Automorphism Groups of Regular Elements with Von-Neumann Inverses of Local near-rings admitting Frobenius Derivations

Abstract

This paper presents the classi_cation of the invariant subgroups of the automorphism groups of the regular elements obtained from _nite local near-rings, the appropriate algebraic structure to study non-linear functions on _nite groups. Just as rings of matrices operate on vector spaces, near-rings operate on groups. In this paper, we construct classes of zero symmetric local near-ring of characteristic p_k; k = 1; 2 ; k _ 3 admitting frobenius derivations, characterize the structures of the cyclic groups generated by the regular elements R(N) as well as the structures and the orders of the automorphism groups Aut(R(N)) of the regular elements.

Subject Classi_cation:16N60, 16W25, 16Y30.
Keywords: Regular Elements, Von-Neumann Inverses, Zero symmetric local near-rings

1 Introduction

Near-rings with identity are the appropriate algebraic structures used to study non-linear functions on _nite groups. Let N be a zero symmetric near-ring from the classes of near-rings considered in this paper. Suppose R(N) is the set of all the regular elements in N , then every element x 2 R(N) is such that x = 0 or x 2 N_: The automorphism groups of R(N) denoted Aut(R(N)) is the set whose elements are the automorphisms _ : R(N) ! R(N) and where the group operation is composition of automorphisms. Thus the group structure of Aut(R(N)) is obtained as a subgroup of the symmetric groups Sym(R(N)):

Idealization modules, rings and near-rings is perhaps one of the new approaches in ring classi_cation problems up to isomorphism. The works in[1] determined constructions of idealized completely primary _nite rings of characteristic p_n : n = 1; 2; 3 and determined the structures of the unit groups R_ . This was however neither extended to the automorphisms of R_ or the Von-Neumann regular elements V RN(R) nor to the dual results in near-rings. Osba, Henriksen and Osama [2] conducted a classi_cation survey

on combining local and Von-Neumann Regular Rings as a basis upon which the regularity properties of rings and their ideals could be explored. The rings studied in [2] were _nite and their Von-Neumann inverses gave some asymptotic patterns. Their _ndings demonstrated how to combine the Von- Neumann inverses of classes of rings such as the power series rings and the ring of integers. The study was however not extended to the automorphisms of the said algebraic structures. In a closely related research, the study on regular elements of Galois rings can be attributed to Osama and Emad [3] where they characterized the regular elements in the ring of integers modulo n, Z_n . Furthermore, they studied the arithmetic functions denoted as V (n) and determined the relationship between V (n) and the Euler's phi function, '(n) . This gave an extension of the ring theoretic algebra employed in counting the regular elements of Z_n to the number theoretic methodologies. For instance, the research revealed that if a is a regular element in Z_n , then $a_{(\square 1)}$ _ $a_{'(n)\square 1}$ (mod n). They proposed a criterion for getting the possible

Von Neumann inverses in the set of regular elements of Z_n and explored the asymptotic properties of V (n) . Their _ndings did not consider extensions and idealization using maximal submodules of $Z_n 8n$ 2 Z. Closely related works can also be seen in Osba et al [4] and Oduor, Omamo and Musoga[5]. To obtain a classi_cation of algebraic structures,
it is imperative to obtain group structures that are isomorphic to those structures, the automorphism groups. Therefore, this paper gives a complete classi_cation of the regular
elements R(N) of some classes of zero-symmetric local near-rings which admit two classes
of derivations, that is a generalized frobenius derivation in Construction I and an identity frobenius automorphism in construction II.

2 Local Near-Ring of Characteristics p; p_2 and p_k : k _ 3 admitting frobenius derivations

2.1 Construction I

Let R_0 = GN(p_{kr}; p_k) be a Galois near-ring of order p_{kr} and characteristic p_k and let
M = hu_ii : i = 1; _ _ _ h be an h□dimensional near-module of R_0 so that the ordered
pair (N; +) = (R_0 _M; +) is a group. On N; let
$p_k u_i$ =
Y_k
i=1
u_i = 0
and $u_i r_0$ = $(r_0)_{d_i} u_i$ when k = 1; 2 where r_0 2 R_0; k; r are invariants and d_i a Frobenius derivation associated with elements of M and given by; $d_i(u_i) = (u_i)_p$. Let J be a near-ideal of M satisfying the condition that whenever u_i; u_j 2 J ; we have u_i _ u_j 2 J or
u_i _ u_j = 0 . If $\_i$ are any units of R_0 , then we can see that the elements of N = R_0_M
are of the form: x = r_0+
P_h
i=1 $\_i u_i$. In fact, if x = r_0+
P_h
i=1 $\_i u_i$ and y = s_0+
P_h
i=1 $\_i u_i$

are any two elements of N , then we have their product as:
x _ y =
_
r_0 +
X_h
i=1
$\_i u_i$
_
_
_
s_0 +
X_h

