Some soft Compatible maps and Common Fixed point Theorem in soft S-metric spaces

Abstract: In this paper, we introduce a new concept like $(\alpha)-$soft compatible maps, $(\beta)-$soft compatible maps, soft compatible map of type-I and soft compatible map of type-II in soft S-metric spaces. Finally, by the influence of these new concepts we will establish common fixed point theorem for four soft self maps on a complete soft S-metric space.

Keywords: Soft compatible mappings, Soft S-metric Space, Common fixed soft point.

MSC: 47H10, 54H25

1. Introduction

Russian mathematician Molodtsov [11] in 1999, introduced Soft set theory. He proposed the soft set as a completely generic mathematical tool for modeling uncertainties. Soft set theory has a rich potential for application in many directions. In 2003, Maji *et al.* [10] presented an application of soft sets in a decision-making problem and also introduced theory of soft sets. Babitha and Sunil [3] introduced the idea of soft set relation and function and discussed some related concepts. Sezgin and Atagun [12] and others modified the work of Maji *et al.* [10] and gave some new results. In 2012, Das and Samanta [7] introduced soft real set and soft real number and studied some of their basic properties (see [8]). They also introduced the notion of soft metric space.

Banach construction Principle [4], is one of the main pillars of the theory of metric fixed points. Many researchers investigated the Banach fixed point theorem in many directions and presented generalizations, extensions, and applications of their findings. This principle was also extended in the field of soft theory. Wardowski D. [15] in 2013, established the results on a soft mapping and its fixed points. Yazar *et al.* [16] introduced soft contractive mappings on soft metric spaces and prove some fixed point theorems of soft contractive mappings. In 2016, Mujahid Abbas *et al.* [1] introduced fixed point theory of soft metric. There are also numerous generalizations of soft metric spaces to prove the fixed point theorem therein. Some of these generalizations are as follows: In 2016 Guler *et al.* [9] proved fixed point theorem in soft G metric space, Altıntas and Taskopru [2] defined the soft cone metric space and obtained soft versions of some fixed point results. The definition of soft b-metric was given by Wadkar *et al.* [13] in 2017. They also established some fixed point results in the framework of soft b-metric spaces (see [14]). In 2018, Aras *et al.* [5] introduced soft S-metric spaces and also discussed its important properties. They proved some results on soft mapping with a soft contractive condition (see [6]).

We now recollect some basic definition, properties and results in soft metric spaces.

Definition 1.1 [12] A pair (F, E) is called a soft set over a given universal set X, if and only if F is a mapping from a set of parameters E (each parameter could be a word or a sentence) into the power set of X denoted by $P(X)$, that is, $F: E \to P(X)$. Clearly, a soft set over X is a parameterized family of subsets of the given universe X.

Definition 1.2 [10] A soft set (F, E) over X is said to be a null soft set denoted by $\widetilde{\Phi}$, if for all $e \in E$, $F(e) = \phi$.

Definition 1.3 [10] A soft set (F, E) over X is said to be an absolute soft set denoted by $\tilde{X}$ if for all $e \in E, F(e) = X$.

Definition 1.4 [6] Let $\mathbb{R}$ be the set of real numbers and $\mathcal{B}(\mathbb{R})$ the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F: E \to \mathcal{B}(\mathbb{R})$ is called a soft real set. If a real soft set is a singleton soft set, it will be called a soft real number and denoted by $\bar{r}$, $\bar{s}$, $\bar{t}$ etc. $\bar{0}$ and $\bar{1}$ are the soft real numbers where $\bar{0}(e) = 0$, $\bar{1}(e) = 1$, for all $e \in E$ respectively.

Definition 1.5 [6] (Properties of Soft Real Numbers): Let $\tilde{r}$, $\tilde{s}$ be two soft real numbers. Then the following statements hold:

(i) $\tilde{r} \leq \tilde{s}$ if $\tilde{r}(e) \tilde{\leq} \tilde{s}(e)$ for all e ∈ E;

(ii) $\tilde{r} \geq \tilde{s}$ if $\tilde{r}(e) \tilde{\geq} \tilde{s}(e)$ for all e ∈ E;

(iii) $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(e) \tilde{<} \tilde{s}(e)$ for all e ∈ E;

(iv) $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(e) \tilde{>} \tilde{s}(e)$ for all e ∈ E.

Definition 1.6 [6] A soft set (F, E) over X is said to be a soft point if there is exactly one $e \in E$ such that F(e) ={u}, for some $u \in \tilde{X}$ and F(e′) = ϕ, ∀ e′ ∈ E−{e}. It will be denoted by F_e^u or $\hat{u}_e$.

The soft point $\hat{u}_e$ is said to be belonging to the soft set (F, E), denoted by $\hat{u}_e \in (F, E)$, if $\hat{u}_e(e) \in F(e)$, i.e., $\{u\} \subseteq F(e)$.

Definition 1.7 [3] A soft S-metric on $\tilde{X}$ is a mapping $S: SP(\tilde{X}) \times SP(\tilde{X}) \times SP(\tilde{X}) \rightarrow \mathbb{R}(E)^*$ which satisfies the following conditions:

$(\bar{S_1})$ $S(\hat{u}_a, \hat{v}_b, \hat{w}_c) \geqq \bar{0}$;

$(\bar{S_2})$ $S(\hat{u}_a, \hat{v}_b, \hat{w}_c) = \bar{0}$, if and only if $\hat{u}_a = \hat{v}_b = \hat{w}_c$;

$(\bar{S_3})$ $S(\hat{u}_a, \hat{v}_b, \hat{w}_c) \leqq S(\hat{u}_a, \hat{u}_a, \hat{t}_d) + S(\hat{v}_b, \hat{v}_b, \hat{t}_d) + S(\hat{w}_c, \hat{w}_c, \hat{t}_d)$,

for all $\hat{u}_a, \hat{v}_b, \hat{w}_c, \hat{t}_d \in SP(\tilde{X})$, then the soft set $\tilde{X}$ along with the soft S-metric is called soft S-metric space and denoted by $(\tilde{X}, S, E)$.

