Decomposition with the Mixed Model in Time Series Analysis using Buys-Ballot Procedure

Abstract: This article provides a general overview of the decomposition with the mixed model. The decomposition of such series into various components requires a method that can adequately estimate and investigate the trend parameters, seasonal indices and residual component of the series. In this article, the Buys-Ballot method of decomposition of time series is discussed with emphasis on the mixed model. The analysis indicates that, the estimated and computed trend parameters, seasonal indices and the residual components are listed in Table 6. Therefore, the residual mean obtained is 0.9749, while the variance is 0.0047. Hence, the fitted mixed decomposition model becomes

$$\hat{X}_t = (\ 2.9749 - 0.0016t\ )\hat{S}_t\ .$$

Keywords: Buy-Ballot method, time series decomposition, mixed model, transformation, linear trend component.

1 **Introduction**

One of tasks regularly used in time series analysis is the decomposition of a given time series into its various components. The classical decomposition procedure is equally known procedure of decomposing time series. Its applications is usually predicated on time series models. As the literature reveals, classical decomposition procedure has attracted so much research attention. The aims of the classical decomposition procedure have been mentioned in several studies. The advantages of the classical decomposition procedure are; it is used to investigate the presence of trend, seasonal, cyclical and error components in time series analysis. Time series analysis involve the separation of an observed series into components consisting trend (long term direction), seasonal (calendar related movements), cyclical (long term oscillations) and irregular (short term fluctuations).

The three time series models most commonly used are the

Additive Model: $X_t = T_t + S_t + C_t + I_t$ (1)

Multiplicative Model: $X_t = T_t \times S_t \times C_t \times I_t$ (2)

Mixed Model: $X_t = T_t \times S_t \times C_t + I_t$ (3)

For short term period in which cyclical and trend components are jointly combined Chatfield [1] and the observed time series $\left(X_t,\ t = 1, 2, ..., n\right)$ can be decomposed into the trend-cycle component $\left(M_t\right)$, seasonal component $\left(S_t\right)$ and the irregular component $\left(e_t\right)$. Therefore, the decomposition models are

Additive Model:

$$X_t = M_t + S_t + e_t \tag{4}$$

Multiplicative Model:

$$X_t = M_t \times S_t \times e_t \tag{5}$$

and Mixed Model

$$X_t = M_t \times S_t + e_t. \tag{6}$$

In this article, we observe that, any of the additive or multiplicative or mixed model may be used to effect the decomposition of a time series. The procedure of decomposition has involved the four basic components which make up a time series analysis. Also, we should emphasize that it is not an invariable rule for all components to be available. If yearly time series is confronted, there can be no seasonal component. Similarly, for short term period, the cyclical component can be ignored. In both cases one of the steps in the decomposition of time series outlined below may be omitted. In descriptive method of time series decomposition, the first step will normally be to estimate and then to eliminate trend-cycle (M_t) for each time period from the original data either by subtraction or division. The resulting time series after elimination the trend-cycle (M_t) is the de-trended series and expresses the effects of the season and irregular components. The de-trended series is expressed mathematically as:

$$X_t - \hat{M}_t \quad (7)$$

for the additive model or

$$X_t \,/\, \hat{M}_t \quad (8)$$

for the multiplicative model or

$$X_t \,/\, \hat{M}_t \quad (9)$$

for the mixed model

The seasonal effect is obtained by estimating the average of the de-trended series at each season. The de-trended, de-seasonalized series is obtained as

$$X_t - \hat{M}_t - \hat{S}_t$$

(10)

for the additive model, or

$$X_t \,/\, \left(\hat{M}_t \, \hat{S}_t\right)$$

(11)

for the multiplicative model

$$X_t \,/\, \left(\hat{M}_t \, \hat{S}_t\right)$$

(12)

for the mixed model

On when to use any of the three time series models, Chatfield [1] observed that, when the seasonal indices in direct proportion to the mean, then the seasonal indices is be multiplicative model shown in equation (2) may be applied. Additive model given in equation (1) is used, if the seasonal indices stays roughly the same size, regardless of the mean level. Nwogu, *et al*, [2] and Dozie, *et al*, [3] provided a test for choice of model based on Chi-Square distribution. Although time series data does not satisfy all the assumptions of most common statistical test, the Chi-Square test appears to be the most efficient among them. The proposed test is able to distinguish between the mixed and multiplicative models with a high degree of confidence.

