

Bayesian Estimation of Exponentiated Rayleigh Distribution under Symmetric and Asymmetric loss functions

Abstract

Exponentiated Rayleigh distribution for complete samples is the subject of the paper's Bayesian analysis. The form parameter of the distribution has been expressed by the Bayes estimators under a variety of priors and loss functions. For the purposes of posterior analysis, the gamma and Jeffrey priors have been assumed. Under six different loss functions, the Bayes estimation was obtained "Squared error, Linear exponential, Precautionary, Entropy, De Groot and non-Linear exponential loss functions". Finding a good estimator of the unidentified shape parameter is the study's main goal. The simulation study was done to examine the Mean Square Error (MSE) for the effectiveness of different estimators.

Keywords: Bayesian Estimation, MLE, Bayes estimate, Exponentiated Rayleigh Distribution, Loss function, Prior, Posterior,

1. INTRODUCTION

For modelling data, Burr (1942) introduced twelve different forms of cumulative distribution function. Among them Burr Type X and Burr Type XII received the maximum attention, where Surles and Padgett (2001) observed that the Burr Type X distribution (Exponentiated Rayleigh distribution) can be used quite effectively in modeling strength data and also modeling general lifetime data. Several aspects of the ER distribution were studied in literature, see for example Sartawi and Abu-Salih (1991), Jaheen (1995), Ahmed, Fakhry and Jaheen (1997), Karam and Jbur (2014), Feroze and Aslam (2012) and Sindhu and Aslam (2014). The cumulative distribution function (CDF) and the probability density function (pdf) of the ER distribution with shape parameter ($\theta > 0$) are respectively as follows: (1994)

$$F(x, \theta) = (1 - e^{-x^{-2}})^{\theta} ; x > 0; \theta > 0 \quad (1)$$

$$f(x; \theta) = 2\theta x e^{x^2} (1 - e^{-x^2})^{(\theta-1)} ; x > 0; \theta > 0 \quad (2)$$

The random number has been generated by inverse transformation method, which is for uniform random U:

$$X_i = \left(-\ln(1 - U_i^{1/\theta})\right)^{0.5} ; \quad i = 1, 2, \dots, n \quad (3)$$

Studying a particular phenomenon via the challenge of estimating the unknown parameters in statistical distributions is one of the significant problems that individuals who are interested in applied statistics face. This study examines the estimations of the ER distribution's unknown shape parameter. The ER distribution is a significant distribution in operations research and statistics, and it is used in a number of fields including biology, agriculture, and other sciences. Considering the Bayesian analysis of the unknown parameters with various priors (informative and non-informative) and loss functions for complete samples is the main goal of this.

