

A Study on Generalized Blaise Numbers

Abstract. In this paper, we introduce and investigate the generalized Blaise sequences and we deal with, in detail, two special cases, namely, Blaise and Blaise-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Blaise, Blaise-Lucas and Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan, modified Jacobsthal-Padovan numbers.

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1. Introduction

Adjusted Jacobsthal-Padovan sequence $\{K_n\}_{n \geq 0}$ (OEIS: A159287, [7]), Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) sequence $\{L_n\}_{n \geq 0}$ (OEIS: A072328, [7]), Jacobsthal-Padovan sequence $\{Q_n\}_{n \geq 0}$ (OEIS: A159284, [7]), and modified Jacobsthal-Padovan sequence $\{M_n\}_{n \geq 0}$ are defined, respectively, by the third-order recurrence relations

$$K_{n+3} = K_{n+1} + 2K_n, \quad K_0 = 0, K_1 = 1, K_2 = 0, \quad (1.1)$$

$$L_{n+3} = L_{n+1} + 2L_n, \quad L_0 = 3, L_1 = 0, L_2 = 2, \quad (1.2)$$

$$Q_{n+3} = Q_{n+1} + 2Q_n, \quad Q_0 = 1, Q_1 = 1, Q_2 = 1, \quad (1.3)$$

$$M_{n+3} = M_{n+1} + 2M_n, \quad M_0 = 3, M_1 = 1, M_2 = 3. \quad (1.4)$$

The sequences $\{Q_n\}_{n \geq 0}$, $\{L_n\}_{n \geq 0}$, $\{K_n\}_{n \geq 0}$ and $\{M_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} K_{-n} &= -\frac{1}{2}K_{-(n-1)} + \frac{1}{2}K_{-(n-3)}, \\ L_{-n} &= -\frac{1}{2}L_{-(n-1)} + \frac{1}{2}L_{-(n-3)}, \\ Q_{-n} &= -\frac{1}{2}Q_{-(n-1)} + \frac{1}{2}Q_{-(n-3)}, \\ M_{-n} &= -\frac{1}{2}M_{-(n-1)} + \frac{1}{2}M_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, \dots$ respectively. Therefore, recurrences (1.1)-(1.4) hold for all integer n . For more information on Jacobsthal-Padovan sequence, see [14] and [1].

Now, we define two sequences related to Adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers. Blaise and Blaise-Lucas numbers are defined as

$$B_n = B_{n-2} + 2B_{n-3} + 1, \quad \text{with } B_0 = 0, B_1 = 1, B_2 = 1, \quad n \geq 3,$$

and

$$C_n = C_{n-2} + 2C_{n-3} - 2, \quad \text{with } C_0 = 4, C_1 = 1, C_2 = 3, \quad n \geq 3,$$

respectively. The first few values of Blaise and Blaise-Lucas numbers are

$$0, 1, 1, 2, 4, 5, 9, 14, 20, 33, 49, 74, 116, 173, \dots$$

and

$$4, 1, 3, 7, 3, 11, 15, 15, 35, 43, 63, 111, 147, 235, \dots$$

respectively. The sequences $\{B_n\}$ and $\{C_n\}$ satisfy the following fourth order linear recurrences:

$$\begin{aligned} B_n &= B_{n-1} + B_{n-2} + B_{n-3} - 2B_{n-4}, & B_0 = 0, B_1 = 1, B_2 = 1, B_3 = 2, & n \geq 4, \\ C_n &= C_{n-1} + C_{n-2} + C_{n-3} - 2C_{n-4}, & C_0 = 4, C_1 = 1, C_2 = 3, C_3 = 7, & n \geq 4. \end{aligned}$$

There are close relations between Blaise, Blaise-Lucas and Adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers. For example,

they satisfy the following interrelations:

$$\begin{aligned}
 2B_n &= K_{n+2} + K_{n+1} + 2K_n - 1, \\
 2C_n &= -K_{n+2} + 6K_{n+1} + K_n + 2, \\
 52B_n &= 2L_n + 9L_{n+1} + 10L_{n+2} - 26, \\
 C_n &= L_n + 1, \\
 2B_n &= Q_{n+2} - 1, \\
 2C_n &= -3Q_{n+2} + 2Q_{n+1} + 7Q_n + 2, \\
 46B_n &= 8M_{n+2} + 5M_{n+1} - 2M_n - 23, \\
 46C_n &= -5M_{n+2} - 6M_{n+1} + 53M_n + 46,
 \end{aligned}$$

and

$$\begin{aligned}
 2K_n &= -B_{n+2} + 3B_n + 1, \\
 52K_n &= 9C_{n+2} - C_{n+1} - 6C_n - 2, \\
 4L_n &= -3B_{n+2} + 14B_{n+1} - 9B_n + 1, \\
 2L_n &= C_{n+3} - C_{n+1}, \\
 2Q_n &= -B_{n+2} + 2B_{n+1} + B_n + 1, \\
 26Q_n &= 4C_{n+2} + C_{n+1} + 6C_n - 11, \\
 2M_n &= -B_{n+2} + 6B_{n+1} - 3B_n + 1, \\
 26M_n &= 3C_{n+2} + 4C_{n+1} + 24C_n - 31.
 \end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Blaise, Blaise-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized (r, s, t, u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n \geq 0}$ (or shortly $\{W_n\}_{n \geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \quad (1.5)$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [2,4,5,6,9,11,12,15,16]. The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, \dots$ when $u \neq 0$. Therefore, recurrence (1.5) holds for all integers n .

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \tag{1.6}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= r, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\ \alpha\beta\gamma\delta &= -u. \end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

THEOREM 1. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) For all integers n , Binet's formula of generalized Tetranacci numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{p_2\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{p_3\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{p_4\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)} \tag{1.7}$$

where

$$\begin{aligned} p_1 &= W_3 - (\beta + \gamma + \delta)W_2 + (\beta\gamma + \beta\delta + \gamma\delta)W_1 - \beta\gamma\delta W_0, \\ p_2 &= W_3 - (\alpha + \gamma + \delta)W_2 + (\alpha\gamma + \alpha\delta + \gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha + \beta + \delta)W_2 + (\alpha\beta + \alpha\delta + \beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - (\alpha + \beta + \gamma)W_2 + (\alpha\beta + \alpha\gamma + \beta\gamma)W_1 - \alpha\beta\gamma W_0. \end{aligned}$$

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (1.6) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (1.6) for non-negative integers is valid for all integers n (see [3]).