2 Methodology.

The Buys-Ballot estimates of the row, column and overall means for the mixed model and their expected values derived by Dozie [4] are shown in equations (13), (14), (15), (16), (17) and (18) for linear trending curve.

$$\bar{X}_{i.} = \left[a - bs + bsi\right] + bC_1 + \bar{e}_{i.}$$

(13)

$$\bar{X}_{.j} = \left[a + b\left(\frac{n-s}{2}\right) + bj\right] \times S_j + \bar{e}_{.j}$$

(14)

$$\bar{X}_{..} = a + b\left(\frac{n-s}{2}\right) + bC_1 + \bar{e}_{..}$$

(15)

$$E\left(\bar{X}_{i.}\right) = \left(a - bs + bsi + \frac{b}{s}\sum_{j=1}^{s} jS_j\right) \quad (16)$$

$$E\left(\bar{X}_{.j}\right) = \left[a + b\left(\frac{n-s}{2}\right) + bj\right] \times S_j$$

(17)

$$E\left(\bar{X}_{..}\right) = a + b\left(\frac{n-s}{2}\right) + \frac{b}{s}\sum_{j=1}^{s} jS_j$$

(18)

$$\hat{a} = \bar{X}_{..} - b\left(\frac{n-s}{2}\right) + bC_1 + \bar{e}_{..}$$

(19)

$$S_j = \frac{\bar{X}_{.j}}{a + b\left(\frac{n-s}{2}\right) + bj}$$

(20)

Where $C_I = \frac{b}{s}\sum_{j=1}^{s} jS_j$, $\bar{e}_{i.} = \frac{1}{s}\sum_{j=1}^{s} e_{ij}$, $\bar{e}_{.j} = \frac{1}{m}\sum_{i=1}^{m} e_{ij}$, $\bar{e}_{..} = \frac{1}{m}\sum_{i=1}^{m} \bar{e}_{i.}$,

$E\left(\bar{e}_{i.}\right) = 0, \quad E\left(\bar{e}_{.j}\right) = 0 \quad E\left(\bar{e}_{..}\right) = 0$

For complete account of Buy-Ballot procedure, see the works of Wei [5], Iwueze *et.al* [6] Nwogu *et.al* [2], Dozie *et.al* [3], Dozie and Ijomah [7], Dozie and Nwanya [8], Dozie [4], Dozie and Uwaezuoke [9] and Dozie and Ibebuogu [10]. Dozie and Ihekuna [11] Akpanta and Iwueze [12].

2.1 Basic Properties of Means and Variances for Mixed Model

$(i)\ \frac{\bar{X}_{.j}}{\bar{X}_{..}}$ (21)

$(ii)\ \sum_{j=1}^{s}\left(\frac{\bar{X}_{.j}}{\bar{X}_{..}}\right) = s$ (22)

$(iii)\ \hat{\sigma}^2\left(\frac{\bar{X}_{.j}}{\bar{X}_{..}}\right)$ (23)

For equation (21) the overall means $(\bar{X}_{..})$ and the seasonal means $\left(\bar{X}_{.j},\ j = 1,2,...,s\right)$ of the Buys-Ballot table are used to assess seasonal indices as a ratios $\left(\frac{\bar{X}_{.j}}{\bar{X}_{..}}\right)$. For equation (22) the periodic means mimic the shape of the trending series of the original time series data and contain seasonal component in $C_1 = \sum_{j=1}^{s} jS_j$

For equation (23) the ratios of the seasonal means and overall means is used to assess the series with seasonal indices.

