Original Research Article

Global smooth solutions of the modified Landau-Lifshitz-Bloch equation

Abstract

The Landau-Lifshitz-Bloch equation is an important dynamical model for studying the magnetization of ferromagnets above Curie temperature. This article shows the global existence and uniqueness of smooth solutions of the modified Landau-Lifshitz-Bloch equation.

AMS Subject Classification (2020): 35B35; 35Q55

Key Words: Landau-Lifshitz-Bloch equation, global smooth solutions

1 Introduction

The object of this short paper is to study the cauchy problem of the modified Landau-Lifshitz-Bloch (LLB) equation

$$u_t = \Delta u + u \times \Delta u - k(1 + \mu |u|^{2\theta})u, \tag{1.1}$$

in the whole space $\mathbb{R}^d$ for $d = 2, 3$, with the parameters $k, \mu > 0$ and $\theta \geq 1$. The unknown vector $u \in \mathbb{R}^3$ represents the magnetization vector. Compared to the standard Landau-Lifshitz equation [6], the restriction $u \in \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$, is relaxed. This system should be supplemented with initial data

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{1.2}$$

The Landau-Lifshitz-Bloch equation can be derived from the Landau-Lifshitz equation or the Gilbert equation taking into account the effect of thermal excitation [1–3]. This equation has attracted a lot attention in the literature and a large amount of work has been done on the LLB equation, such as the domain wall motion in antiferromagnets [4], two stochastic forms of the LLB equation [5] and the LLB equation with quantum spin number [11]. The LLB equation has a wide range of applications such as laser, magnetic recording, magnetization switching, control domain walls, etc [1]. The existence of global weak solutions was studied by Le in [8] recently and the existence of a smooth solution to the LLB equation in $\mathbb{R}^2$ when $\theta = 1$ was studied by Li *et al* in [9]. In a special case, the second author studied the global well-posedness of the Landau-Lifshitz-Bloch equation with a helicity term [12]. In this paper, we continue to study the general case when $\theta \geq 1$ is a real number. Since the local well-posedness is standard and can be obtained by applying the same method as in [7, 10], we can state the main result in the following two theorems.

Theorem 1.1. *Let the dimension* $d = 2$ *and* $u_0 \in H^m (m \geq 2)$*, then for any* $T > 0$*, there exists a unique global smooth solution* $u(t, \cdot)$ *of the LLB equation* (1.1)-(1.2) *that satisfies*

$$\partial_t^j \partial_x^\alpha u \in L^\infty([0, T]; L^2(\mathbb{R}^2)) \quad \text{and} \quad \partial_t^k \partial_x^\beta u \in L^2([0, T]; L^2(\mathbb{R}^2)), \tag{1.3}$$

where $2j + |\alpha| \leq m$ *and* $2k + |\beta| \leq m + 1$.

---

[*]

Theorem 1.2. *Let the dimension $d = 3$, the initial value $u_0 \in H^m (m \geq 2)$, and assume that $||u_0||_{H^2}$ is small enough, then for any $T > 0$ there exists a unique solution of equation (1.1)-(1.2) that satisfies*

$$\partial_t^j \partial_x^\alpha u \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \quad \text{and} \quad \partial_t^k \partial_x^\beta u \in L^2([0,T]; L^2(\mathbb{R}^3)), \tag{1.4}$$

where $2j + |\alpha| \leq m$, and $2k + |\beta| \leq m + 1$.

In the next section, we prove the above two theorems by a series of lemmas.

2 Proof of the results

Lemma 2.1. *Assume $d = 2, 3$ and $u_0 \in H^m (m \geq 2)$, then the smooth solution of* (1.1)-(1.2) *satisfies the estimates*

$$||u(\cdot,t)||_{L^2}^2 + 2\int_0^t ||\nabla u(\cdot,s)ds + 2k\int_0^t (1+\mu|u|^{2\theta})|u|^2(\cdot,s)ds = ||u_0||_{L^2}^2, \tag{2.5}$$

$$||\nabla u(\cdot,t)||_{L^2}^2 + \int_0^t ||\Delta u(\cdot,s)||_{L^2}^2 ds \leq C(||u_0||_{H^2}), \quad \text{and} \tag{2.6}$$

$$||u(\cdot,t)||_{L^\infty} \leq C||u_0||_{H^2}, \qquad \forall t \geq 0. \tag{2.7}$$

Proof. Taking the inner product of (1.1) with u and then integrating over $\mathbb{R}^d$, we get

$$\int_{\mathbb{R}^d} u_t \cdot u dx = \int_{\mathbb{R}^d} \Delta u \cdot u dx + \int_{\mathbb{R}^d} (u \times \Delta u) \cdot u dx - k\int_{\mathbb{R}^d} [(1+\mu|u|^{2\theta})u] \cdot u dx. \tag{2.8}$$

By integration by parts and the fact that $(u \times \Delta u) \cdot u = 0$ for any $u \in \mathbb{R}^3$, we obtain (2.5).