Lemma 1.8 [3] Let $(\tilde{X}, S, E)$ is a soft S-metric space. Then we have

$$S(\hat{u}_a, \hat{u}_a, \hat{v}_b) = S(\hat{v}_b, \hat{v}_b, \hat{u}_a).$$

Definition 1.9 [4] A soft sequence $\{\hat{u}_{a_n}^n\}$ in $(\tilde{X}, S, E)$ converges to $\hat{v}_b$ if and only if $S(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{v}_b) \rightarrow \bar{0}$ as $n \rightarrow \infty$ and we denote this by $\lim_{n\rightarrow\infty} \hat{u}_{a_n}^n = \hat{v}_b$.

Definition 1.10 [4] A soft sequence $\{\hat{u}_{a_n}^n\}$ in $(\tilde{X}, S, E)$ is called a Cauchy sequence if for $\tilde{\varepsilon} > \bar{0}$, there exists $n_0 \in \mathbb{N}$ such that $S(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_m}^m) < \tilde{\varepsilon}$, for each $m, n \geq n_0$.

Definition 1.11 [4] A soft S-metric space $(\tilde{X}, S, E)$ is said to be complete if every Cauchy sequence is converging to some soft point of $(\tilde{X}, S, E)$.

Definition 1.3 [3] Let $(\tilde{X}, S, E)$ and $(\tilde{Y}, S', E')$ be two soft S-metric spaces. The mapping $f_\varphi: (\tilde{X}, S, E) \rightarrow (\tilde{Y}, S', E')$ is a soft mapping, where $f: \tilde{X} \rightarrow \tilde{Y}$ and $\varphi: E \rightarrow E'$ are two mappings.

Definition 1.12 [4] Let $f_\varphi: (\tilde{X}, S, E) \rightarrow (\tilde{Y}, S', E')$ be a soft mapping from soft S-metric space $(\tilde{X}, S, E)$ to a soft S-metric space $(\tilde{Y}, S', E')$. Then f_φ is soft continuous at a soft point $\hat{u}_a \in SP(\tilde{X})$ if and only if $(f, \varphi)(\{\hat{u}_{a_n}^n\}) \rightarrow (f, \varphi)(\hat{u}_a)$.

Definition 1.14 [4] Let $(\tilde{X}, S, E)$ be a complete soft S-metric space. A map $f_\varphi: (\tilde{X}, S, E) \rightarrow (\tilde{X}, S, E)$ is said to be a soft contraction mapping if there exists a soft real number $\bar{k} \in \mathbb{R}(E)$, $\bar{0} \leq \bar{k} < \bar{1}$ (where $\mathbb{R}(E)$ denotes the soft real number set) such that

$$S((T, \varphi)(\hat{u}_a), (T, \varphi)(\hat{u}_a), (T, \varphi)(\hat{v}_b)) \leq \bar{k}\ S(\hat{u}_a, \hat{u}_a, \hat{v}_b),$$

for all $\hat{u}_a, \hat{v}_b \in SP(\tilde{X})$.

2. Main Result

In this section, a generalization of (α) −soft compatible maps, (β) −soft compatible maps, soft compatible maps of type-I and soft compatible maps of type-II are introduced in soft S-metric spaces. Also, we have presented a series of Proposition in order to the pertinent results. Finally, we have proved common fixed point theorem for four soft continuous self maps on a complete soft S-metric space.

Definition 2.1. Suppose $f_\psi, g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ are two soft mappings. Then f_ψ and g_φ are known as soft compatible if

$$\lim_{n\to\infty} S\big(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n})\big) = \bar{0},$$

or

$$\lim_{n\to\infty} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n})\big) = \bar{0},$$

where $\{\hat{u}^n_{a_n}\}$ is a soft sequence in $\tilde{X}$ which satisfies

$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b$, for any $\hat{v}_b \in SP(\tilde{X})$.

Definition 2.2. Suppose $f_\psi, g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ are two soft mappings. Then f_ψ and g_φ are known as $(\alpha) -$ soft compatible mappings if

$$\lim_{n\to\infty} S\big(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), g_\varphi g_\varphi(\hat{u}^n_{a_n})\big) = \bar{0},$$

where $\{\hat{u}^n_{a_n}\}$ is a soft sequence in $\tilde{X}$ which satisfies

$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b$, for any $\hat{v}_b \in SP(\tilde{X})$.

Definition 2.3. Suppose $f_\psi, g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ are two soft mappings. Then f_ψ and g_φ are known as $(\beta) -$ soft compatible mappings if

$$\lim_{n\to\infty} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), f_\psi f_\psi(\hat{u}^n_{a_n})\big) = \tilde{0},$$

where $\{\hat{u}^n_{a_n}\}$ is a soft sequence in $\tilde{X}$ which satisfies

$\lim_{n\to\infty} f_\psi(\hat{u}_{a_n}^n) = \lim_{n\to\infty} g_\varphi(\hat{u}_{a_n}^n) = \hat{v}_b$, for any $\hat{v}_b \in SP(\tilde{X})$.

Definition 2.4. Suppose $f_\psi, g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ are two soft mappings. Then f_ψ and g_φ are known as soft compatible of type-I if

$$\underline{\lim_{n\to\infty}} S\big(f_\psi g_\varphi(\hat{u}_{a_n}^n), f_\psi g_\varphi(\hat{u}_{a_n}^n), \hat{v}_b\big) \tilde{\geq} S\big(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), \hat{v}_b\big),$$

where $\{\hat{u}_{a_n}^n\}$ is a soft sequence in $\tilde{X}$ that satisfies

$\lim_{n\to\infty} f_\psi(\hat{u}_{a_n}^n) = \lim_{n\to\infty} g_\varphi(\hat{u}_{a_n}^n) = \hat{v}_b$, for any $\hat{v}_b \in SP(\tilde{X})$.