Next, we consider two special cases of the generalized (r, s, t, u) sequence $\{W_n\}$ which we call them (r, s, t, u) -Fibonacci and (r, s, t, u) -Lucas sequences. (r, s, t, u) -Fibonacci sequence $\{G_n\}_{n \geq 0}$ and (r, s, t, u) -Lucas sequence $\{H_n\}_{n \geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$G_{n+4} = rG_{n+3} + sG_{n+2} + tG_{n+1} + uG_n, \tag{1.8}$$

$$G_0 = 0, G_1 = 1, G_2 = r, G_3 = r^2 + s,$$

$$H_{n+4} = rH_{n+3} + sH_{n+2} + tH_{n+1} + uH_n, \tag{1.9}$$

$$H_0 = 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t.$$

The sequences $\{G_n\}_{n \geq 0}$ and $\{H_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)} - \frac{s}{u}G_{-(n-2)} - \frac{r}{u}G_{-(n-3)} + \frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)} - \frac{s}{u}H_{-(n-2)} - \frac{r}{u}H_{-(n-3)} + \frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, \dots$ respectively. Therefore, recurrences (1.8) and (1.9) hold for all integers n .

For all integers n , (r, s, t, u) -Fibonacci and (r, s, t, u) -Lucas numbers (using initial conditions in (1.8) or (1.9)) can be expressed using Binet's formulas as in the following corollary.

COROLLARY 2. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) Binet's formula of (r, s, t, u) -Fibonacci and (r, s, t, u) -Lucas numbers are*

$$G_n = \frac{\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^{n+2}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^{n+2}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^{n+2}}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

respectively.

Proof. Take $W_n = G_n$ and $W_n = H_n$ in Theorem 1, respectively. \square

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n .

LEMMA 3. *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized (r, s, t, u) sequence $\{W_n\}_{n \geq 0}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{1 - rz - sz^2 - tz^3 - uz^4}. \tag{1.10}$$

Proof. For a proof, see Soykan [9, Lemma 1]. \square

The following theorem presents Simson's formula of generalized (r, s, t, u) sequence (generalized Tetranacci sequence) $\{W_n\}$.

THEOREM 4 (Simson's Formula of Generalized (r, s, t, u) Numbers). *For all integers n , we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{1.11}$$

Proof. (1.11) is given in Soykan [8]. \square

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

THEOREM 5. For $n \in \mathbb{Z}$, for the generalized Tetranacci sequence (or generalized (r, s, t, u) -sequence or 4-step Fibonacci sequence) we have the following:

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_nW_{2n} - 3H_n^2W_n + 3H_{2n}W_n + W_0H_n^3 + 2W_0H_{3n} - 3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n} - H_nW_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [10, Theorem 1]. \square

Using Theorem 5, we have the following corollary, see Soykan [10, Corollary 4].

COROLLARY 6. For $n \in \mathbb{Z}$, we have

(a): $2(-u)^{n+4}G_{-n} = -(3ru^2 + t^3 - 3stu)^2G_n^3 - (2su - t^2)^2G_{n+3}^2G_n - (-rt^2 - tu + 2rsu)^2G_{n+2}^2G_n - (-st^2 + 2s^2u + 4u^2 + rtu)^2G_{n+1}^2G_n + 2(3ru^2 + t^3 - 3stu)((-2su + t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1})G_n^2 + 2(2su - t^2)(-rt^2 - tu + 2rsu)G_{n+3}G_{n+2}G_n + 2(2su - t^2)(-st^2 + 2s^2u + 4u^2 + rtu)G_{n+3}G_{n+1}G_n - 2(-st^2 + 2s^2u + 4u^2 + rtu)(-rt^2 - tu + 2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su + t^2)G_{2n+3}G_n + u^2(-rt^2 - tu + 2rsu)G_{2n+2}G_n + u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n+1}G_n - 2u^2(2su - t^2)G_{2n}G_{n+3} + 2u^2(-rt^2 - tu + 2rsu)G_{2n}G_{n+2} + 2u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n}G_{n+1} - 3u^2(3ru^2 + t^3 - 3stu)G_{2n}G_n.$

(b): $H_{-n} = \frac{1}{6}(-u)^{-n}(H_n^3 + 2H_{3n} - 3H_{2n}H_n).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 4$ in Theorem 5,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n} + 6H_nG_{2n} - 3H_n^2G_n + 3H_{2n}G_n), \tag{1.12}$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}(H_n^3 + 2H_{3n} - 3H_{2n}H_n), \tag{1.13}$$

respectively.

If we define the square matrix A of order 4 as

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix}$$

then we get the following Theorem.

THEOREM 7. *For all integers m, n , we have*

(a): $B_n = A^n$, i.e.,

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

(b): $U_1 A^n = A^n U_1$.

(c): $U_{n+m} = U_n B_m = B_m U_n$.

Proof. For the proof, see Soykan [9, Theorem 19]. \square

THEOREM 8. *For all integers m, n , we have*

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \tag{1.14}$$

Proof. For the proof, see Soykan [9, Theorem 20]. \square

In the next sections, we present new results.

2. Generalized Blaise Sequence

In this paper, we consider the case $r = 1, s = 1, t = 1, u = -2$. A generalized Blaise sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n \geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = W_{n-1} + W_{n-2} + W_{n-3} - 2W_{n-4} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero.

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)} + \frac{1}{2}W_{-(n-3)} - \frac{1}{2}W_{-(n-4)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (2.1) holds for all integers n .