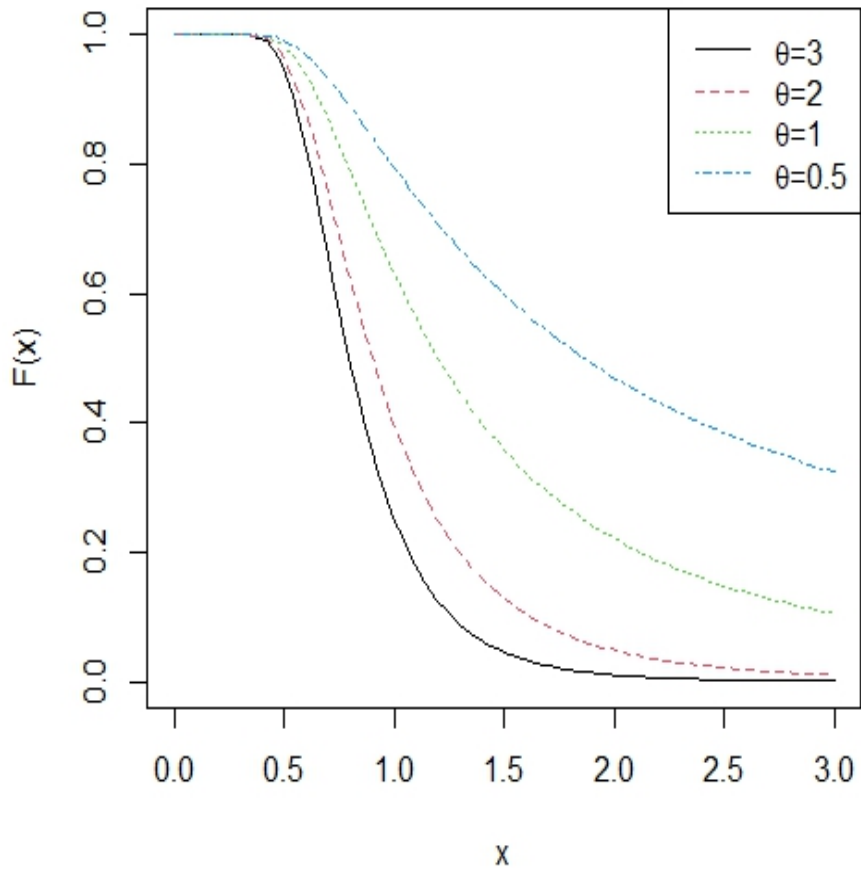


Figure 1. plot of cumulative density function at different value of parameter

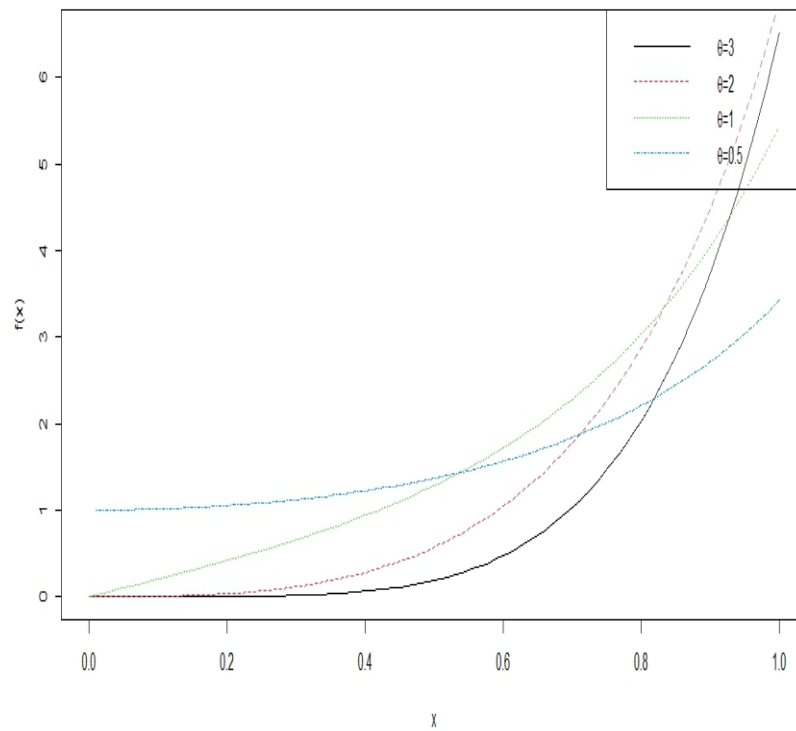


Figure 2. plot of probability density function at different value of parameter

2. LIKLEHOOD FUNCTION

Let X_1, X_2, \dots, X_n be a random sample from ER distribution with shape parameter $\theta > 0$. Therefore, the likelihood function of θ , from (1), as follows as:

$$L(\theta|data) = \prod_{i=1}^n f(x_i, \theta) \tag{4}$$

$$L(\theta|data) = \prod_{i=1}^n \left(2\theta x e^{-x_i^2 + \ln(1-x_i^2)^{-1}} e^{-\theta \ln(1-e^{-x_i^2})^{-1}} \right) \tag{5}$$

$$L(\theta|data) = q_1 \theta^n e^{-\theta [\sum_{i=1}^n \ln(1-e^{-x_i^2})^{-1}]} \tag{6}$$

Such that $q_1 = 2^n e^{(\sum_{i=1}^n \ln x_i - \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \ln(1-e^{-x_i^2})^{-1})}$

3. Bayesian Estimators using Different Prior and Loss Functions

In this section Bayesian Estimators of the shape parameter for four different prior functions and under six different loss functions has been determined.