Definition 2.5. Suppose $f_\psi, g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ are two soft mappings. Then f_ψ and g_φ are known as soft compatible of type-II if

$$\underline{\lim_{n\to\infty}} S\big(g_\varphi f_\psi(\hat{u}_{a_n}^n), g_\varphi f_\psi(\hat{u}_{a_n}^n), \hat{v}_b\big) \tilde{\geq} S\big(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), \hat{v}_b\big),$$

where $\{\hat{u}_{a_n}^n\}$ is a soft sequence in $\tilde{X}$ that satisfies

$\lim_{n\to\infty} f_\psi(\hat{u}_{a_n}^n) = \lim_{n\to\infty} g_\varphi(\hat{u}_{a_n}^n) = \hat{v}_b$, for any $\hat{v}_b \in SP(\tilde{X})$.

Proposition 2.6. Consider two soft mappings $f_\psi, g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ such that g_φ is soft continuous. Then f_ψ and g_φ are soft compatible if and only if they are $(\alpha) -$ soft compatible.

Proof: Let $\{u_{a_n}^n\}$ is a soft sequence in $\tilde{X}$ and suppose that f_ψ and g_φ are soft compatible such that

$\lim_{n\to\infty} f_\psi(\hat{u}_{a_n}^n) = \lim_{n\to\infty} g_\varphi(\hat{u}_{a_n}^n) = \hat{v}_b$,

for any $\hat{v}_b \in \tilde{X}$. Then, by triangle inequality, we have

$$S\big(f_\psi g_\varphi(\hat{u}_{a_n}^n), f_\psi g_\varphi(\hat{u}_{a_n}^n), g_\varphi g_\varphi(\hat{u}_{a_n}^n)\big) \tilde{\leq} 2\, S\big(f_\psi g_\varphi(\hat{u}_{a_n}^n), f_\psi g_\varphi(\hat{u}_{a_n}^n), g_\varphi f_\psi(\hat{u}_{a_n}^n)\big)$$
$$+ S\big(g_\varphi g_\varphi(\hat{u}_{a_n}^n), g_\varphi f_\psi(\hat{u}_{a_n}^n), g_\varphi f_\psi(\hat{u}_{a_n}^n)\big),$$

it follows that

$$\lim_{n\to\infty} S\big(f_\psi g_\varphi(\hat{u}_{a_n}^n), f_\psi g_\varphi(\hat{u}_{a_n}^n), g_\varphi g_\varphi(\hat{u}_{a_n}^n)\big) = \bar{0}\,.$$

Hence, f_ψ and g_φ are (α) – soft compatible.

Conversely, suppose that f_{ψ} and g_{φ} are (α) – soft compatible. Then we have

$$S\big(f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n}), g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n})\big) \widetilde{\leqq} 2\, S\big(f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n}), g_{\varphi}g_{\varphi}(\hat{u}^{n}_{a_n})\big)$$

$$+ S\big(g_{\varphi}g_{\varphi}(\hat{u}^{n}_{a_n}), g_{\varphi}g_{\varphi}(\hat{u}^{n}_{a_n}), g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n})\big),$$

which implies that

$$S\big(f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n}), g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n})\big) = \bar{0}\,.$$

Thus, f_{ψ} and g_{φ} are soft compatible. ■

Proposition 2.7. Consider two soft mappings $f_{\psi}, g_{\varphi} : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ such that f_{ψ} is soft continuous. Then f_{ψ} and g_{φ} are soft compatible if and only if they are (β) – soft compatible.

Proof: Let $\{u^{n}_{a_n}\}$ is a soft sequence in $\tilde{X}$ and suppose that f_{ψ} and g_{φ} are soft compatible such that

$$\lim_{n\to\infty} f_{\psi}(\hat{u}^{n}_{a_n}) = \lim_{n\to\infty} g_{\varphi}(\hat{u}^{n}_{a_n}) = \hat{v}_{b},$$

for any $\hat{v}_{b} \in \tilde{X}$. Then, by triangle inequality, we have

$$S\big(g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}f_{\psi}(\hat{u}^{n}_{a_n})\big) \widetilde{\leqq} 2\, S\big(g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n})\big)$$

$$+ S\big(f_{\psi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n})\big),$$

it follows that

$$\lim_{n\to\infty} S\big(f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n}), g_{\varphi}g_{\varphi}(\hat{u}^{n}_{a_n})\big) = \bar{0}\,.$$

Hence, f_{ψ} and g_{φ} are (β) – soft compatible.

Conversely, suppose that f_{ψ} and g_{φ} are (β) – soft compatible. Then we have

$$S\big(g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n})\big) \widetilde{\leqq} 2\, S\big(g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}f_{\psi}(\hat{u}^{n}_{a_n})\big)$$

$$+ S\big(f_{\psi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n})\big),$$

which implies that

$$S\big(g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), g_{\varphi}f_{\psi}(\hat{u}^{n}_{a_n}), f_{\psi}g_{\varphi}(\hat{u}^{n}_{a_n})\big) = \bar{0}\,.$$

Thus, f_ψ and g_φ are soft compatible. ■

Proposition 2.8. Consider two soft mappings $f_\psi, g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ such that g_φ is soft continuous. If f_ψ and g_φ are (α) – soft compatible then they are soft compatible of type I.

Proof: Let $\{\hat{u}^n_{a_n}\}$ be a soft sequence in $\tilde{X}$ and suppose that f_ψ and g_φ are (α) – soft compatible such that

$$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b,$$

for any $\hat{v}_b \,\epsilon\, \tilde{X}$. Since $g_\varphi$be soft continuous map, we have

$$S\big(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), g_\varphi g_\varphi(\hat{u}^n_{a_n})\big) = \bar{0} = S\big(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), g_\varphi(\hat{v}_b)\big)$$

and

$$S\big(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), \hat{v}_b\big) \lesseqqgtr \lim_{n\to\infty}[2\ S\big(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), f_\psi g_\varphi(\hat{u}^n_{a_n})\big) +$$

$$S\big(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b\big)]$$

$$= \lim_{n\to\infty} S\big(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b\big).$$

Hence it follows that

$$\underline{\lim_{n\to\infty}}\, S\big(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b\big) \gtreqqless S\big(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), \hat{v}_b\big).$$

Above inequality holds for every choice of the sequence $\{\hat{u}^n_{a_n}\}$ in $\tilde{X}$ with corresponding to $\hat{v}_b$ in $\tilde{X}$ and hence f_ψ and g_φ are soft compatible of type-I. ■

Proposition 2.9. Consider two soft mappings $f_\psi, g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ such that f_ψ is soft continuous. If f_ψ and g_φ are (β) – soft compatible then they are soft compatible of type-II.