Characteristic equation of $\{W_n\}$ is

$$z^4 - z^3 - z^2 - z + 2 = (z^3 - z - 2)(z - 1) = 0$$

whose roots are

$$\begin{aligned}\alpha &= \sqrt[3]{1 + \frac{\sqrt{78}}{9}} + \sqrt[3]{1 - \frac{\sqrt{78}}{9}} \simeq 1.521379706804568, \\ \beta &= \omega \sqrt[3]{1 + \frac{\sqrt{78}}{9}} + \omega^2 \sqrt[3]{1 - \frac{\sqrt{78}}{9}}, \\ \gamma &= \omega^2 \sqrt[3]{1 + \frac{\sqrt{78}}{9}} + \omega \sqrt[3]{1 - \frac{\sqrt{78}}{9}}, \\ \delta &= 1,\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}\alpha + \beta + \gamma + \delta &= 1, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -1, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\ \alpha\beta\gamma\delta &= 2.\end{aligned}$$

Note also that

$$\begin{aligned}\alpha + \beta + \gamma &= 0, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 2.\end{aligned}$$

The first few generalized Blaise numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Blaise numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{2}(W_0 + W_1 + W_2 - W_3)$
2	W_2	$\frac{1}{4}(3W_0 + 3W_1 - W_2 - W_3)$
3	W_3	$\frac{1}{8}(9W_0 + W_1 + W_2 - 3W_3)$
4	$W_1 - 2W_0 + W_2 + W_3$	$\frac{1}{16}(11W_0 + 11W_1 + 3W_2 - 9W_3)$
5	$2W_2 - W_1 - 2W_0 + 2W_3$	$\frac{1}{32}(33W_0 + 17W_1 - 7W_2 - 11W_3)$
6	$W_2 - 4W_0 + 4W_3$	$\frac{1}{64}(67W_0 + 19W_1 + 11W_2 - 33W_3)$
7	$4W_2 - 8W_0 + 5W_3$	$\frac{1}{128}(105W_0 + 89W_1 + W_2 - 67W_3)$
8	$5W_2 - 3W_1 - 10W_0 + 9W_3$	$\frac{1}{256}(283W_0 + 107W_1 - 29W_2 - 105W_3)$
9	$6W_2 - W_1 - 18W_0 + 14W_3$	$\frac{1}{512}(497W_0 + 225W_1 + 73W_2 - 283W_3)$
10	$13W_2 - 4W_1 - 28W_0 + 20W_3$	$\frac{1}{1024}(947W_0 + 643W_1 - 69W_2 - 497W_3)$
11	$16W_2 - 8W_1 - 40W_0 + 33W_3$	$\frac{1}{2048}(2233W_0 + 809W_1 - 47W_2 - 947W_3)$
12	$25W_2 - 7W_1 - 66W_0 + 49W_3$	$\frac{1}{4096}(3851W_0 + 2139W_1 + 339W_2 - 2233W_3)$
13	$42W_2 - 17W_1 - 98W_0 + 74W_3$	$\frac{1}{8192}(8129W_0 + 4529W_1 - 615W_2 - 3851W_3)$

Note that the sequences $\{B_n\}$ and $\{C_n\}$ which are defined in the section Introduction, are the special cases of the generalized Blaise sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$ in this section as well. Blaise sequence $\{B_n\}_{n \geq 0}$ and Blaise-Lucas sequence $\{C_n\}_{n \geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}
 B_n &= B_{n-1} + B_{n-2} + B_{n-3} - 2B_{n-4}, & B_0 = 0, B_1 = 1, B_2 = 1, B_3 = 2, & n \geq 4, \\
 C_n &= C_{n-1} + C_{n-2} + C_{n-3} - 2C_{n-4}, & C_0 = 4, C_1 = 1, C_2 = 3, C_3 = 7, & n \geq 4.
 \end{aligned}$$

The sequences $\{B_n\}_{n \geq 0}$ and $\{C_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
 B_{-n} &= \frac{1}{2}B_{-(n-1)} + \frac{1}{2}B_{-(n-2)} + \frac{1}{2}B_{-(n-3)} - \frac{1}{2}B_{-(n-4)} \\
 C_{-n} &= \frac{1}{2}C_{-(n-1)} + \frac{1}{2}C_{-(n-2)} + \frac{1}{2}C_{-(n-3)} - \frac{1}{2}C_{-(n-4)}
 \end{aligned}$$

for $n = 1, 2, 3, \dots$ respectively.

Next, we present the first few values of the Blaise and Blaise-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
B_n	0	1	1	2	4	5	9	14	20	33	49	74	116	173
B_{-n}	0	0	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{3}{8}$	$-\frac{9}{16}$	$-\frac{11}{32}$	$-\frac{33}{64}$	$-\frac{67}{128}$	$-\frac{105}{256}$	$-\frac{283}{512}$	$-\frac{497}{1024}$	$-\frac{947}{2048}$
C_n	4	1	3	7	3	11	15	15	35	43	63	111	147	235
C_{-n}	4	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{19}{8}$	$\frac{1}{16}$	$\frac{51}{32}$	$\frac{89}{64}$	$\frac{43}{128}$	$\frac{417}{256}$	$\frac{451}{512}$	$\frac{745}{1024}$	$\frac{2971}{2048}$	$\frac{2929}{4096}$	$\frac{8243}{8192}$

Theorem 1 can be used to obtain the Binet formula of generalized Blaise numbers. Using these (the above) roots and the recurrence relation, Binet’s formula of generalized Blaise numbers can be given as follows:

THEOREM 9. (Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n , Binet’s formula of generalized Blaise numbers is

$$W_n = \frac{(\alpha W_3 - \alpha(1 - \alpha)W_2 + (-\alpha^2 + 2)W_1 - 2W_0)\alpha^n}{2\alpha^2 + 4\alpha - 6} + \frac{(\beta W_3 - \beta(1 - \beta)W_2 + (-\beta^2 + 2)W_1 - 2W_0)\beta^n}{2\beta^2 + 4\beta - 6} + \frac{(\gamma W_3 - \gamma(1 - \gamma)W_2 + (-\gamma^2 + 2)W_1 - 2W_0)\gamma^n}{2\gamma^2 + 4\gamma - 6} - \frac{W_3 - W_1 - 2W_0}{2}$$

Blaise and Blaise-Lucas numbers can be expressed using Binet’s formulas as follows:

COROLLARY 10. (Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n , Binet’s formulas of Blaise and Blaise-Lucas numbers are

$$B_n = \frac{\alpha^{n+3}}{2\alpha^2 + 4\alpha - 6} + \frac{\beta^{n+3}}{2\beta^2 + 4\beta - 6} + \frac{\gamma^{n+3}}{2\gamma^2 + 4\gamma - 6} - \frac{1}{2}$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1$$

respectively.