Types of loss functions: -

If $\hat{\theta}$ represent of estimator for the shape parameter θ , then for:

1. Squared error loss function (SELF): SELF defined as:(2011)

$$L(\hat{\theta}, \theta) = c(\hat{\theta} - \theta)^2; \hat{\theta}_{self} = E(\theta) \tag{7}$$

2. Linear exponential loss function: the loss function defined as: (2011)

$$L(\hat{\theta}, \theta) = (e^{c(\hat{\theta}-\theta)} - c(\hat{\theta} - \theta) - 1); \hat{\theta}_{linex} = -\frac{1}{c} \ln E(e^{-c\theta}) \tag{8}$$

3. Precautionary loss function: the Precautionary loss function defined as: (2012)

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}}; \hat{\theta}_{plf} = \sqrt{E(\theta^2)} \tag{9}$$

4. Entropy loss function (ELF): ELF defined as: (2011)

$$L(\hat{\theta}, \theta) = \left(\frac{\hat{\theta}}{\theta}\right)^t - t \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1; \hat{\theta}_{elf} = (E(\theta^{-t}))^{-\frac{1}{t}} \tag{10}$$

5. De Groot loss function (DLF): DLF defined as: (2011)

$$L(\hat{\theta}, \theta) = \left(\frac{(\hat{\theta} - \theta)}{\hat{\theta}}\right)^2; \hat{\theta}_{dlf} = \frac{E(\theta^2)}{E(\theta)} \tag{11}$$

6. Non- Linear exponential loss functions is defined as:

$$L(\hat{\theta}, \theta) = (e^{c(\hat{\theta}-\theta)} + c(\hat{\theta} - \theta)^2 - c(\hat{\theta} - \theta) - 1); \hat{\theta}_{Nlinex} = \frac{1}{c+1} \ln E(e^{-c\theta}) - 2E(\theta) \tag{12}$$

3.1 The Posterior distributions with different priors

For the random variable X, the posterior density function of the shape parameter θ is:

$$P(\theta|\underline{x}) = \frac{L(\theta|\underline{x}).p(\theta)}{\int_0^\infty L(\theta|\underline{x}).p(\theta)d\theta} \tag{13}$$

For ‘‘Bayesian estimation’’, we describe two distinct posterior distributions under complete samples, and two distinct prior distributions for the shape parameter.

The Jeffery’s prior, for the parameter (θ) is:

$$p(\theta) = \frac{1}{\theta}; \theta > 0 \tag{14}$$

Then under the assumption of this prior distribution and by equation (13), the posterior distribution will be as:

$$p_j(\theta|\underline{x}) = \frac{q_1 \theta^n e^{-\theta [\sum_{i=1}^n \ln(1-e^{-x_i^2})^{-1}]} \frac{1}{\theta}}{\int_0^\infty q_1 \theta^n e^{-\theta [\sum_{i=1}^n \ln(1-e^{-x_i^2})^{-1}]} \frac{1}{\theta} d\theta} \tag{15}$$

$$p_j(\theta|\underline{x}) = \frac{\left(\sum_{i=1}^n \ln(1 - e^{-x_i^2})\right)^n}{\Gamma(n)} \theta^{n-1} e^{-\theta\left(\sum_{i=1}^n \ln(1 - e^{-x_i^2})\right)^{-1}} \quad \theta > 0 \quad (16)$$

The Gamma prior, considered to be:

$$p(\theta) = \frac{b^a \theta^{a-1}}{\Gamma(a)} e^{-\theta b}, \quad \theta > 0, \quad a, b > 0 \quad (17)$$

By equation (13) the posterior distribution under the assumption Gamma prior is:

$$p_j(\theta|\underline{x}) = \frac{q_1 \theta^n e^{-\theta[\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1}]} \frac{b^a \theta^{a-1}}{\Gamma(a)} e^{-\theta b}}{\int_0^\infty q_1 \theta^n e^{-\theta[\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1}]} \frac{b^a \theta^{a-1}}{\Gamma(a)} e^{-\theta b} d\theta} \quad (18)$$

$$p_j(\theta|\underline{x}) = \frac{\left(\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b\right)^{n+a}}{\Gamma(n + a)} \theta^{n+a-1} e^{-\theta[\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b]}; \quad \theta > 0 \quad (19)$$

3.2 Bayesian Estimators under Jeffrey Prior using the six different loss functions:

$$\hat{\theta}_{SELF} = \frac{n}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1}} \quad (20)$$

$$\hat{\theta}_{linex} = \frac{1}{c} \ln \left(\frac{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1}}{\ln(1 - e^{-x_i^2})^{-1} + c} \right) \quad (21)$$

$$\hat{\theta}_{plf} = \sqrt{\frac{n(n+1)}{(\ln(1 - e^{-x_i^2})^{-1})^2}} \quad (22)$$

$$\hat{\theta}_{elf} = \frac{\Gamma(n)^{1/t}}{\Gamma(n-t)} \frac{1}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1}} \quad (23)$$

$$\hat{\theta}_{DLF} = \frac{(n+1)}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1}} \quad (24)$$

$$\hat{\theta}_{Nlinex} = \frac{\left(\ln \left(\frac{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1}}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + c} \right) \right)^n - 2 \frac{n}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1}}}{c + 2} \quad (25)$$