Proof: Let $\{\hat{u}^n_{a_n}\}$ be a soft sequence in $\tilde{X}$ and suppose that f_ψ and g_φ are (β) – soft compatible such that

$$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b,$$

for any $\hat{v}_b \,\epsilon\, \tilde{X}$. Since $g_\varphi$be soft continuous map, we have

$$S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), f_\psi f_\psi(\hat{u}^n_{a_n})\big) = \bar{0} = S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), f_\psi(\hat{v}_b)\big)$$

and

$$S\big(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), \hat{v}_b\big) \lesseqqgtr \lim_{n\to\infty}[2\ S\big(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), g_\varphi f_\psi(\hat{u}^n_{a_n})\big) +$$

$$S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big)]$$

$$= \lim_{n\to\infty} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big).$$

Hence it follows that

$$\underline{\lim_{n\to\infty}} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big) \gtreqqless S\big(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), \hat{v}_b\big).$$

Above inequality holds for every choice of the sequence $\{\hat{u}^n_{a_n}\}$ in $\tilde{X}$ with corresponding to $\hat{v}_b$ in $\tilde{X}$ and hence f_ψ and g_φ are soft compatible of type-II. ■

Proposition 2.10. Consider two soft mappings $f_\psi,\ g_\varphi :\ (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ such that g_φ be a soft continuous map. If f_ψ and g_φ are soft compatible of type-I and for every soft sequence $\{\hat{u}^n_{a_n}\}$ in $\tilde{X}$ we have

$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b$, for some $\hat{v}_b\ in\ \tilde{X}$, it follows that

$\lim_{n\to\infty} f_\psi g_\varphi\big(\hat{u}^n_{a_n}\big) = \hat{v}_b$, then it is (α) – soft compatible.

Proof: Let $\{\hat{u}^n_{a_n}\}$ is a soft sequence in $\tilde{X}$ and suppose that f_ψ and g_φ are soft compatible of type-I such that

$$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b,$$

for some $\hat{v}_b \in \tilde{X}$. Since g_φ be soft continuous map, we have

$$S\big(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), \hat{v}_b\big) \lesseqqgtr \underline{\lim_{n\to\infty}} S\big(f_\psi g_\varphi\big(\hat{u}^n_{a_n}\big), f_\psi g_\varphi\big(\hat{u}^n_{a_n}\big), \hat{v}_b\big)$$

and

$$S\big(g_\varphi g_\varphi\big(\hat{u}^n_{a_n}\big), g_\varphi g_\varphi\big(\hat{u}^n_{a_n}\big), \hat{v}_b\big) \lesseqqgtr \underline{\lim_{n\to\infty}}[2\,S\Big(g_\varphi g_\varphi\big(\hat{u}^n_{a_n}\big), g_\varphi g_\varphi\big(\hat{u}^n_{a_n}\big), g_\varphi(\hat{v}_b)\Big)$$

$$+ S(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), \hat{v}_b)]$$

$$= S(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), \hat{v}_b).$$

Thus, we obtain

$$\lim_{n\to\infty} S(g_\varphi g_\varphi(\hat{u}^n_{a_n}), g_\varphi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b) \widetilde{\leq} \underline{\lim}_{n\to\infty} S(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b).$$

Further, we have

$$S\left(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), g_\varphi g_\varphi(\hat{u}^n_{a_n})\right) \widetilde{\leq} 2\, S(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b)$$

$$+ S(g_\varphi g_\varphi(\hat{u}^n_{a_n}), g_\varphi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b).$$

Taking limit inferior as $n \to \infty$, we get

$$\underline{\lim}_{n\to\infty} S\left(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), g_\varphi g_\varphi(\hat{u}^n_{a_n})\right)$$

$$\widetilde{\leq}\ 2\, \underline{\lim}_{n\to\infty} S(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b) + \underline{\lim}_{n\to\infty} S(g_\varphi g_\varphi(\hat{u}^n_{a_n}), g_\varphi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b)$$

$$\widetilde{\leq}\ 2\, \underline{\lim}_{n\to\infty} S(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b) + \underline{\lim}_{n\to\infty} S(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), \hat{v}_b).$$

Thus, $\lim_{n\to\infty} S\left(f_\psi g_\varphi(\hat{u}^n_{a_n}), f_\psi g_\varphi(\hat{u}^n_{a_n}), g_\varphi g_\varphi(\hat{u}^n_{a_n})\right) = \bar{0}$.

Hence f_ψ and g_φ are (α) – soft compatible. ■

Proposition 2.11. Consider two soft mappings $f_\psi,\ g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ such that f_φ be a soft continuous map. If f_ψ and g_φ are soft compatible of type-II and for every soft sequence $\{\hat{u}^n_{a_n}\}$ in $\tilde{X}$ we have

$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b$, for some $\hat{v}_b\ in\ \tilde{X}$, it follows that

$\lim_{n\to\infty} g_\varphi f_\psi(\hat{u}^n_{a_n}) = \hat{v}_b$, then it is (β) – soft compatible.