(iv) It is a product slope of seasonal indices

(v) A constant multiple of the square of the seasonal indices $\left(S_j^2\right)$

2.2 Estimation of Trend Parameters

The expression in equation (13) is $\equiv \alpha + \beta_i$ (24)

where $\hat{a} = \alpha + \hat{b}\left(s - c_1\right)$ (25)

$$\hat{b} = \frac{\beta}{s} \tag{26}$$

When there is no trend and $b = 0$, $\bar{X}_{..} = \hat{a}$ (27)

2.3 Estimation of Seasonal Indices $S_j, \quad \left(j = 1, 2, ..., s\right)$

The expression in equation (14) is $\equiv \left[\alpha + \beta_j\right] \times S_j$ (28)

where $\alpha = a + b\left(\frac{n-s}{2}\right)$ (29)

$\beta = b$ (30)

Hence, $S_j = \dfrac{\bar{X}_{.j}}{a + b\left(\frac{n-s}{2}\right) + bj}$

When there no trend and $b=0$, we obtain from (20)

$$S_j = \frac{\bar{X}_{.j}}{\bar{X}_{..}}$$

(31)

3. **Analysis**

In this section, we demonstrate the application of Buys-Ballot procedure for estimation of linear trend cycle and seasonal component and residual component using real life example. Appendix A shows monthly time series data on St Ambros Hospital in Aba from January, 2010 to December, 2019. The graphs of the time series data registered by the Hospital, Aba are shown in Figures 3.1, 3.2, 3.3 and 3.4. The data was transformed by taking the inverse square root of the one hundred and eight (108) observed values given in Appendix B. From the transformed series, the periodic and seasonal totals, means and standard deviations are obtained in Tables 5 and 6. The seasonal mean of the transformed series was plotted against the seasonal standard deviation in Figure 3.2. Since the time series data shows no evidence of $\hat{b} =$ 0 and $\hat{a} = X_{..}$ = 2.9749. That is, when $b=0$, a is estimated using the overall mean $\bar{X}_{..}$ The seasonal indices are estimated by averaging ratio $(\frac{\bar{X}_{.j}}{\bar{\bar{X}}_{..}})$ of the mixed decomposition model for each season given in Table 5 and plotted in Figure 3.3.

As Figures 3.1, 3.2, 3.3, 3.4 and Appendix A indicate, the time series data is seasonal with evidence of upward trend or downward trend. There is an upsurge of the series in May, June and November and a little drop in March, September and October. The periodic standard deviation are stable while the seasonal standard deviation differ, suggesting that the seasonal indices may be multiplicative or mixed model.

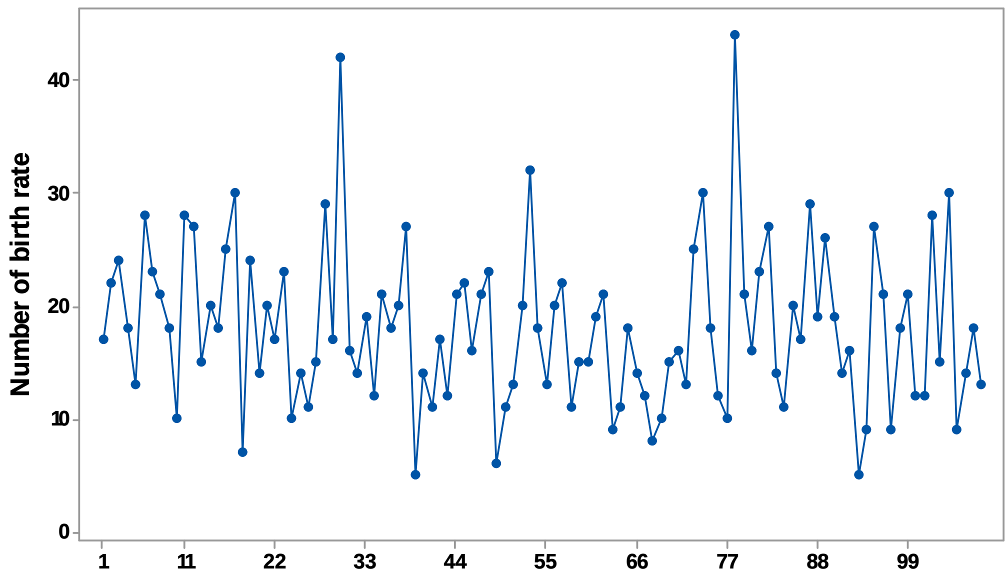

Fig.3.1: Plot of birth rate in Aba between 2011 to 2019

3.1 **Estimates of Trend and Seasonal Indices**

Trend and seasonal components are given as:

$$\bar{X}_{.j} = 2.8981 - 0.0016j$$

(32)