3.3 Bayesian Estimators under Gamma Prior using the six different loss functions:

$$\hat{\theta}_{SELF} = \frac{n + a}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b} \quad (26)$$

$$\hat{\theta}_{linex} = \frac{1}{c} \ln \left(\frac{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b}{\ln(1 - e^{-x_i^2})^{-1} + c + b} \right)^{(n+a)} \quad (27)$$

$$\hat{\theta}_{plf} = \sqrt{\frac{(n+a)(n+a+1)}{(\ln(1 - e^{-x_i^2})^{-1} + b)^2}}$$

$$\hat{\theta}_{elf} = \frac{\Gamma(n+a)^{1/t}}{\Gamma(n+a-t)} \frac{1}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b} \quad (28)$$

$$\hat{\theta}_{DLF} = \frac{(n+a+1)}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b} \quad (29)$$

$$\hat{\theta}_{N\text{Linex}} = \frac{\left(\ln \left(\frac{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b + c} \right)^{n+a} - 2 \frac{n}{\sum_{i=1}^n \ln(1 - e^{-x_i^2})^{-1} + b} \right)}{c + 2} \quad (30)$$

4. SIMULATION RESULTS AND CONCLUSIONS

In this section we mainly perform some simulation experiments to observe the behaviour of the different Bayes estimators for the shape parameter proposed in the previous section (3), using different sample sizes “n= 10, 20, 30, 50, 75 and 100”and for two distinct parameters values “θ= 1.5 and 3”, by applying the Monte Carlo simulation to compare the performance of these estimators using the mean squared error (MSE). All results are computed by R program based on replications and for two different put “a = 3, b= 0.8, c= 1; a = 4, b= 1.2, c= 1”.

The mean and MSE values for the Bayesian estimator, are recorded in tables (1), (2), (3) and (4), and summarized generally in table (5).

Table (1): The mean value of $\hat{\theta}$, when $\theta = 1.5$

n	Jeffery prior					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	1.6566	1.7356	1.8124	1.5118	1.4047	1.6125
20	1.5573	1.5860	1.6253	1.4871	1.4301	1.5464
30	1.5502	1.5657	1.6017	1.5105	1.4622	1.5345
50	1.5264	1.5312	1.5462	1.5022	1.4713	1.5144
75	1.5219	1.5217	1.5320	1.5043	1.4811	1.5132
100	1.5191	1.5262	1.5245	1.5032	1.4761	1.5112

n	Gamma prior					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	1.8812	1.9512	2.0247	1.7456	1.6615	1.8357
20	1.6832	1.7274	1.7453	1.6231	1.5634	1.6606
30	1.6452	1.6597	1.67082	1.5843	1.5407	1.6328
50	1.5782	1.5832	1.6182	1.5523	1.5376	1.5632
75	1.5472	1.5638	1.5679	1.5413	1.5246	1.5539
100	1.5452	1.5532	1.5509	1.5234	1.5232	1.5423

n	Gamma prior					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	1.9054	1.6345	2.1420	1.7657	1.7032	1.8632
20	1.7034	1.5604	1.7676	1.6345	1.6245	1.7682
30	1.6401	1.5456	1.6875	1.6208	1.5934	1.6456
50	1.5789	1.5345	1.6556	1.5753	1.5443	1.5965
75	1.5556	1.5245	1.5236	1.5532	1.5265	1.5534
100	1.5434	1.5156	1.5123	1.5256	1.5134	1.5431

Table (2): The mean value of $\hat{\theta}$, when $\theta = 3$

n	Jeffery prior					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	3.3243	3.4652	3.5247	2.8184	2.8125	3.1517
20	3.1145	3.1924	3.3458	2.7857	2.7823	3.1386
30	3.0764	3.1778	3.2764	2.6219	2.7225	3.0261
50	3.0621	3.1231	3.1289	2.6708	2.7706	3.0226
75	3.0423	3.0645	3.0643	2.5823	2.5669	3.0227
100	3.0351	3.0545	3.0565	2.4596	2.5636	3.0200