Proof: Let $\{\hat{u}^n_{a_n}\}$ is a soft sequence in $\tilde{X}$ and suppose that f_ψ and g_φ are soft compatible of type-II such that

$$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b,$$

for some $\hat{v}_b \in \tilde{X}$. Since f_φ be soft continuous map, we have

$$S\big(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{v}_b\big) \tilde{\leqq} \underline{\lim}_{n\to\infty} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big)$$

and

$$S\big(f_\varphi f_\varphi(\hat{u}^n_{a_n}), f_\varphi f_\varphi(\hat{u}^n_{a_n}), \hat{v}_b\big) \tilde{\leqq} \underline{\lim}_{n\to\infty} [2\, S\left(f_\varphi f_\varphi(\hat{u}^n_{a_n}), f_\varphi f_\varphi(\hat{u}^n_{a_n}), f_\varphi(\hat{v}_b)\right)$$

$$+ S\big(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{v}_b\big)]$$

$$= S\big(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{v}_b\big).$$

Thus, we obtain

$$\lim_{n\to\infty} S\big(f_\varphi f_\varphi(\hat{u}^n_{a_n}), f_\varphi f_\varphi(\hat{u}^n_{a_n}), \hat{v}_b\big) \tilde{\leqq} \underline{\lim}_{n\to\infty} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big).$$

Further, we have

$$S\left(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), f_\varphi f_\varphi(\hat{u}^n_{a_n})\right) \tilde{\leqq} 2\, S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big)$$

$$+ S\big(f_\varphi f_\varphi(\hat{u}^n_{a_n}), f_\varphi f_\varphi(\hat{u}^n_{a_n}), \hat{v}_b\big).$$

Taking limit inferior as $n \to \infty$, we get

$$\underline{\lim}_{n\to\infty} S\left(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), f_\varphi f_\varphi(\hat{u}^n_{a_n})\right)$$

$$\tilde{\leqq}\ 2\, \underline{\lim}_{n\to\infty} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big) + \underline{\lim}_{n\to\infty} S\big(f_\varphi f_\varphi(\hat{u}^n_{a_n}), f_\varphi f_\varphi(\hat{u}^n_{a_n}), \hat{v}_b\big)$$

$$\tilde{\leqq}\ 2\, \underline{\lim}_{n\to\infty} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big) + \underline{\lim}_{n\to\infty} S\big(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), \hat{v}_b\big).$$

Thus, $\lim_{n\to\infty} S\left(g_\varphi f_\psi(\hat{u}^n_{a_n}), g_\varphi f_\psi(\hat{u}^n_{a_n}), f_\varphi f_\varphi(\hat{u}^n_{a_n})\right) = \bar{0}$.

Hence f_ψ and g_φ are $(\alpha) -$ soft compatible. ■

Proposition 2.12. Let $f_\psi,\ g_\varphi : (\tilde{X}, S, E) \to (\tilde{X}, S, E)$ be two soft mappings. Suppose that f_ψ and g_φ are soft compatible of type-I (resp. of type-II) $f_\psi(\hat{v}_b) = g_\varphi(\hat{v}_b)$ for some $\hat{v}_b\ in\ \tilde{X}$, then

$$S\left(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), g_\varphi g_\varphi(\hat{v}_b)\right) \cong\!\!\!\!\le S\left(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), f_\psi g_\varphi(\hat{v}_b)\right)$$

$$\left(\text{resp.}, S\left(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), f_\psi f_\psi(\hat{v}_b)\right) \cong\!\!\!\!\le S\left(g_\varphi(\hat{v}_b), g_\varphi(\hat{v}_b), g_\varphi f_\psi(\hat{v}_b)\right)\right).$$

Proof: Let $\{\hat{u}^n_{a_n}\}$ be a soft sequence in $\tilde{X}$ defined as $\hat{u}^n_{a_n} = \hat{v}_b$ for $n = 1, 2, \ldots$ and

$f_\psi(\hat{v}_b) = g_\varphi(\hat{v}_b)$ for some $\hat{v}_b \ in\ \tilde{X}$. Then we have

$\lim_{n\to\infty} f_\psi(\hat{u}^n_{a_n}) = \lim_{n\to\infty} g_\varphi(\hat{u}^n_{a_n}) = \hat{v}_b$.

Suppose f_ψ and g_φ are soft compatible of type-I, then

$$S\left(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), g_\varphi g_\varphi(\hat{v}_b)\right) \cong\!\!\!\!\le 2\, S\left(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), f_\psi g_\varphi(\hat{v}_b)\right)$$

$$+S\left(f_\psi g_\varphi(u^n_{a_n}), f_\psi g_\varphi(u^n_{a_n}), g_\varphi g_\varphi(u^n_{a_n})\right)$$

$$\cong\!\!\!\!\le S\left(f_\psi(\hat{v}_b), f_\psi(\hat{v}_b), f_\psi g_\varphi(\hat{v}_b)\right).$$

Theorem 2.13. Suppose that $f_\psi,\ g_\varphi,\ R_\phi, and\ T_\xi$ are four soft self mappings defined on a complete soft S-metric space $(\tilde{X}, S, E)$.

$(\bar{a})$ Let $f_\psi(\tilde{X}, S) \subseteq R_\phi(\tilde{X}, S),\ g_\varphi(\tilde{X}, S) \subseteq T_\xi(\tilde{X}, S)$, then there exists a constant $\bar{k} \in (\bar{0}, \bar{1})$ such that for each $\hat{u}_a, \hat{v}_b, \hat{w}_c \in SP(\tilde{X})$ we have

$$S(f_\psi(\hat{u}_a), f_\psi(\hat{v}_b), g_\varphi(\hat{w}_c)) \cong\!\!\!\!\le \tilde{k} \max\{S(R_\phi(\hat{u}_a), R_\phi(\hat{v}_b), T_\xi(\hat{w}_c)), S(f_\psi(\hat{u}_a), f_\psi(\hat{u}_a), R_\phi(\hat{u}_a)),$$

$$S(g_\varphi(\hat{w}_c), g_\varphi(\hat{w}_c), T_\xi(\hat{w}_c)), S(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), T_\xi(\hat{w}_c)),$$

$$S(g_\varphi(\hat{w}_c), g_\varphi(\hat{w}_c), R_\phi(\hat{u}_a))\}, \quad (2.1)$$

Also, if $f_\psi,\ g_\varphi,\ R_\phi, and\ T_\xi$ satisfying any one conditions provided below:

$(\bar{b})$ T_ξ is soft continuous and the pair (f_ψ, R_ϕ) and (g_φ, T_ξ) are soft compatible of type-I.