Note that for all integers n , adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers can be expressed using Binet’s formulas as

$$\begin{aligned} K_n &= \frac{1}{(\alpha - \beta)(\alpha - \gamma)}\alpha^{n+1} + \frac{1}{(\beta - \alpha)(\beta - \gamma)}\beta^{n+1} + \frac{1}{(\gamma - \alpha)(\gamma - \beta)}\gamma^{n+1}, \\ L_n &= \alpha^n + \beta^n + \gamma^n, \\ M_n &= \frac{(3\alpha + 1)}{(\alpha - \beta)(\alpha - \gamma)}\alpha^{n+1} + \frac{(3\beta + 1)}{(\beta - \alpha)(\beta - \gamma)}\beta^{n+1} + \frac{(3\gamma + 1)}{(\gamma - \alpha)(\gamma - \beta)}\gamma^{n+1}, \\ Q_n &= \frac{(\alpha + 1)}{(\alpha - \beta)(\alpha - \gamma)}\alpha^{n+1} + \frac{(\beta + 1)}{(\beta - \alpha)(\beta - \gamma)}\beta^{n+1} + \frac{(\gamma + 1)}{(\gamma - \alpha)(\gamma - \beta)}\gamma^{n+1}, \end{aligned}$$

respectively, see Soykan [14] for more details. So, by using Binet’s formulas of Blaise, Blaise-Lucas and Adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

LEMMA 11. For all integers n , the following equalities (identities) are true:

(a):

- $K_{n+1} = B_{n+1} - B_n$.
- $2K_n = B_{n+3} - B_{n+2} - B_{n+1} + B_n$.
- $2B_{n+4} = 5K_{n+2} + 9K_{n+1} + 6K_n - 1$.
- $2B_n = K_{n+2} + K_{n+1} + 2K_n - 1$.

- $2K_n = -B_{n+2} + 3B_n + 1.$

(b):

- $52K_{n+3} = 12C_{n+3} + 17C_{n+2} - 11C_{n+1} - 18C_n.$
- $52K_n = C_{n+3} - 2C_{n+1} + 9C_{n+2} - 8C_n.$
- $C_{n+4} = 6K_{n+2} + 2K_{n+1} + 4K_n + 1.$
- $2C_n = -K_{n+2} + 6K_{n+1} + K_n + 2.$
- $52K_n = 9C_{n+2} - C_{n+1} - 6C_n - 2.$
- $C_{n+1} + 6C_n = 18K_{n+1} + 2K_n + 7.$

(c):

- $L_{n+3} = 2B_{n+2} + 4B_{n+1} - 6B_n.$
- $4L_n = B_{n+3} - 3B_{n+2} + 13B_{n+1} - 11B_n.$
- $52B_{n+4} = 30L_{n+2} + 53L_{n+1} + 58L_n - 26.$
- $52B_n = 2L_n + 9L_{n+1} + 10L_{n+2} - 26.$
- $4L_n = -3B_{n+2} + 14B_{n+1} - 9B_n + 1.$
- $4(10B_{n+1} - 9B_n) = 14L_n + 3L_{n+1} - 2.$

(d):

- $2L_{n+3} = 3C_{n+3} - C_{n+1} - 2C_n.$
- $2L_n = C_{n+3} - C_{n+1}.$
- $C_{n+4} = L_{n+2} + 2L_{n+1} + 1.$
- $C_n = L_n + 1.$
- $L_n = C_n - 1.$

(e):

- $Q_{n+3} = B_{n+3} + B_{n+1} - 2B_n.$
- $2Q_n = B_{n+3} - B_{n+2} + B_{n+1} - B_n.$
- $2B_{n+4} = Q_{n+2} + 4Q_{n+1} + 4Q_n - 1.$
- $2B_n = Q_{n+2} - 1.$
- $2Q_n = -B_{n+2} + 2B_{n+1} + B_n + 1.$
- $2B_{n+1} = Q_{n+1} + 2Q_n - 1.$

(f):

- $52Q_{n+3} = 41C_{n+3} + 18C_{n+2} - 17C_{n+1} - 42C_n.$
- $52Q_n = 11C_{n+3} + 8C_{n+2} - 9C_{n+1} - 10C_n.$
- $C_{n+4} = 4Q_{n+2} + 2Q_{n+1} - 4Q_n + 1.$
- $2C_n = -3Q_{n+2} + 2Q_{n+1} + 7Q_n + 2.$
- $26Q_n = 4C_{n+2} + C_{n+1} + 6C_n - 11.$
- $3C_{n+1} + 2C_n = 8Q_{n+1} - 2Q_n + 5.$

(g):

- $M_{n+3} = B_{n+3} + 2B_{n+2} + 3B_{n+1} - 6B_n.$
- $2M_n = B_{n+3} - B_{n+2} + 5B_{n+1} - 5B_n.$
- $46B_{n+4} = 16M_{n+2} + 33M_{n+1} + 42M_n - 23.$
- $46B_n = 8M_{n+2} + 5M_{n+1} - 2M_n - 23.$
- $2M_n = -B_{n+2} + 6B_{n+1} - 3B_n + 1.$
- $2(8B_{n+1} - 5B_n) = M_{n+1} + 6M_n - 3.$

(h):

- $52M_{n+3} = 99C_{n+3} + 20C_{n+2} - 29C_{n+1} - 90C_n.$
- $52M_n = 31C_{n+3} + 6C_{n+2} - 23C_{n+1} - 14C_n.$
- $23C_{n+4} = 18M_{n+2} + 40M_{n+1} - 16M_n + 23.$
- $46C_n = -5M_{n+2} - 6M_{n+1} + 53M_n + 46.$
- $26M_n = 3C_{n+2} + 4C_{n+1} + 24C_n - 31.$
- $6C_n - 5C_{n+1} = 8M_n - 6M_{n+1} + 1.$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n .

LEMMA 12. *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Blaise sequence $\{W_n\}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - W_0)z + (W_2 - W_1 - W_0)z^2 + (W_3 - W_2 - W_1 - W_0)z^3}{1 - z - z^2 - z^3 + 2z^4}.$$

Proof. Take $r = 1, s = 1, t = 1, u = -2$ in Lemma 3.

The previous lemma gives the following results as particular examples.

COROLLARY 13. *Generating functions of Blaise and Blaise-Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} B_n z^n &= \frac{z}{1 - z - z^2 - z^3 + 2z^4}, \\ \sum_{n=0}^{\infty} C_n z^n &= \frac{4 - 3z - 2z^2 - z^3}{1 - z - z^2 - z^3 + 2z^4}, \end{aligned}$$

respectively.

3. Simson Formulas

Now, we present Simson's formula of generalized Blaise numbers.