n	Gamma prior (a= 3, b = 0.8)					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	3.3456	2.8732	3.4729	2.9598	2.9556	3.2162
20	3.2593	2.8978	3.3543	2.9372	2.9618	3.1032
30	3.1208	2.9343	3.2467	2.9784	2.9738	3.0432
50	3.0878	2.9991	3.1343	3.0023	3.0056	3.0634
75	3.0695	2.9922	3.0881	3.0092	3.0084	3.0457
100	3.0414	2.9821	3.0652	3.0076	3.0073	3.0465

n	Gamma prior (a= 4, b = 1.2)					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	2.9675	2.9013	2.9292	2.9598	2.9556	3.2162
20	3.1278	2.8323	3.2567	2.9372	2.9618	3.1032
30	3.1149	2.9324	3.2076	2.9784	2.9738	3.0432
50	3.0643	2.9435	3.1465	3.0023	3.0056	3.0634
75	3.0351	2.9772	3.0881	3.0923	3.0084	3.0457
100	3.0484	2.9822	3.0652	3.0772	3.0073	3.0465

Table (3): The MSE value of $\hat{\theta}$, when $\theta = 1.5$

n	Jeffrey's prior					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	0.3343	0.2302	0.4765	0.2187	0.2343	0.2934
20	0.1345	0.1153	0.1627	0.10834	0.1134	0.1243
30	0.0807	0.0786	0.1041	0.0797	0.0792	0.0831
50	0.0486	0.5674	0.0534	0.0468	0.0245	0.0533
75	0.0323	0.3493	0.3391	0.0310	0.0302	0.0332
100	0.0241	0.2321	0.2383	0.2354	0.0232	0.0245
n	Gamma prior (a= 4, b = 1.2)					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	0.4323	0.2141	0.6014	0.2808	0.2543	0.3754
20	0.1621	0.1054	0.2062	0.1243	0.1162	0.1489
30	0.1204	0.0802	0.1323	0.0932	0.0853	0.1032
50	0.0573	0.0453	0.0637	0.0549	0.0483	0.0549
75	0.0352	0.0304	0.0365	0.0343	0.0323	0.0421
100	0.0244	0.0234	0.0275	0.0256	0.0231	0.0245

n	Gamma prior (a= 4, b = 1.2)					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	0.4327	0.2145	0.6013	0.2918	0.2532	0.3767
20	0.1623	0.1052	0.2067	0.1242	0.1160	0.1476
30	0.1101	0.0801	0.1322	0.0912	0.0857	0.1034
50	0.0574	0.0463	0.0651	0.0507	0.0478	0.0541
75	0.0351	0.0306	0.0386	0.0334	0.0312	0.0321
100	0.0251	0.0224	0.0275	0.0297	0.0251	0.0242

Table (4): The MSE value of $\hat{\theta}$, when $\theta = 3$

n	Jeffrey's prior					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	1.3456	1.6709	1.9305	0.6836	0.9347	1.0532
20	0.5326	0.5867	0.6467	0.3875	0.4587	0.4357
30	0.3189	0.3343	0.3568	0.2557	0.2861	0.2856
50	0.2018	0.2096	0.2204	0.1743	0.1865	0.1904
75	0.1306	0.1388	0.1395	0.1183	0.1228	0.1256
100	0.0972	0.0976	0.0971	0.0876	0.0912	0.0912
n	Gamma prior (a= 3, b = 0.8)					
	B_{SELF}	B_{PLF}	B_{DLF}	B_{LINEX}	B_{ELF}	B_{NLINEX}
10	0.8503	1.0097	1.2108	0.4493	0.5734	0.6532
20	0.4245	0.4609	0.5213	0.2956	0.3453	0.3657
30	0.2867	0.3078	0.3284	0.2205	0.2423	0.2556

50	0.1987	0.1997	0.2121	0.1612	0.1715	0.1784
75	0.1346	0.1308	0.1353	0.1122	0.1176	0.1216
100	0.0932	0.0961	0.0976	0.0845	0.0861	0.0812

According to tables (3) and (4), as sample size rises, the rate at which estimates approach the true value of the shape parameter increases. Based on the best estimates from experiments 3 and 4, it can be concluded that the performance of the Bayes estimator with the Gamma prior and the linear exponential loss function is preferable to that of other priors with other priors and for various sample sizes.

5. CONCLUSIONS

The above study makes the suggestion that when $\alpha=3$, the performance of the Bayes estimator under the Gamma prior with linear loss function, records full appearance "for all sample sizes," as the best prior distribution, and utilising various loss functions for the entire data set.

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