$i=1$
_$_i u_i$
_
= $r_0 s_0$ +
X$_h$
$i=1$
n
_$_i(r_0 + p_k R_0)_{d_i}$ + _$_i(s_0 + p_k R_0)_{d_i}$
o
u_i:

It has been shown in[6] that (N;+; _) with the product given in the construction is a left (respective right) local near-ring whose unique maximal ideal is the Jordan ideal, J(N) .

2.2 Construction II

Let R_o = GN(p_{kr}; p_k) . Let i = 1; :::; h and u_i 2 Z_L(N) and M=< u_i >.
Then,
N = R_o _M= R_o _
X$_h$
$i=1$
(R_o=pR_o)$_i$
is a group with respect to addition.
On N , let
(r_o; r_1; :::; r_h)(s_o; s_1; :::; s_h) = ($r_o s_o$; $r_o s_1$ + $r_1 s_o$; :::; $r_o s_h$ + $r_h s_o$)_
where _ is the identity Frobenius automorphism. The multiplication turns N into a local zero symmetric near-ring with identity (1; 0; :::; 0) .
Indeed N = R_o _M is commutative since _ is the identity Frobenius automorphism.
Proposition 2.1. Consider N = GN(p_{kr}; p_k) where k _ 3 . Then, charN = p_k and:
(i). Z_L(N) = pR_o _
P$_h$
$_{i=1}$(R_o=pR_o)$_i$
(ii). (Z_L(N))$_{k□1}$ = $p_{k□1}R_o$ 6= (0)
(iii). (Z_L(N))$_k$ = (0) .
Lemma 2.1. Let N = GN(p_{kr}; p_k)_M where p is prime k and r are positive integers and M is a h-dimensional module over N . Then if h = 0 ,
(i) R(N) _=
(1 + Z(N)) [f0g and
(ii) j R(N) j= (p$_{(k□1)r}$)(p$_r$ □ 1) + 1

Proof. Let a 2 R(N) _=
(1 + Z(N)) . Then a is invertible or 0. But N is local means
that a is regular i.e. a 2 R(N) .
Thus R(N) _ [< a > _1 + Z(N))] [f0g..(i)
Conversely, let a 2 R(N) . Then by de_nition 9 an element b 2 R(N) such that
a = a$_2$b) a(1 □ ab) = 0 .
If a 2 (N_) then 1 □ ab = 0) ab = 1 .
Hence b is a Von Neumann inverse of a. If is not a member of N_ then ab is not a member of N_ but ab = aabb = a$_2$b$_2$ = abab = (ab)$_2$.
Since N commutes) ab = (ab)$_2$) ab(1 □ ab) = 0 .

Now) 1□ab is a unit and ab = 0 so that a = 0 because b is its Von Neumann inverse.
[f< a > _1 + Z(N)g [f0g] _ R(N) ....(ii)
Combining (i) and (ii) gives
R(N) _=
[1 + Z(N)] [f0g
= < a > _[1 + Z(N)] [f0g
Next,
N_ = (N_=1 + Z(N)) _ 1 + Z(N)
_=
< a > _[1 + Z(N)]
= $Z_{pr□1}$ _ [1 + Z(N)]
But
j [1 + Z(N)] j = j Z(N) j
= $p_{(k□1)r}$
Therefore j N_ j= (p_r □ 1)($p_{(k□1)r}$)
But R(N) = N_ [f0g j R(N) j= (p_r □ 1)($p_{(k□1)r}$) + 1 as required.
Theorem 2.1. Let N be the near-ring constructed in I and II and R(N) be the set of all the regular elements. Then
(i).
R(N) =
_
$Z_{pr□1}$ _ (Z_r
$_p$)$_h$ [f0g; CharN = p;
$Z_{pr□1}$ _ (Z_r
$_p$)$_{h+1}$ [f0g; CharN = p_2.
(ii).
R(N) =
_
$Z_{2r□1}$ _ Z_2 _ $Z_{2k□}$___________$_2$ _ $Z_{r□1}$
$_{2k□1}$ _ (Z_2)$_h$ [f0g; p = 2;
$Z_{pr□1}$ _ Z_r
$_{pk□1}$ _ (Z_r
$_p$)$_h$ [f0g; p 6= 2 : CharN = p_k : k _ 3.
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Proof. Let $_1$; :::; $_r$ 2 F_q with $_1$ = 1 such that _ $_1$; :::; _ $_r$ form a basis for F_q regarded as a vector space over its prime subnear-_eld F_q where q = p_r for any prime p and a positive integer r .
(i) [case 1.] Let charN = p
It can be observed that for every l = 1; :::; r and 1 _ i _ h, 1 + $_lu_i$ 2 1 + Z(N) and
(1 + $_lu_1$)$_p$ = (1 + $_lu_1$ + $_lu_2$)$_p$
= :::
= (1 + $_lu_1$ + $_lu_2$ + ::: + $_lu_h$)$_p$
= 1:
That is, y_p = 1 8 y 2 1 + Z(N) .
Now for the positive integers a_{1l}; a_{2l}; :::; a_{hl} with a_{1l} _ p; a_{2l} _ p; :::; a_{hl} _ p; we notice that