THEOREM 14 (Simson's Formula of Generalized Blaise Numbers). *For all integers n , we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = 2^{n-3}(W_3 - W_1 - 2W_0)(W_3^3 + 2W_2^3 - W_1^3 - 4W_0^3 + (-3W_2 - W_1 + W_0)W_3^2 + 2(W_3 - W_1 - 3W_0)W_2^2 + (5W_3 + 3W_2 - 5W_0)W_1^2 + 4(W_2 + 2W_1)W_0^2 - 2W_1W_2W_3 + 4W_0W_2W_3 - 6W_0W_1W_3).$$

Proof. Take $r = 1, s = 1, t = 1, u = -2$ in Theorem 4. \square

The previous theorem gives the following results as particular examples.

COROLLARY 15. *For all integers n , the Simson's formulas of Blaise and Blaise-Lucas numbers are given*

as

$$\begin{vmatrix} B_{n+3} & B_{n+2} & B_{n+1} & B_n \\ B_{n+2} & B_{n+1} & B_n & B_{n-1} \\ B_{n+1} & B_n & B_{n-1} & B_{n-2} \\ B_n & B_{n-1} & B_{n-2} & B_{n-3} \end{vmatrix} = 2^{n-1},$$

$$\begin{vmatrix} C_{n+3} & C_{n+2} & C_{n+1} & C_n \\ C_{n+2} & C_{n+1} & C_n & C_{n-1} \\ C_{n+1} & C_n & C_{n-1} & C_{n-2} \\ C_n & C_{n-1} & C_{n-2} & C_{n-3} \end{vmatrix} = -13 \times 2^{n+2},$$

respectively.

4. Some Identities

In this section, we obtain some identities of Blaise and Blaise-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{B_n\}$.

LEMMA 16. *The following equalities are true:*

- (a): $16W_n = (11W_0 + 11W_1 + 3W_2 - 9W_3)B_{n+5} + (7W_0 - 9W_1 - W_2 + 3W_3)B_{n+4} - (17W_0 + W_1 + 9W_2 - 11W_3)B_{n+3} - (33W_0 + 17W_1 - 7W_2 - 11W_3)B_{n+2}.$
- (b): $8W_n = (9W_0 + W_1 + W_2 - 3W_3)B_{n+4} - (3W_0 - 5W_1 + 3W_2 - W_3)B_{n+3} - (11W_0 + 3W_1 - 5W_2 - W_3)B_{n+2} - (11W_0 + 11W_1 + 3W_2 - 9W_3)B_{n+1}.$
- (c): $4W_n = (3W_0 + 3W_1 - W_2 - W_3)B_{n+3} - (W_0 + W_1 - 3W_2 + W_3)B_{n+2} - (W_0 + 5W_1 + W_2 - 3W_3)B_{n+1} - (9W_0 + W_1 + W_2 - 3W_3)B_n.$
- (d): $2W_n = (W_0 + W_1 + W_2 - W_3)B_{n+2} + (W_0 - W_1 - W_2 + W_3)B_{n+1} - (3W_0 - W_1 + W_2 - W_3)B_n - (3W_0 + 3W_1 - W_2 - W_3)B_{n-1}.$
- (e): $W_n = W_0B_{n+1} + (W_1 - W_0)B_n + (W_2 - W_1 - W_0)B_{n-1} + (W_3 - W_1 - W_2 - W_0)B_{n-2}.$

Proof. Note that all the identities hold for all integers n . We prove (a). To show (a), writing

$$W_n = a \times B_{n+5} + b \times B_{n+4} + c \times B_{n+3} + d \times B_{n+2}$$

and solving the system of equations

$$W_0 = a \times B_5 + b \times B_4 + c \times B_3 + d \times B_2$$

$$W_1 = a \times B_6 + b \times B_5 + c \times B_4 + d \times B_3$$

$$W_2 = a \times B_7 + b \times B_6 + c \times B_5 + d \times B_4$$

$$W_3 = a \times B_8 + b \times B_7 + c \times B_6 + d \times B_5$$

we find that $16a = 11W_0 + 11W_1 + 3W_2 - 9W_3$, $16b = 7W_0 - 9W_1 - W_2 + 3W_3$, $16c = 11W_3 - W_1 - 9W_2 - 17W_0$, $16d = 7W_2 - 17W_1 - 33W_0 + 11W_3$. The other equalities can be proved similarly. \square

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{C_n\}$.

LEMMA 17. *The following equalities are true:*

$$\text{(a): } 104W_n = -(56W_0 + 27W_1 + 2W_2 - 33W_3)C_{n+5} - 2(5W_0 - 2W_1 - 4W_2 + W_3)C_{n+4} + (60W_0 + 15W_1 + 4W_2 - 27W_3)C_{n+3} + 2(55W_0 + 30W_1 - 5W_2 - 28W_3)C_{n+2}.$$

$$\text{(b): } 104W_n = -(66W_0 + 23W_1 - 6W_2 - 31W_3)C_{n+4} + 2(2W_0 - 6W_1 + W_2 + 3W_3)C_{n+3} + (54W_0 + 33W_1 - 12W_2 - 23W_3)C_{n+2} + 2(56W_0 + 27W_1 + 2W_2 - 33W_3)C_{n+1}.$$

$$\text{(c): } 104W_n = -(62W_0 + 35W_1 - 8W_2 - 37W_3)C_{n+3} - 2(6W_0 - 5W_1 + 3W_2 - 4W_3)C_{n+2} + (46W_0 + 31W_1 + 10W_2 - 35W_3)C_{n+1} + 2(66W_0 + 23W_1 - 6W_2 - 31W_3)C_n.$$

$$\text{(d): } 104W_n = -(74W_0 + 25W_1 - 2W_2 - 45W_3)C_{n+2} - 2(8W_0 + 2W_1 - 9W_2 - W_3)C_{n+1} + (70W_0 + 11W_1 - 4W_2 - 25W_3)C_n + 2(62W_0 + 35W_1 - 8W_2 - 37W_3)C_{n-1}.$$

$$\text{(e): } 104W_n = -(90W_0 + 29W_1 - 20W_2 - 47W_3)C_{n+1} - 2(2W_0 + 7W_1 + W_2 - 10W_3)C_n + (50W_0 + 45W_1 - 14W_2 - 29W_3)C_{n-1} + 2(74W_0 + 25W_1 - 2W_2 - 45W_3)C_{n-2}.$$

Now, we give a few basic relations between $\{B_n\}$ and $\{C_n\}$.