$(\bar{c})$ R_ϕ is soft continuous and the pair (f_ψ, R_ϕ) and (g_φ, T_ξ) are soft compatible of type-I.

$(\bar{d})$ f_ψ is soft continuous and the pair (f_ψ, R_ϕ) and (g_φ, T_ξ) are soft compatible of type-II.

$(\bar{e})$ g_φ is soft continuous and the pair (f_ψ, R_ϕ) and (g_φ, T_ξ) are soft compatible of type-II.

Then $f_\psi,\ g_\varphi,\ R_\phi, and\ T_\xi$ have a unique common fixed soft point in $\tilde{X}$.

Proof: Let $\hat{u}^0_{a_0} \epsilon\, SP(\tilde{X})$. From $(\bar{a})$ we can construct a soft sequence $\{\hat{v}^n_{b_n}\}$ in $SP(\tilde{X})$ such that

$T_\xi(\hat{u}^{2n+1}_{a_{2n+1}}) = f_\psi(\hat{u}^{2n}_{a_{2n}}) = \hat{v}^{2n}_{b_{2n}}$ and $R_\phi(\hat{u}^{2n+1}_{a_{2n+1}}) = g_\varphi(\hat{u}^{2n+2}_{a_{2n+2}}) = \hat{v}^{2n+1}_{b_{2n+1}}$.

Then from (2.1), we have

$$S\left(f_\psi(\hat{u}^{2n}_{a_{2n}}), f_\psi(\hat{u}^{2n}_{a_{2n}}), g_\varphi(\hat{u}^{2n+1}_{a_{2n+1}})\right)$$

$$\widetilde{\leq}\ \bar{k}\max\{S\left(R_\phi(\hat{u}^{2n}_{a_{2n}}), R_\phi(\hat{u}^{2n}_{a_{2n}}), T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\right), S\left(f_\psi(\hat{u}^{2n}_{a_{2n}}), f_\psi(\hat{u}^{2n}_{a_{2n}}), R_\phi(\hat{u}^{2n}_{a_{2n}})\right),$$

$$S\left(g_\varphi(\hat{u}^{2n+1}_{a_{2n+1}}), g_\varphi(\hat{u}^{2n+1}_{a_{2n+1}}), T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\right), S\left(f_\varphi(\hat{u}^{2n}_{a_{2n}}), f_\varphi(\hat{u}^{2n}_{a_{2n}}), T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\right),$$

$$S\left(g_\varphi(\hat{u}^{2n+1}_{a_{2n+1}}), g_\varphi(\hat{u}^{2n+1}_{a_{2n+1}}), R_\phi(\hat{u}^{2n}_{a_{2n}})\right)\},$$

which implies that

$$S(\hat{v}^{2n}_{b_{2n}}. \hat{v}^{2n}_{b_{2n}}, \hat{v}^{2n+1}_{b_{2n+1}})$$

$$\widetilde{\leq}\ \bar{k}\max\{S(\hat{v}^{2n-1}_{b_{2n-1}}, \hat{v}^{2n-1}_{b_{2n-1}}, \hat{v}^{2n}_{b_{2n}}), S(\hat{v}^{2n}_{b_{2n}}, \hat{v}^{2n}_{b_{2n}}, \hat{v}^{2n-1}_{b_{2n-1}}), S(\hat{v}^{2n+1}_{b_{2n+1}}, \hat{v}^{2n+1}_{b_{2n+1}}, \hat{v}^{2n}_{b_{2n}}),$$

$$S(\hat{v}^{2n}_{b_{2n}}, \hat{v}^{2n}_{b_{2n}}, \hat{v}^{2n}_{b_{2n}}), S(\hat{v}^{2n+1}_{b_{2n+1}}, \hat{v}^{2n+1}_{b_{2n+1}}, \hat{v}^{2n-1}_{b_{2n-1}})$$

$$\widetilde{\leq}\ \bar{k}\max\{S(\hat{v}^{2n-1}_{b_{2n-1}}, \hat{v}^{2n-1}_{b_{2n-1}}, \hat{v}^{2n}_{b_{2n}}), S(\hat{v}^{2n}_{b_{2n}}, \hat{v}^{2n}_{b_{2n}}, \hat{v}^{2n+1}_{b_{2n+1}})\}.$$

Suppose that $n > m$ for some $n, m\ \epsilon\ \mathbb{N}$, then we have

$$S(\hat{v}^n_{b_n}, \hat{v}^n_{b_n}, \hat{v}^m_{b_m})\ \widetilde{\leq}\ 2\,S(\hat{v}^n_{b_n}, \hat{v}^n_{b_n}, \hat{v}^{n+1}_{b_{n+1}}) + 2\,S(\hat{v}^{n+1}_{b_{n+1}}, \hat{v}^{n+1}_{b_{n+1}}, \hat{v}^{n+2}_{b_{n+2}})$$

$$+ \cdots + S(\hat{v}^{m-1}_{b_{m-1}}, \hat{v}^{m-1}_{b_{m-1}}, \hat{v}^m_{b_m})$$

$$\widetilde{\leq}\ 2\left(\bar{k}^m + \bar{k}^{m-1} + \cdots + \bar{k}^{n-1}\right) S(\hat{v}^1_{b_1}, \hat{v}^1_{b_1}, \hat{v}^0_{b_0})$$

$$\widetilde{\leq}\ 2\,\bar{k}^m\ \frac{1-\bar{k}^{(n-m)}}{1-\bar{k}}\, S(\hat{v}^1_{b_1}, \hat{v}^1_{b_1}, \hat{v}^0_{b_0}) \to \bar{0}, \text{ as } m \to \infty.$$

It follows that $\{\hat{v}^n_{b_n}\}$ is a Cauchy Sequence. Since $(\tilde{X}, S, E)$ is a complete soft S-metric space, so there is some $\widehat{w}_c\ \epsilon\ SP(\tilde{X})$ such that $\hat{v}^n_{b_n} \to \widehat{w}_c$.