Y$_r$

$_{l=1}$

f(1 + _$_l$u$_1$)$_{a_{1l}}$g _

Y$_r$

$_{l=1}$

f(1 + _$_l$u$_1$ + _$_l$u$_2$)$_{a_{1l}}$g _

::: _

Y$_r$

$_{l=1}$

f(1 + _$_l$u$_1$ + _$_l$u$_2$ + ::: + _$_l$u$_h$)$_{a_{1l}}$g = 1

will imply that a$_{il}$ = p; 8l = 1; :::r and 1 _ i _ h:

Let

S$_{1l}$ = f(1 + _$_l$u$_1$)$_{a_1}$: a$_1$ = 1; :::; pg

S$_{2l}$ = f(1 + _$_l$u$_1$ + _$_l$u$_2$)$_{a_2}$: a$_2$ = 1; :::; pg

...

S$_{hl}$ = f(1 + _$_l$u$_1$ + ::: + _$_l$u$_h$)$_{a_h}$: a$_h$ = 1; :::; pg:

Then, S$_{1l}$; S$_{2l}$; :::; S$_{hl}$ are all cyclic subgroups of 1 + Z(N) and they are each of order p hence 1 + Z(N) is abelian and each element a 2 1 + Z(R) is such that a$_p$ = 1:

Now,

Y$_r$

$_{l=1}$

j< 1 + _$_l$u$_1$ >j _

Y$_r$

$_{l=1}$

j< 1 + _$_l$u$_1$ + _$_l$u$_2$ >j :::

Y$_r$

$_{l=1}$

j< 1 + _$_l$u$_1$ + _$_l$u$_2$ + ::: + _$_l$u$_h$ >j = p$_{hr}$:

The intersection of any pair of the cyclic subgroups S$_{1l}$; S$_{2l}$; :::; S$_{hl}$ gives the identity group and the product of the hr subgroups S$_{1l}$:::S$_{hl}$ exhausts 1 + Z(R) .

But

R(N) =< a > o(1 + Z(N)) [f0g = Z$_{pr□1}$ _ (Z$_r$

$_p$)$_h$ [f0g

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such that o(a) = p$_r$ □ 1 . This settles the case 1.

(ii.)[case 2] Let p$_2$ = charN

For every l = 1; :::; r; (1 + p_$_l$)$_p$ = 1;

(1 + _$_l$u$_1$)$_{p_2}$ = 1; (1 + _$_l$u$_1$ + _$_l$u$_2$)$_{p_2}$ = 1; :::; (1 + _$_l$u$_1$ + ::: + _$_l$u$_h$)$_{p_2}$ = 1) . For positive integers a$_l$; b$_{1l}$; :::; b$_{hl}$ with a$_l$ _ p; b$_{1l}$ _ p$_2$; :::; b$_{hl}$ _ p$_2$. It is clear that

Y$_r$

$_{l=1}$

(1 + p_$_l$)$_{a_l}$:

Y$_r$

$_{l=1}$

(1 + _$_l$u$_1$)$_{b_{1l}}$ :::::

Y$_r$

$_{l=1}$

(1 + _$_l$u$_1$ + _$_l$u$_2$ + ::: + _$_l$u$_h$)$_{b_{hl}}$ = 1

) al = p; b$_{1l}$ = p$_2$; :::; b$_{hl}$ = p$_2$ for every L = 1; 2; :::; r and 1 _ i _ h.