LEMMA 18. *The following equalities are true:*

$$104B_n = 37C_{n+5} + 8C_{n+4} - 35C_{n+3} - 62C_{n+2},$$

$$104B_n = 45C_{n+4} + 2C_{n+3} - 25C_{n+2} - 74C_{n+1},$$

$$104B_n = 47C_{n+3} + 20C_{n+2} - 29C_{n+1} - 90C_n,$$

$$104B_n = 67C_{n+2} + 18C_{n+1} - 43C_n - 94C_{n-1},$$

$$104B_n = 85C_{n+1} + 24C_n - 27C_{n-1} - 134C_{n-2},$$

and

$$\begin{aligned}
 16C_n &= B_{n+5} + 37B_{n+4} - 19B_{n+3} - 51B_{n+2}, \\
 8C_n &= 19B_{n+4} - 9B_{n+3} - 25B_{n+2} - B_{n+1}, \\
 4C_n &= 5B_{n+3} - 3B_{n+2} + 9B_{n+1} - 19B_n, \\
 2C_n &= B_{n+2} + 7B_{n+1} - 7B_n - 5B_{n-1}, \\
 C_n &= 4B_{n+1} - 3B_n - 2B_{n-1} - B_{n-2}.
 \end{aligned}$$

5. Relations Between Special Numbers

In this section, we present identities on Blaise, Blaise-Lucas numbers and adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers. We know from Lemma 11 that

$$\begin{aligned}
 52B_n &= 2L_n + 9L_{n+1} + 10L_{n+2} - 26, \\
 2C_n &= -3Q_{n+2} + 2Q_{n+1} + 7Q_n + 2.
 \end{aligned}$$

Note also that from Lemma 16 and Lemma 17, we have the formulas of W_n as

$$\begin{aligned}
 4W_n &= (3W_0 + 3W_1 - W_2 - W_3)B_{n+3} - (W_0 + W_1 - 3W_2 + W_3)B_{n+2} \\
 &\quad - (W_0 + 5W_1 + W_2 - 3W_3)B_{n+1} - (9W_0 + W_1 + W_2 - 3W_3)B_n, \\
 104W_n &= -(62W_0 + 35W_1 - 8W_2 - 37W_3)C_{n+3} - 2(6W_0 - 5W_1 + 3W_2 - 4W_3)C_{n+2} \\
 &\quad + (46W_0 + 31W_1 + 10W_2 - 35W_3)C_{n+1} + 2(66W_0 + 23W_1 - 6W_2 - 31W_3)C_n.
 \end{aligned}$$

Using the above identities, we obtain relation of generalized Blaise numbers in the following forms (in terms of Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) and Jacobsthal-Padovan numbers):

LEMMA 19. *For all integers n , we have the following identities:*

$$\begin{aligned}
 \text{(a): } 52W_n &= (4W_3 - 3W_2 + 5W_1 - 6W_0)L_{n+2} + (W_3 + 9W_2 - 2W_1 - 8W_0)L_{n+1} + 2(3W_3 + W_2 - 6W_1 + \\
 &\quad 2W_0)L_n - 26W_3 + 26W_1 + 52W_0. \\
 \text{(b): } 2W_n &= (W_1 - W_0)Q_{n+2} + (W_2 - W_1)Q_{n+1} + (W_3 - W_2 - W_1 + W_0)Q_n - W_3 + W_1 + 2W_0.
 \end{aligned}$$

6. On the Recurrence Properties of Generalized Blaise Sequence

Taking $r = 1, s = 1, t = 1, u = -2$ in Theorem 5, we obtain the following Proposition.

PROPOSITION 20. *For $n \in \mathbb{Z}$, generalized Blaise numbers (the case $r = 1, s = 1, t = 1, u = -2$) have the following identity:*

$$W_{-n} = \frac{2^{-n-1}}{3} (-6W_{3n} + 6C_n W_{2n} - 3C_n^2 W_n + 3C_{2n} W_n + W_0 C_n^3 + 2W_0 C_{3n} - 3W_0 C_n C_{2n}).$$

From the above Proposition 20 (or by taking $G_n = B_n$ and $H_n = C_n$ in (1.12) and (1.13) respectively), we have the following corollary which gives the connection between the special cases of generalized Blaise sequence at the positive index and the negative index: for Blaise and Blaise-Lucas numbers: take $W_n = B_n$ with $B_0 = 0, B_1 = 1, B_2 = 1, B_3 = 2$ and take $W_n = C_n$ with $C_0 = 4, C_1 = 1, C_2 = 3, C_3 = 7$, respectively. Note that in this case $H_n = C_n$.

COROLLARY 21. For $n \in \mathbb{Z}$, we have the following recurrence relations:

(a): Blaise sequence:

$$B_{-n} = \frac{2^{-n-1}}{3}(-6B_{3n} + 6C_n B_{2n} - 3C_n^2 B_n + 3C_{2n} B_n).$$

(b): Blaise-Lucas sequence:

$$C_{-n} = \frac{2^{-n-1}}{3}(C_n^3 + 2C_{3n} - 3C_{2n} C_n).$$

We can also present the formulas of B_{-n} and C_{-n} in the following forms.

COROLLARY 22. For $n \in \mathbb{Z}$, we have the following recurrence relations:

(a): $B_{-n} = \frac{2^{-n-5}}{3}(-96B_{3n} + 24(5B_{n+3} - 3B_{n+2} + 9B_{n+1} - 19B_n)B_{2n} - 3(5B_{n+3} - 3B_{n+2} + 9B_{n+1} - 19B_n)^2 B_n + 12(5B_{2n+3} - 3B_{2n+2} + 9B_{2n+1} - 19B_{2n})B_n).$

(b):

(i): $2^{n+1}B_{-n} = -K_n^2 - K_{n-1}^2 - 2K_{n-2}^2 + (K_{n+2} - 6K_{n+1} - 6K_{n-1} + 2K_{n-2})K_n + (K_{n+1} - 12K_{n-2})K_{n-1} + 2K_{2n} + 2K_{2n-2} + 4K_{2n-4} - 2^n.$

(ii): $2^{n+2}C_{-n} = -K_n^2 - 12K_{n-1}^2 + 4K_{n-2}^2 + (K_{n+2} - 6K_{n+1} - 72K_{n-1} - 4K_{n-2})K_n + 12(K_{n+1} + 2K_{n-2})K_{n-1} + 2K_{2n} + 24K_{2n-2} - 8K_{2n-4} + 2^{n+2}.$

(c):

(i): $13B_{-n} = \frac{1}{2^{n+2}}(L_n^2 + 9L_{n-1}^2 + 20L_{n-2}^2 - L_{2n} - 9L_{2n-2} - 20L_{2n-4} - 13 \times 2^{n+1}).$