Using (25), (26) and (20), we obtain,

$$\hat{b} = -0.0016, \quad \hat{a} = 2.9749 \text{ and } \hat{S}_j = \frac{\hat{X}_{.j}}{2.8981 - 0.0016j}$$

The computational method for the Buys-Ballot estimates for the trend values obtained in Table 2. Observe that, the estimation method requires only the periodic means ($\bar{X}_{i.}$) for computation of the estimates. These means are extracted from the 9 periods of the Buys-Ballot table laid out in Appendix B. As Table 2 indicates, the Buys-Ballot estimates of the trend parameters are $\hat{a} =$ 2.9749 and $\hat{b} =$ -0.0016. The season means ($\bar{X}_{.j}$) required for the computation of the seasonal indices are based on the 9 periods of the Buys-Ballot table shown in Appendix A. The estimates of the seasonal indices are $\hat{S}_1 = 0.9332$, $\hat{S}_2 =$ 1.0114,

$\hat{S}_3 = 0.9391$, $\hat{S}_4 = 0.9769$, $\hat{S}_5 = 0.9840$, $\hat{S}_6 = 1.0556$, $\hat{S}_7 = 0.9627$, $\hat{S}_8 = 0.9777$, $\hat{S}_9 = 0.9346$, $\hat{S}_{10} = 0.9111$, $\hat{S}_{11} = 1.0362$, $\hat{S}_{12} = 0.9611$, for mixed model.

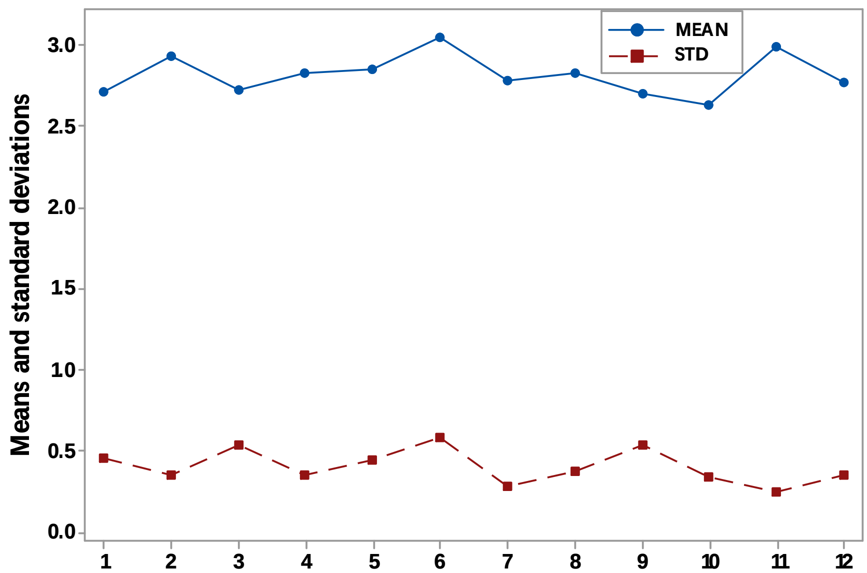

Fig.3.2: Seasonal means and standard deviations of birth rate

Table 1: **Estimates of Seasonal Indices**

j	$\bar{X}_{.j}$	$\hat{S}_j$
1	2.7030	0.9332
2	2.9280	1.0114
3	2.7170	0.9391
4	2.8250	0.9769
5	2.8440	0.9840
6	3.0490	1.0556
7	2.7792	0.9627
8	2.8210	0.9777
9	2.6950	0.9346
10	2.6260	0.9111
11	2.9847	1.0362
12	2.7670	0.9611