Set
T_l = f1 + p_$_l$)$_a$: a = 1; :::; pg
S_{1l} = f1 + _$_l$u$_1$)$_{b_1}$: b$_1$ = 1; :::; p$_2$g
...
S_{hl} = f(1 + _$_l$u$_1$ + _$_l$u$_2$+; :::;+_$_l$u$_h$)$_{b_h}$: b$_h$ = 1; :::; p$_2$g
T$_l$; S$_1$; :::; S$_h$ are all cyclic subgroups of the group 1 + Z(N) and they are of the orders indicated by their de_nitions.
Since
Y$_r$
l=1
j< 1 + p_$_1$ >j :::::
Y$_r$
l=1
j< 1 + _$_l$u$_1$ + ::: + _$_l$u$_h$ >j= p$_{(2h+1)r}$;
and the intersection of any pair of the cyclic subgroups T$_l$; :::; S$_h$ gives an identity group, the product of the (h + 1)r subgroups T$_l$; S$_1$; :::; S$_h$ is direct and exhausts 1 + Z(N):
Thus according to case 1 , we have
R(N) =< _ > _(1 + Z) [f0g
R(N) = Z$_{pr□1}$ _ (Z$_r$
$_p$)$_{h+1}$ [f0g
(iii) [Case 3]. Let char N = p$_k$: k _ 3 . We provide the general case using p = odd.
Notice that every l = 1; :::; r; (1 + p_$_1$)$_{pk□1}$ = 1
(1 + _$_l$u$_1$)$_{pk}$ = 1; :::; (1 + p_$_L$u$_1$ + _$_l$u$_2$ + ::: + _$_l$u$_n$)$_{pk}$ = 1 .
Let a$_l$; b$_{1l}$; :::; b$_{hl}$ 2 Z$_+$ with a$_l$ _ p$_{k□1}$, b$_{il}$ _ p$_k$: 1 _ i _ h. We notice that
Y$_r$
l=1
f(1 + p_$_L$)$_{aL}$g _
Y$_r$
l=1
f(1 + _$_l$u$_1$)$_{b1l}$g _
Y$_r$
l=1
f(1 + _$_l$u$_1$ + _$_l$u$_2$ + ::: + _$_l$u$_h$)g = 1

which implies that a$_l$ = p$_{k□1}$, b$_{1l}$ = p$_k$ = _ _ _ = b$_{hl}$ = p$_k$. Set
T$_l$ = < f(1 + p_$_l$)$_a$ j a = 1; :::; p$_{k□1}$g >
S$_{1l}$ = < f(1 + _$_l$u$_1$)$_{b_1}$ j b$_1$ = 1; _ _ _ ; p$_k$g >
...
S$_{hl}$ = < f(1 + _$_l$u$_1$ + _ _ _ + _$_l$u$_n$)$_{b_h}$ j b$_h$ = 1; _ _ _ ; p$_k$g >
The sets de_ned are all cyclic subgroups of the group 1+Z(N) and they are of the indicated orders. Furthermore, the intersection of any pair of the cyclic subgroups indicated gives an identity group and the product of the (h + 1)r subgroups gives:
j T$_l$ _ S$_{1L}$ _ S$_{hl}$ j= p$_{k((h+1)r)□1}$ exhausting 1 + Z(N) .
Thus 1 + Z(N) _=
Z$_r$
$_{pk□1}$ _ (Z$_r$
$_p$)$_h$:
Therefore

R(N) =< _ > _(1 + (Z(N))) [f0g
= $Z_{pr□1}$ _ Z_r
$_{pk□1}$ _ (Z_r
$_p$)$_h$ [f0g:

Theorem 2.2. Let N = R_o _ M where r = 1 and p -prime, k 2 Z_+ . If M =
R_0=pR_0 _ ::: _ R_0=pR_0 . Let r_0 2 R(R_0) then, its Von-Neumann inverse is
r$_{□1}$
$_0$ = r$_{pk□pk□1□1}$
$_0$ and (r_0; :::; r_h)$_{□1}$ = (r$_{pk□pk□1□1}$;□r_1t0r$_{□1}$
$_0$; :::;□r_ht0r$_{□1}$
$_0$)

Theorem 2.3. Let N be a near-ring of construction I and II, R(N) be the set of all the regular elements including 0. Then if
Aut : R(N) ! R(N) is an automorphism:
(i) when charN = p , then Aut(R(N)) _=
[($Z_{pr□1}$)$_\_$ _ GL_{hr}(GN(p_r; p))][M where
M= x 2 R(N) : Aut(x) = 0
(ii) when charN = p_2 , then, Aut(R(N)) _=
[($Z_{pr□1}$)$_\_$ _ $GL_{(h+1)r}$(GN(p_{2r}; p_2))][M
(iii) when
charN = p_k : k _ 3 , then,
Aut(R(N)) _=
[($Z_{pr□1}$)$_\_$ _ $GL_{(k□1)r}$(GN(p_{kr}; p_k))] _ GL_{hr}(GN(p_{kr}; p_k))][M.

