sqrt{\pi G_N\rho}} \qquad (24)$$

5. Discussion

This is our new result in equation no. (24) which supports the static like universe. Einstein [2, 3, 12] considered this solution to be proof of his hypothesis that with sufficiently high density it should be possible to be the closed universe. In the above result we have the radius of the universe R given by the density of matter ρ, with the result that the larger value of ρ the smaller value of R. However if Λ is a given universal constant like G, both ρ and Rare determined in terms of Λ. In 1917 there was very little information available about ρ, from which Λ could be determined. The value of $R\approx10^{26}-10^{28}cm$, as a result, anything said back then is only of historical interest. We take ρ as $10^{-31}gmcm^{-3}$ as we get a rough estimate of mass density in the form of galaxies $R\cong10^{28}-10^{29}cm$ and $\Lambda\cong10^{-58}cm^{-2}$. The effect of a Λ term on the perihelion shift of mercury is considered by Islam [6, 9, 19] as $|\Lambda|\langle10^{-42}cm^{-2}$. The term Λ introduces a repulsion force between two bodies that grow in proportion to their distance apart. The above value is too small to make a discernible difference from the standard general relativity prediction [10, 11] (that is $\Lambda=0$) in any of the solar system lists. As a result, the local gravity tests posed no threat to the Einstein universe. However, the model lasted only a decade, for reason of the observable prediction and supports of expanding universe.

Since $\rho\rangle0$, we must have k=1, and therefore $\Lambda\rangle0$. The value of Λ which makes the universe stable is just

$\Lambda_E = 4\pi G\rho$. The model we have just described is called the Einstein universe. This universe is static [2, 3, 12] (but unfortunately unstable, as one can show), with positive curvature and curvature radius $R_E = \Lambda_E^{-1/2} = 1/(4\pi G\rho)^{1/2}$.

6. Conclusion

The cosmological constant is the inverse of the scale factor squared, which is a fixed value in a closed universe. It was introduced by Albert Einstein to describe the virialized cluster's final radius. Two scenarios for the universe are suggested based on the results - one with a large amount of nonbaryonic matter and a zero cosmology constant and another where all matter is baryonic. In equation no (24) we take as get a rough estimate of mass density in the form of galaxies. The above value is too small to make a discernible difference from the standard general relativity prediction in any of the solar system lists. Since $\rho > 0$, we must have E, and therefore $\Lambda > 0$. The value of Λ which makes the universe stable is just $\Lambda_E = \frac{4\pi G\rho}{c^2}$ .The model we have just described is called the Einstein universe. This universe is static (but unfortunately unstable, as one can show), with positive curvature and curvature radius $R_E = \Lambda_E^{-1/2} = c/(4\pi G\rho)^{1/2}$. After the discovery of the expansion of the universe is the late 1920's there has no longer any reason to seek, static solution to the field equations. The motivation which has led Einstein to introduce his cosmological constant term before subsided. Einstein subsequently regarded the Λ term as the biggest mistake he had made in his life. Since then, however Λ has not died but has been the subject to much interest and series study on both conceptual and observational grubs.

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Appendix

$$T_\nu^\mu(dust, p=o) = \rho diag\,(1,0,0,0)$$

$$T_\nu^\mu(p=o) = \rho diag\,(1,0,0,0)$$

$$T_\nu^\mu(\Lambda) = \frac{1}{4\pi GR^2} diag\,(1,0,0,0)$$

$$T_\nu^\mu(\Lambda) = (4\pi G)^{-1}\Lambda\delta_\nu^\mu$$

$$T_\nu^\mu(\Lambda) = \left(\frac{\pi}{32M^2R^3}\right)\delta_\nu^\mu \qquad (25)$$

We can define new transformation of coordinates $\bar{x}^i = \delta_j^i R x^j$ (26)

Since R is a constant, the coordinate differentials satisfying $d\bar{x}^i = \delta^i_j R dx^j$ as a result of which the line element (1) takes the form,

$$ds^2 = dt^2 - \left[1 + \frac{r^2}{4}\right]^{-2} \delta_{ij} dx^i dx^j \qquad (27)$$

$$d\bar{x}^i = \delta^i_j R dx^j \qquad r^2 = \delta_{ij} x^i x^j$$

$$d\bar{x}^j = \delta^j_i R dx^i \qquad \bar{r}^2 = \delta_{ij} \bar{x}^i \bar{x}^j = \delta_{ij} R^2 x^i$$

$$ds^2 = dt^2 - \left[1 + \frac{\pi^2 \bar{r}^2}{16 R^3 \pi^2}\right]^{-2} \delta_{ij} d\bar{x}^i d\bar{x}^j \qquad (28)$$

Also, under the transformation in equation (26) $T^{\mu}_{\nu}(dust)$ is left unchanged and to be sure $T^{\mu}_{\nu}(\Lambda)$ is invariant under all coordinate transformation. Thus the transformation given by the equation (26) does not affect the values given for the source tensors in equation (25) and (26). It follows, therefore that in the limiting case $M \to \infty$, $T^{\mu}_{\nu}(dust) \to 0$ and $T^{\mu}_{\nu}(\Lambda) \to 0$, while $g_{\mu\nu} = \eta_{\mu\nu}$. Thus as indicated above, Minkowski space time can be viewed as a limiting case of a static, spherical universe [13, 17, 18] in which the background mass is infinite, but the mass density and cosmological term are finite valueless.