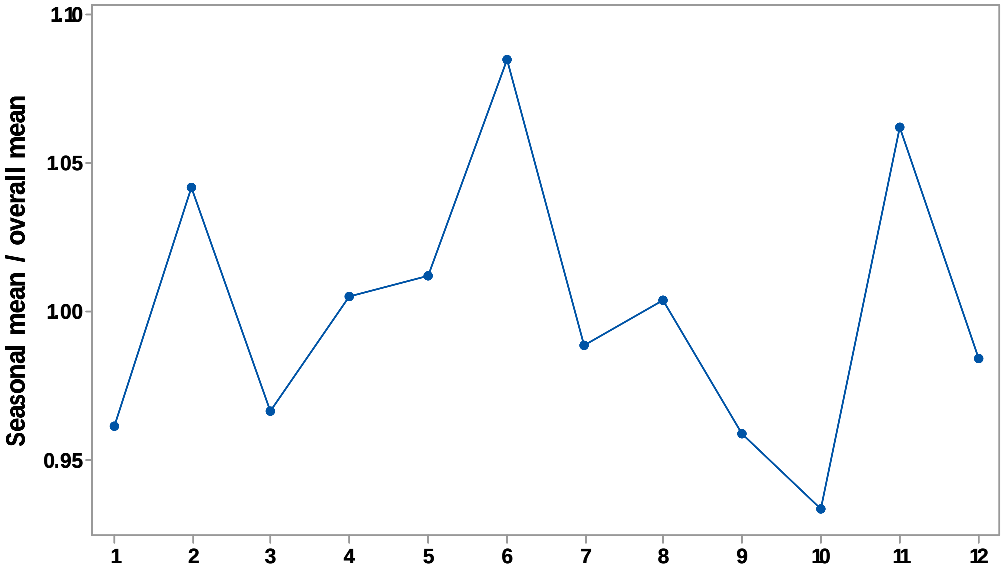

Fig.3.3: Seasonal means and overall means

Table 2: Estimates of trend and seasonal indices

Parameter	Mixed model
$\hat{a}$	2.9749
$\hat{b}$	-0.0016
$\hat{S}_1$	0.9332
$\hat{S}_2$	1.0114
$\hat{S}_3$	0.9391
$\hat{S}_4$	0.9769
$\hat{S}_5$	0.9840
$\hat{S}_6$	1.0556
$\hat{S}_7$	0.9627
$\hat{S}_8$	0.9777
$\hat{S}_9$	0.9346
$\hat{S}_{10}$	0.9111
$\hat{S}_{11}$	1.0362

$\hat{S}_{12}$	0.9611
$\sum_{j=1}^{s}\hat{S}_j$	12.0000

Note: mixed decomposition model satisfies

$$\left(\sum_{j=1}^{s} S_j = s\right) \text{ as in (12)}$$

Table 3**: Row totals, means and standard deviations**

Periods i	r_i	Linear trend cycle T_i	$\bar{X}_i$	$\sigma_{i.}$
1	9	35.89	2.99	0.32
2	9	34.24	2.85	0.41
3	9	34.46	2.87	0.38
4	9	33.34	2.78	0.46
5	9	32.57	2.71	0.43
6	9	31.03	2.59	0.30
7	9	35.37	2.95	0.45
8	9	33.90	2.83	0.50
9	9	32.84	2.74	0.39

$$n = \sum_{j=1}^{r} c_j = \sum_{i=1}^{c} r_i = total\ \ number\ \ of\ \ observation$$

Where,

r_i = Number of observation in the rth row

c_j = Number of observation in the jth column.