Further, suppose that condition $(\bar{b})$ holds. Then, since the pair (g_φ, T_ξ) is soft compatible of type (I) and T_ξ is soft continuous, we have

$$S\big(T_\xi(\widehat{w}_c), T_\xi(\widehat{w}_c), \widehat{w}_c\big) \lessapprox \varliminf_{n\to\infty} S\big(g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}}), g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}}), \widehat{w}_c\big) \tag{2.2}$$

and $T_\xi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}}) \to T_\xi(\widehat{w}_c)$ (2.3)

By taking $\hat{u}_a, \hat{v}_b = \hat{u}^{2n}_{a_{2n}}$ and $\widehat{w}_c = T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})$ in (2.1), we obtain

$$S\left(f_\psi(\hat{u}^{2n}_{a_{2n}}), f_\psi(\hat{u}^{2n}_{a_{2n}}), g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\right)$$

$$\lessapprox \bar{k}\max\{ S\big(R_\phi(\hat{u}^{2n}_{a_{2n}}), R_\phi(\hat{u}^{2n}_{a_{2n}}), T_\xi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\big), S\left(f_\psi(\hat{u}^{2n}_{a_{2n}}), f_\psi(\hat{u}^{2n}_{a_{2n}}), R_\phi(\hat{u}^{2n}_{a_{2n}})\right),$$

$$S\left(g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}}), g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}}), T_\xi T_\xi((\hat{u}^{2n+1}_{a_{2n+1}})\right),$$

$$S\big(f_\varphi(\hat{u}^{2n}_{a_{2n}}), f_\varphi(\hat{u}^{2n}_{a_{2n}}), T_\xi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\big), S\big(g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}}), g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}}), R_\phi(\hat{u}^{2n}_{a_{2n}})\big)\}, \tag{2.4}$$

Now, taking limit inferior on both the side equation (2.4) we obtain

$$\varliminf_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\big)$$

$$\lessapprox \bar{k}\max\{ S\left(\widehat{w}_c, \widehat{w}_c, T_\xi(\widehat{w}_c)\right), S(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c), \varliminf_{n\to\infty} S\left(T_\xi(\widehat{w}_c), T_\xi(\widehat{w}_c), g_\varphi T_\xi(\widehat{w}_c)\right),$$

$$S\left(\widehat{w}_c, \widehat{w}_c, T_\xi(\widehat{w}_c)\right), \varliminf_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\big)\}$$

$$\lessapprox \bar{k}\max\{ S\left(\widehat{w}_c, \widehat{w}_c, T_\xi(\widehat{w}_c)\right), \varliminf_{n\to\infty} \left[2\, S\big(T_\xi(\widehat{w}_c), T_\xi(\widehat{w}_c), \widehat{w}_c\big) + S\left(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\right)\right],$$

$$\varliminf_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\big)\}$$

$$\lessapprox \bar{k}\max\{ \varliminf_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\big), 3\ \varliminf_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\big),$$

$$S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}^{2n+1}_{a_{2n+1}})\big)\},$$

which implies that

$$\lim_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}_{a_{2n+1}}^{2n+1})\big) \lesseqgtr \bar{k} \lim_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}_{a_{2n+1}}^{2n+1})\big)$$

$$\precsim \bar{k} \lim_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}_{a_{2n+1}}^{2n+1})\big),$$

a contradiction.

Therefore

$$\lim_{n\to\infty} S\big(\widehat{w}_c, \widehat{w}_c, g_\varphi T_\xi(\hat{u}_{a_{2n+1}}^{2n+1})\big) = \bar{0},$$

Thus, form (2.2) we get

$$S\big(T_\xi(\widehat{w}_c), T_\xi(\widehat{w}_c), \widehat{w}_c\big) = \bar{0},$$

which further implies

$$T_\xi(\widehat{w}_c) = \widehat{w}_c. \tag{2.5}$$

Now by taking $\hat{u}_a, \hat{v}_b = \hat{u}_{a_{2n}}^{2n}$ and $\widehat{w}_c = \widehat{w}_c$ in (2.1), we get

$$S\left(f_\psi(\hat{u}_{a_{2n}}^{2n}), f_\psi(\hat{u}_{a_{2n}}^{2n}), g_\varphi(\widehat{w}_c)\right)$$

$$\lesseqgtr \bar{k} \max\{ S\big(R_\phi(\hat{u}_{a_{2n}}^{2n}), R_\phi(\hat{u}_{a_{2n}}^{2n}), T_\xi(\widehat{w}_c)\big), S\left(f_\psi(\hat{u}_{a_{2n}}^{2n}), f_\psi(\hat{u}_{a_{2n}}^{2n}), R_\phi(\hat{u}_{a_{2n}}^{2n})\right),$$

$$S\left(g_\varphi(\widehat{w}_c), g_\varphi(\widehat{w}_c), T_\xi(\widehat{w}_c)\right), S\big(f_\varphi(\hat{u}_{a_{2n}}^{2n}), f_\varphi(\hat{u}_{a_{2n}}^{2n}), T_\xi(\widehat{w}_c)\big),$$

$$S\big(g_\varphi(\widehat{w}_c), g_\varphi(\widehat{w}_c), R_\phi(\hat{u}_{a_{2n}}^{2n})\big)\}. \tag{2.6}$$

Taking limit as $n \to \infty$ in (2.6), we obtain

$$S\left(\widehat{w}_c, \widehat{w}_c, g_\varphi(\widehat{w}_c)\right) \lesseqgtr \bar{k}\, S\big(g_\varphi(\widehat{w}_c), g_\varphi(\widehat{w}_c), \widehat{w}_c\big) = \bar{k}\, S\left(\widehat{w}_c, \widehat{w}_c, g_\varphi(\widehat{w}_c)\right),$$

a contradiction, since $\bar{k} \epsilon (\bar{0}, \bar{1})$.