Proof. Proof of (i.) By enumeration, R(N) =< a > _(1 + Z_L(N)) [f0g where
< a >= $Z_{pr□1}$. Since gcd(p_r □ 1; j 1 + Z_L(N) j) = 1 , we have that
Aut($Z_{pr□1}$ _ 1 + Z_L(N)) _=
Aut($Z_{pr□1}$ _ Aut(1 + Z(N))
But,
Aut($Z_{pr□1}$) = ($Z_{pr□1}$)$_\_$
which is a permutation group whose order coincides with the order of 1 + Z_L(N):
Next, de_ne a zero automorphism to be the set
4 = fx 2 R(N)g : Aut(x) = 0:
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Then clearly 4 = fAut(0) = 0_ng .
When charN = p;
R(N) = [$Z_{pr□1}$ _ (Z_r
$_p$)$_h$] [f0g
which implies that
AutR(N) _=
[($Z_{pr□1}$)$_\_$ _ GL_{hr}(F_p)] [4:
This proves (i). The conditions (ii) and (iii) follow from the proof of (i) with modi_cations.

Theorem 2.4. Let N be a zero symmetric local near-rings from the class of near-rings of constructions I and II. Then:
(i)
j Aut(R(N)) j = ['(p_r □ 1) _
Y$_{hr}$
$_{k=1}$

$(p_k − p_{k−1})] + 1$

when charN = p

(ii)

j Aut(R(N)) j = ['(p_r − 1) _

$_{(h}Y_{+1)r}$

k=1

$(p_k − p_{k−1})] + 1$

when charN = p_2

(iii)

j Aut(R(N)) j = ['(p_r − 1) _

$_{(k}Y_{−1)r}$

k=1

$(p_k − p_{k−1})$ _

Y_{hr}

k=1

$(p_k − p_{k−1})] + 1$

when charN = p_k : k _ 3

Proof. (i) Let charN = p

By de_nition of '(n) attributed to Osama and Emad [[3]],

j ($Z_{p_r−1}$)_ j= '(p_r − 1)

and since Aut($Z_{p_r−1}$) =j ($Z_{p_r−1}$)_ j= '(p_r − 1) the pre_x of right hand side to (i) is clear.

From the previous theorem, Aut(1 + Z_L(N) = GL_{hr}(Z_p) . Thus, we need to _nd all the elements of

GL_{hr}(Z_p)

in the endomorphism, End(1+Z_L(N)) and calculate the distinct ways of extending such an element to an endomorphism. So we need all such matrices that are invertible modulo

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p .

Let R_p 2 End(1 + Z_L(N)) then the number of matrices A 2 R_p that are invertible modulo p are upper block triangular matrices whose number can be given as:

]A =

Y_n

k=1

$(p_k − p_{k−1})$

Now when charR = p , n = hr therefore

]A =

Y_{hr}

k=1

$[p_k − p_{k−1}]$

this means that

j $Z_{p_r−1}$ _ GL_{hr}(Z_p)) j = '(p_r − 1) _

Y_{hr}

k=1

$[p_k − p_{k−1}]$

Finally 0 2 R(N) and Aut(0) = 0 . Now j Aut(0) j=j f0g j= 1 thus

j AutR(N) j = j [Aut($Z_{p_r−1}$) _ Aut(GL_{hr}(Z_p))] j + j Autf0g j

= ['(p_r − 1) _

Y_{hr}
$_{k=1}$
$(p_k \square\ p_{k\square 1})] + 1$
as required. The proofs to (ii) and (iii) are similar to proofs of (i) with modi_cations on the orders of $GL_n(Z_p)$

3 Conclusion

This study has determined the structures of the regular elements R(N) of idealized local near-rings with derivations as cyclic groups using the Fundamental Theorem of Finitely generated Abelian Groups. The algebraic structures of R(N) have been completely classi_ed as subgroups of Sym(N_) using the automorphisms groups Aut(R(N)): The research reveals unique algebraic structures.

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