Table 4**: Seasonal totals, means and standard deviations**

Seasons j	c_j	Linear trend cycle $T._j$	$\bar{X}_{.j}$	$\sigma_{.j}$
1	12	24.32	2.70	0.45
2	12	26.35	2.93	0.35
3	12	24.45	2.72	0.54
4	12	25.42	2.83	0.34
5	12	25.60	2.84	0.44
6	12	27.44	3.05	0.58
7	12	25.01	2.78	0.27
8	12	25.39	2.82	0.37
9	12	24.26	2.70	0.53
10	12	23.63	2.63	0.33
11	12	26.86	2.98	0.25

12	12	24.91	2.77	0.34
Overall Total	144			

Table 5: **Estimates of Seasonal Indices**

j	$\bar{X}_{.j}$	$\frac{\bar{X}_{.j}}{\bar{X}_{..}}$
1	2.7030	0.9614
2	2.9280	1.0413
3	2.7170	0.9664
4	2.8250	1.0048
5	2.8440	1.0115
6	3.0490	1.0844
7	2.7792	0.9885
8	2.8210	1.0033
9	2.6950	0.9585
10	2.6260	0.9334
11	2.9847	1.0616
12	2.7670	0.9841

The estimated trend line for these data is

$\hat{T}_t = 2.9749 - 0.0016t$, with t = 1 in 2011 and estimated trend values given in Table 2. The estimates of the residuals obtained by dividing the original series by $\hat{M}_t$ *and* $\hat{S}_t$

Therefore, the residual mean obtained is 0.9749, while the variance is 0.0047. Hence, the fitted model becomes

$$\hat{X}_t = (2.9749 - 0.0016t)\hat{S}_t$$

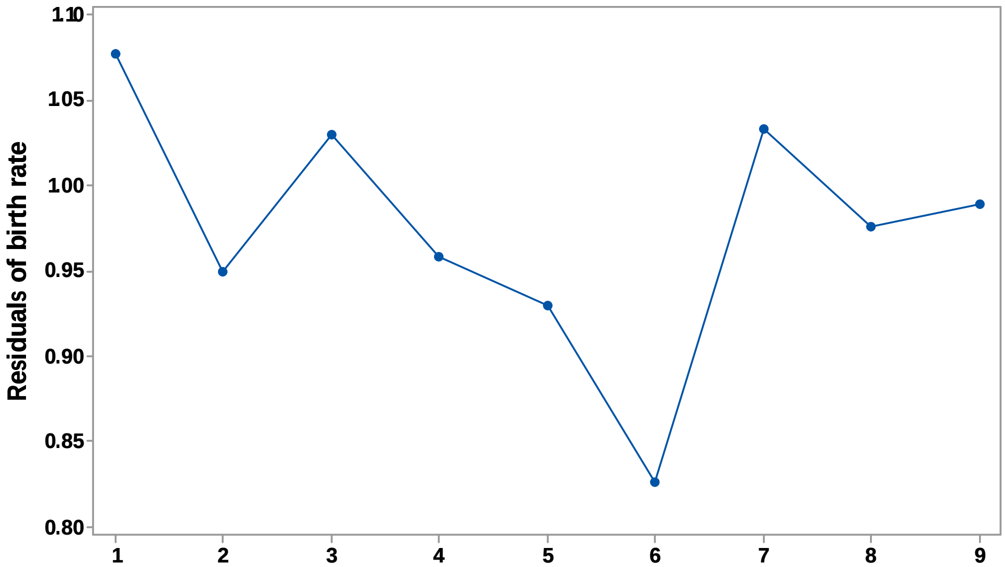

Fig.3.4: Residuals of birth rate, between 2011 to 2019

Table 6: **Estimates of Trend, Seasonal Indices and Irregular Component**

Year	t	Y_t	$\hat{T}_t$	$\hat{S}_t$	$\hat{Y}_t = \hat{T}_t \times \hat{S}_t$	$\hat{R}_t = \frac{Y_t}{\hat{Y}_t}$
2010	1	2.9909	2.9733	0.9322	2.7746	1.0780
2011	2	2.8540	2.9717	1.0114	3.0056	0.9496
2012	3	2.8720	2.9701	0.9391	2.7892	1.0297
2013	4	2.7790	2.9685	0.9769	2.8999	0.9586
2014	5	2.7140	2.9669	0.9840	2.9194	0.9298
2015	6	2.5855	2.9653	1.0556	3.1302	0.8260
2016	7	2.9470	2.9637	0.9627	2.8532	1.0329
2017	8	2.8250	2.9621	0.9777	2.8960	0.9755
2018	9	2.7360	2.9605	0.9346	2.7669	0.9888