Thus, we get $g_\varphi(\widehat{w}_c) = \widehat{w}_c$. (2.7)

Further, as we have $g_\varphi(\tilde{X}, S) \subseteq T_\xi(\tilde{X}, S)$, so there exists a soft point $\hat{y}_d$ in $SP(\tilde{X})$ such that

$$g_\varphi(\widehat{w}_c) = R_\phi(\hat{y}_d).$$

Again by (2.1), we get

$$S\big(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), \widehat{w}_c\big) \widetilde{\leqq} \bar{k} \max\{ S\big(R_\phi(\hat{y}_d), R_\phi(\hat{y}_d), \widehat{w}_c\big), S\left(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), R_\phi(\hat{y}_d)\right),$$

$$S(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c), S\big(f_\varphi(\hat{y}_d), f_\varphi(\hat{y}_d), \widehat{w}_c\big), S\left(\widehat{w}_c, \widehat{w}_c, R_\phi(\hat{y}_d)\right)\},$$

which implies that

$S\big(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), \widehat{w}_c\big) \widetilde{\leqq} \bar{k}\, S\big(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), \widehat{w}_c\big) \widetilde{<} S\big(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), \widehat{w}_c\big)$, as $\bar{k} \in (\bar{0}, \bar{1})$,

a contradiction and hence

$$f_\psi(\hat{y}_d) = \widehat{w}_c\ .$$

Given that the pair (f_ψ, R_ϕ) is soft compatible of type-I, so we have

$$f_\psi(\hat{y}_d) = R_\phi(\hat{y}_d) = \widehat{w}_c.$$

From Proposition 2.12, we have

$$S\left(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), R_\phi R_\phi(\hat{y}_d)\right) \widetilde{\leqq} \bar{k} S\left(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), f_\psi R_\phi(\hat{y}_d)\right)$$

and so

$$S\left(\widehat{w}_c, \widehat{w}_c, R_\phi(\widehat{w}_c)\right) \widetilde{\leqq} \bar{k}\, S\left(\widehat{w}_c, \widehat{w}_c, f_\psi(\widehat{w}_c)\right). \tag{2.8}$$

Again using equation (2.1) we obtain

$$S\big(f_\psi(\widehat{w}_c), f_\psi(\widehat{w}_c), \widehat{w}_c\big) \widetilde{\leqq} \bar{k} \max\{ S\big(R_\phi(\widehat{w}_c), R_\phi(\widehat{w}_c), \widehat{w}_c\big), S\left(f_\psi(\widehat{w}_c), f_\psi(\widehat{w}_c), R_\phi(\widehat{w}_c)\right),$$

$$S(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c), S\big(f_\varphi(\widehat{w}_c), f_\varphi(\widehat{w}_c), \widehat{w}_c\big), S\left(\widehat{w}_c, \widehat{w}_c, R_\phi(\widehat{w}_c)\right)\},$$

$$\widetilde{\leqq} \bar{k}\, S\big(f_\psi(\widehat{w}_c), f_\psi(\widehat{w}_c), \widehat{w}_c\big) \widetilde{<} S\big(f_\psi(\widehat{w}_c), f_\psi(\widehat{w}_c), \widehat{w}_c\big),$$

a contradiction and hence

$$f_\psi(\widehat{w}_c) = \widehat{w}_c.$$

Also, from (2.8) we obtain

$R_\phi(\widehat{w}_c) = \widehat{w}_c.$

Hence, we obtain $f_\psi(\widehat{w}_c) = g_\varphi(\widehat{w}_c) = R_\phi(\widehat{w}_c) = T_\xi(\widehat{w}_c) = \widehat{w}_c$, which implies that $\widehat{w}_c$ is the fixed soft point of $f_\psi,\ g_\varphi,\ R_\phi, and\ T_\xi$.

For uniqueness consider another fixed soft point $\hat{y}_d$ in $SP(\tilde{X})$ then from (2.1) we get

$$S\big(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), g_\varphi(\widehat{w}_c)\big) \precsim \bar{k} \max\{S\big(R_\phi(\hat{y}_d), R_\phi(\hat{y}_d), T_\xi(\widehat{w}_c)\big), S\big(f_\psi(\hat{y}_d), f_\psi(\hat{y}_d), R_\phi(\hat{y}_d)\big),$$

$$S\big(g_\varphi(\widehat{w}_c), g_\varphi(\widehat{w}_c), T_\xi(\widehat{w}_c)\big), S\big(f_\varphi(\hat{y}_d), f_\varphi(\hat{y}_d), T_\xi(\widehat{w}_c)\big),$$

$$S\big(g_\varphi(\widehat{w}_c), g_\varphi(\widehat{w}_c), R_\phi(\hat{y}_d)\big)\}.$$

$$S(\hat{y}_d, \hat{y}_d, \widehat{w}_c) \precsim \bar{k} \max\{S(\hat{y}_d, \hat{y}_d, \widehat{w}_c)), S(\hat{y}_d, \hat{y}_d, \hat{y}_d), S(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c), S(\hat{y}_d, \hat{y}_d, \widehat{w}_c),$$

$$S(\widehat{w}_c, \widehat{w}_c, \hat{y}_d)\}.$$

which implies that

$$S(\hat{y}_d, \hat{y}_d, \widehat{w}_c) \precsim \bar{k}\, S(\hat{y}_d, \hat{y}_d, \widehat{w}_c).$$

As $\bar{k} \in (\bar{0}, \bar{1})$, we get $S(\hat{y}_d, \hat{y}_d, \widehat{w}_c) = \bar{0}$,

which implies $\widehat{w}_c = \hat{y}_d$. So, w_c is common fixed soft point of $f_\psi,\ g_\varphi,\ R_\phi, and\ T_\xi$.

By the similar argument we can prove other cases $(\bar{c}),\ (\bar{d})\ and\ (\bar{e})$.

This completes the proof.

Conclusions: In this article, we offer a novel notion of soft compatible mappings like (α) −soft compatible maps, (β) −soft compatible maps, soft compatible map of type-I and soft compatible map of type-II in soft S-metric space. Finally, we have established a common fixed point theorem for four soft self maps under the influence of these new concepts on a complete soft S-metric space.

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