(ii): $2^{n+1}C_{-n} = L_n^2 - L_{2n} + 2^{n+1}.$

(d):

(i): $2^{n+2}B_{-n} = 9Q_n^2 + 4Q_{n-1}^2 + 21Q_{n-2}^2 + 6Q_{2n-2} - 4Q_{2n-3} - 6Q_{2n-4} - 12Q_n Q_{n-1} - 30Q_n Q_{n-2} + 20Q_{n-1} Q_{n-2} - 2^{n+1}.$

(ii): $2^{n+4}C_{-n} = 63Q_{n+2}^2 + 64Q_{n+1}^2 + 55Q_n^2 + 36Q_{n-1}^2 - 252Q_{n-2}^2 + 2(-105Q_{n+2} + 46Q_{n+1} + 112Q_{n-1} + 180Q_{n-2})Q_n - 120(Q_{n+1} + 2Q_{n-2})Q_{n-1} - 84Q_{n+1}Q_{n+2} + 42Q_{2n+2} - 28Q_{2n+1} - 18Q_{2n} - 16Q_{2n-1} - 96Q_{2n-2} + 48Q_{2n-3} + 72Q_{2n-4} + 2^{n+4}.$

(e):

(i): $12167B_{-n} = \frac{1}{2^{n+3}}(-75M_{n+2}^2 + 267M_{n+1}^2 - 1811M_n^2 + 19483M_{n-1}^2 + 56816M_{n-2}^2 + 2(+565M_{n+2} + 1128M_{n+1} - 1950M_{n-1} - 9040M_{n-2})M_n - 226(25M_{n+1} + 96M_{n-2})M_{n-1} - 180M_{n+1}M_{n+2} + 6532M_{2n} + 4140M_{2n-1} - 828M_{2n+1} - 4370M_{2n-2} - 690M_{2n+2} + 13248M_{2n-3} - 49312M_{2n-4} - 12167 \times 2^{n+2}).$

$$(ii): 12167C_{-n} = \frac{1}{2^{n+4}}(3975M_{n+2}^2 + 4824M_{n+1}^2 + 185407M_n^2 - 44772M_{n-1}^2 - 71020M_{n-2}^2 + 2(-29945M_{n+2} - 37014M_{n+1} + 6336M_{n-1} + 11300M_{n-2})M_n + 13560(M_{n+1} + 2M_{n-2})M_{n-1} + 9540M_{n+1}M_{n+2} + 36570M_{2n+2} + 43884M_{2n+1} - 171626M_{2n} - 9936M_{2n-1} + 23184M_{2n-2} - 16560M_{2n-3} + 61640M_{2n-4} + 12167 \times 2^{n+4}).$$

Proof. We use the identities, see Soykan [13],

$$K_{-n} = \frac{1}{2^{n+1}}(-K_n^2 + 2K_{2n} + K_{n+2}K_n - 6K_{n+1}K_n),$$

$$L_{-n} = \frac{1}{2^{n+1}}(L_n^2 - L_{2n}),$$

and

$$Q_{-n} = \frac{1}{2^{n+3}}(9Q_{n+2}^2 + 4Q_{n+1}^2 + 21Q_n^2 + 6Q_{2n+2} - 4Q_{2n+1} - 6Q_{2n} - 12Q_{n+2}Q_{n+1} - 30Q_{n+2}Q_n + 20Q_{n+1}Q_n),$$

$$M_{-n} = \frac{1}{529 \times 2^{n+3}}(108M_{n+1}^2 + 75M_{n+2}^2 + 3551M_n^2 + 690M_{2n+2} + 828M_{2n+1} - 3082M_{2n} + 180M_{n+2}M_{n+1} - 1130M_{n+2}M_n - 1356M_{n+1}M_n).$$

We also use the identities

$$2B_n = K_{n+2} + K_{n+1} + 2K_n - 1,$$

$$2C_n = -K_{n+2} + 6K_{n+1} + K_n + 2,$$

$$52B_n = 2L_n + 9L_{n+1} + 10L_{n+2} - 26,$$

$$C_n = L_n + 1,$$

$$2B_n = Q_{n+2} - 1,$$

$$2C_n = -3Q_{n+2} + 2Q_{n+1} + 7Q_n + 2,$$

$$46B_n = 8M_{n+2} + 5M_{n+1} - 2M_n - 23,$$

$$46C_n = -5M_{n+2} - 6M_{n+1} + 53M_n + 46.$$

(a): By using the identity $4C_n = 5B_{n+3} - 3B_{n+2} + 9B_{n+1} - 19B_n$ and Corollary 18, (or by using Corollary 6 (a)), we obtain (a).

(b): Since $2B_n = K_{n+2} + K_{n+1} + 2K_n - 1$, $2C_n = -K_{n+2} + 6K_{n+1} + K_n + 2$ and $K_{-n} = \frac{1}{2^{n+1}}(-K_n^2 + 2K_{2n} + K_{n+2}K_n - 6K_{n+1}K_n)$, we get (b)

(c): Since $52B_n = 2L_n + 9L_{n+1} + 10L_{n+2} - 26$, $C_n = L_n + 1$ and $L_{-n} = \frac{1}{2^{n+1}}(L_n^2 - L_{2n})$, we obtain (c).

(d): Since $2B_n = Q_{n+2} - 1$, $2C_n = -3Q_{n+2} + 2Q_{n+1} + 7Q_n + 2$ and $Q_{-n} = \frac{1}{2^{n+3}}(9Q_{n+2}^2 + 4Q_{n+1}^2 + 21Q_n^2 + 6Q_{2n+2} - 4Q_{2n+1} - 6Q_{2n} - 12Q_{n+2}Q_{n+1} - 30Q_{n+2}Q_n + 20Q_{n+1}Q_n)$, we get (d).

(e): Since $46B_n = 8M_{n+2} + 5M_{n+1} - 2M_n - 23$, $46C_n = -5M_{n+2} - 6M_{n+1} + 53M_n + 46$ and $M_{-n} = \frac{1}{529 \times 2^{n+3}}(108M_{n+1}^2 + 75M_{n+2}^2 + 3551M_n^2 + 690M_{2n+2} + 828M_{2n+1} - 3082M_{2n} + 180M_{n+2}M_{n+1} - 1130M_{n+2}M_n - 1356M_{n+1}M_n)$, we obtain (e). \square

7. Sum Formulas

The following Corollary gives sum formulas of Jacobsthal-Padovan numbers.