4.0 Concluding Remarks

We have outlined the decomposition method with the mixed model and the technique for the estimation and investigation of trend-cycle, seasonal and residual components in time series analysis. This technique is computationally simple when compared with other descriptive techniques. The estimates of the trend-cycle component and seasonal effects are easily computed from periodic and seasonal averages. Hence, the computations are reduce to $\hat{a} = 2.9749$ *and* $\hat{b} = -0.0016$. The residual components of the estimates are obtained only empirically and listed in Table 6. Therefore, the residual mean obtained is 0.9749, while the variance is 0.0047. Hence, the fitted mixed decomposition model becomes

$\hat{X}_t = (\ 2.9749 - 0.0016t\ )\hat{S}_t$. Under acceptable assumption, the article shows that mixed model satisfies $\left(\sum_{j=1}^{s} S_j = s \right)$ as in (12)

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Appendix A: Buys-Ballot table for St Ambros Hospital Aba, Abia State, Nigeria (2011 – 2019)

Year	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sept.	Oct.	Nov.	Dec.	$\overline{X}_{i.}$	$\sigma^2_{i.}$
2011	17	22	24	18	13	28	23	21	18	10	28	27	20.75	33.30
2012	15	20	18	25	30	7	24	14	20	17	23	10	18.58	42.63
2013	14	11	15	29	17	42	16	14	19	12	21	18	19.00	75.09
2014	20	27	5	14	11	17	12	21	22	16	21	23	17.42	37.72
2015	6	11	13	20	32	18	13	20	22	11	15	15	16.33	45.15
2016	19	21	9	11	18	14	12	8	10	15	16	13	13.83	16.88
2017	25	30	18	12	10	44	21	16	23	27	14	11	20.92	95.54
2018	20	17	29	19	26	19	14	16	5	9	27	21	18.50	49.91
2019	9	18	21	12	12	28	15	30	9	14	18	13	16.58	46.63
$\overline{X}_{.j}$	16.11	19.67	16.89	17.78	18.78	24.11	16.67	17.78	16.44	14.56	20.33	16.78		
$\sigma^2_{.j}$	34.61	41.00	54.86	38.94	71.69	156.9	22.50	38.19	44.28	29.28	25.50	34.19		

Appendix B: Transformed series data for St Ambros Hosiptal Aba, Abia State, Nigeria (2011 – 2019)

Year	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sept.	Oct.	Nov.	Dec.	$\overline{X}_{i.}$	$\sigma^2_{i.}$
2011	2.83	3.09	3.18	2.89	2.56	3.33	3.14	3.04	2.89	2.30	3.33	3.30	2.99	0.10
2012	2.71	3.00	2.89	3.22	3.40	1.95	3.18	2.64	3.00	2.83	3.13	2.30	2.85	0.17
2013	2.64	2.40	2.71	3.37	2.83	3.74	2.77	2.64	2.94	2.48	3.04	2.89	2.87	0.14
2014	3.00	3.30	1.61	2.64	2.40	2.83	2.48	3.04	3.09	2.77	3.04	3.14	2.78	0.21
2015	1.79	2.40	2.56	3.00	3.47	2.89	2.56	3.00	3.09	2.40	2.71	2.71	2.71	0.18
2016	2.94	3.04	2.19	2.39	2.89	2.64	2.48	2.08	2.30	2.71	2.77	2.56	2.59	0.09
2017	3.22	3.40	2.89	2.48	2.30	3.78	3.04	2.77	3.14	3.30	2.64	2.40	2.95	0.20
2018	3.00	2.83	3.37	2.94	3.26	2.94	2.64	2.77	1.61	2.20	3.30	3.04	2.83	0.25
2019	2.20	2.89	3.04	2.48	2.48	3.33	2.71	3.40	2.20	2.64	2.89	2.56	2.74	0.15
$\overline{X}_{.j}$	2.70	2.93	2.72	2.83	2.84	3.05	2.78	2.82	2.70	2.63	2.98	2.77		
$\sigma^2_{.j}$	0.20	0.12	0.29	0.12	0.20	0.33	0.08	0.14	0.29	0.11	0.06	0.12		