COROLLARY 23. For $n \geq 0$ we have the following formulas:

- (a): $\sum_{k=0}^n Q_k = \frac{1}{2} (Q_{n+3} + Q_{n+2} - 2)$.
- (b): $\sum_{k=0}^n Q_{2k} = \frac{1}{2} (Q_{2n+1} + 2Q_{2n} - 1)$.
- (c): $\sum_{k=0}^n Q_{2k+1} = \frac{1}{2} (Q_{2n+2} + 2Q_{2n+1} - 1)$.

Proof. It is given in Soykan [14]. \square

The following Corollary presents sum formulas of Blaise and Blaise-Lucas numbers.

COROLLARY 24. For $n \geq 0$, Blaise and Blaise-Lucas numbers have the following properties (in terms of Jacobsthal-Padovan numbers):

- (a):
 - (i): $\sum_{k=0}^n B_k = \frac{1}{4} (3Q_{n+2} + 3Q_{n+1} + 2Q_n - 2(n+4))$.
 - (ii): $\sum_{k=0}^n B_{2k} = \frac{1}{4} (2Q_{2n+2} + Q_{2n+1} + 2Q_{2n} - 2n - 5)$.
 - (iii): $\sum_{k=0}^n B_{2k+1} = \frac{1}{4} (Q_{2n+2} + 4Q_{2n+1} + 4Q_{2n} - 2n - 5)$.
- (b):
 - (i): $\sum_{k=0}^n C_k = Q_{n+1} + 3Q_n + n$.
 - (ii): $\sum_{k=0}^n C_{2k} = -Q_{2n+2} + 2Q_{2n+1} + 2Q_{2n} + n + 1$.
 - (iii): $\sum_{k=0}^n C_{2k+1} = 2Q_{2n+2} + Q_{2n+1} - 2Q_{2n} + n$.

Proof. The proof follows from Corollary 23 and the identities

$$\begin{aligned} 2B_n &= Q_{n+2} - 1, \\ 2C_n &= -3Q_{n+2} + 2Q_{n+1} + 7Q_n + 2. \quad \square \end{aligned}$$

8. Matrices and Identities Related With Generalized Blaise Numbers

If we define the square matrix A of order 4 as

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} B_{n+1} & B_n + B_{n-1} - 2B_{n-2} & B_n - 2B_{n-1} & -2B_n \\ B_n & B_{n-1} + B_{n-2} - 2B_{n-3} & B_{n-1} - 2B_{n-2} & -2B_{n-1} \\ B_{n-1} & B_{n-2} + B_{n-3} - 2B_{n-4} & B_{n-2} - 2B_{n-3} & -2B_{n-2} \\ B_{n-2} & B_{n-3} + B_{n-4} - 2B_{n-5} & B_{n-3} - 2B_{n-4} & -2B_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & W_n + W_{n-1} - 2W_{n-2} & W_n - 2W_{n-1} & -2W_n \\ W_n & W_{n-1} + W_{n-2} - 2W_{n-3} & W_{n-1} - 2W_{n-2} & -2W_{n-1} \\ W_{n-1} & W_{n-2} + W_{n-3} - 2W_{n-4} & W_{n-2} - 2W_{n-3} & -2W_{n-2} \\ W_{n-2} & W_{n-3} + W_{n-4} - 2W_{n-5} & W_{n-3} - 2W_{n-4} & -2W_{n-3} \end{pmatrix}.$$

then we get the following Theorem.

THEOREM 25. *For all integers m, n , we have*

(a): $B_n = A^n$, i.e.,

$$\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & B_n + B_{n-1} - 2B_{n-2} & B_n - 2B_{n-1} & -2B_n \\ B_n & B_{n-1} + B_{n-2} - 2B_{n-3} & B_{n-1} - 2B_{n-2} & -2B_{n-1} \\ B_{n-1} & B_{n-2} + B_{n-3} - 2B_{n-4} & B_{n-2} - 2B_{n-3} & -2B_{n-2} \\ B_{n-2} & B_{n-3} + B_{n-4} - 2B_{n-5} & B_{n-3} - 2B_{n-4} & -2B_{n-3} \end{pmatrix}.$$

(b): $U_1 A^n = A^n U_1$.

(c): $U_{n+m} = U_n B_m = B_m U_n$.

Proof. Take $r = 1, s = 1, t = 1, u = -2$ in Theorem 7. \square

Using the above last Theorem and the identity

$$2B_n = Q_{n+2} - 1,$$

we obtain the following identity for Jacobsthal-Padovan numbers.

COROLLARY 26. *For all integers n , we have the following formula for Jacobsthal-Padovan numbers:*

$$A^n = \frac{1}{2} \begin{pmatrix} Q_{n+3} - 1 & Q_{n+4} - Q_{n+3} & -Q_{n+4} + 2Q_{n+2} + 1 & -2Q_{n+2} + 2 \\ Q_{n+2} - 1 & Q_{n+3} - Q_{n+2} & -Q_{n+3} + 2Q_{n+1} + 1 & -2Q_{n+1} + 2 \\ Q_{n+1} - 1 & Q_{n+2} - Q_{n+1} & -Q_{n+2} + 2Q_n + 1 & -2Q_n + 2 \\ Q_n - 1 & Q_{n+1} - Q_n & -Q_{n+1} + 2Q_{n-1} + 1 & -2Q_{n-1} + 2 \end{pmatrix}.$$

Next, we present an identity for W_{n+m} .

THEOREM 27. *For all integers m, n , we have*

$$W_{n+m} = W_n B_{m+1} + W_{n-1} (B_m + B_{m-1} - 2B_{m-2}) + W_{n-2} (B_m - 2B_{m-1}) - 2W_{n-3} B_m.$$

Proof. Take $r = 1, s = 1, t = 1, u = -2$ in Theorem 8. \square

As particular cases of the above theorem, we give identities for B_{n+m} and C_{n+m} .

COROLLARY 28. *For all integers m, n , we have*

$$B_{n+m} = B_n B_{m+1} + B_{n-1} (B_m + B_{m-1} - 2B_{m-2}) + B_{n-2} (B_m - 2B_{m-1}) - 2B_{n-3} B_m,$$

$$C_{n+m} = C_n B_{m+1} + C_{n-1} (B_m + B_{m-1} - 2B_{m-2}) + C_{n-2} (B_m - 2B_{m-1}) - 2C_{n-3} B_m.$$

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