Taking inner product of (1.1) with $|u|^{(p-2)}u$ for $p \geq 2$, and then integrating the result over $\mathbb{R}^d$, we get

$$\frac{1}{p}\frac{d}{dt}||u(\cdot,t)||_{L^p}^p = -\int_{\mathbb{R}^d} |u|^{p-2}|\nabla u|^2 dx - (p-2)\int_{\mathbb{R}^d} |u|^{p-4}(u \cdot \nabla u)^2 dx - \int_{\mathbb{R}^d} (1+\mu|u|^{2\theta})|u|^p dx \leq 0. \tag{2.9}$$

This implies that

$$||u(\cdot,t)||_{L^p} \leq C||u_0||_{L^p} \leq C||u_0||_{H^2}, \quad \forall p \geq 2, \ t \geq 0, \tag{2.10}$$

by Sobolev embeddings, where the constant C is independent of p. Letting $p \to \infty$, we obtain (2.7).

Taking inner product of (1.1) with Δu, we have

$$\int_{\mathbb{R}^d} \Delta u \cdot u_t dx = \int_{\mathbb{R}^d} |\Delta u|^2 dx - k\int_{\mathbb{R}^d} [(1+\mu|u|^{2\theta})u] \cdot \Delta u dx. \tag{2.11}$$

For the last term, it can be estimated that

$$\left| k\int_0^t \int_{\mathbb{R}^d} [(1+\mu|u(x,s)|^{2\theta})u(x,s)] \cdot \Delta u dx ds \right| \leq \int_0^t ||\Delta u(x,s)||_{L^2}^2 ds + C(||u_0||_{H^2}), \tag{2.12}$$

where the constant C may depend on t. Integrating (2.11) then leads to (2.6). □

Lemma 2.2. *Assume the dimension $d = 2$ and $u_0 \in H^m$ for $m \geq 2$, then the smooth solution of the LLB equation* (1.1)-(1.2) *satisfies*

$$||\Delta u(\cdot,t)||_{L^2}^2 + \int_0^t ||\Delta \nabla u(\cdot,s)||_{L^2}^2 ds \leq C(T; ||u_0||_{H^2}), \qquad \forall T \geq 0, \quad \forall t \in [0,T]. \tag{2.13}$$

In addition, if $m \geq 3$, we have

$$||\Delta \nabla u(\cdot,t)||_{L^2}^2 + \int_0^t ||\Delta^2 u(\cdot,s)||_{L^2}^2 ds \leq C(T; ||u_0||_{H^3}), \quad T \geq 0, \qquad t \in [0,T]. \tag{2.14}$$

Proof. Taking the Laplacian operator Δ on both sides of (1.1), and then taking inner product of the resultant with Δu, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\Delta u_t(x,t)\cdot\Delta u(x,t)dx=&\int_{\mathbb{R}^2}\Delta^2u(x,t)\cdot\Delta u(x,t)dx+2\int_{\mathbb{R}^2}(\nabla u(x,t)\times\nabla\Delta u(x,t))\cdot\Delta u(x,t)dx\\&+\int_{\mathbb{R}^2}(u(x,t)\times\Delta^2u(x,t))\cdot\Delta u(x,t)dx-k\int_{\mathbb{R}^2}\Delta[(1+\mu|u(x,t)|^{2\theta})u(x,t)]\cdot\Delta u(x,t)dx.\end{aligned}\tag{2.15}$$

For the second and the third term on the right, thanks to the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(u(x,t)\times\Delta^2u(x,t))\cdot\Delta u(x,t)dx\right|=&\left|\int_{\mathbb{R}^2}(\nabla u(x,t)\times\nabla\Delta u(x,t))\Delta u(x,t)dx\right|\\\le&2||\nabla u(\cdot,t)||_{L^4}||\Delta u(\cdot,t)||_{L^4}||\Delta\nabla u(\cdot,t)||_{L^2}\\\le&\frac{1}{4}||\nabla\Delta u(\cdot,t)||_{L^2}^2+C(||\nabla u_0||_{H^2})(1+||\Delta u(\cdot,t)||_{L^2}^4).\end{aligned}\tag{2.16}$$

For the last term, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\Delta(|u(x,t)|^{2\theta}u(x,t))\Delta u(x,t)dx\right|\\=&|\int_{\mathbb{R}^2}2(2\theta-2)|u(x,t)|^{2\theta-4}|u(x,t)\cdot\nabla u(x,t)|^2u(x,t)\Delta u(x,t)dx\\&+2\int_{\mathbb{R}^2}|u(x,t)|^{2\theta-2}|\nabla u(x,t)|^2u(x,t)\Delta u(x,t)dx+3\int_{\mathbb{R}^2}|u(x,t)|^{2\theta}|\Delta u(x,t)|^2dx\\&+2\int_{\mathbb{R}^2}|u(x,t)|^{2\theta-2}(u(x,t)\cdot\nabla u(x,t))\nabla u(x,t)\Delta u(x,t)dx\\&+2\theta\int_{\mathbb{R}^2}|u(x,t)|^{2\theta-2}(u(x,t)\cdot\nabla u(x,t))\nabla u(x,t)\Delta u(x,t)dx|\\\le&6\theta||u(\cdot,t)||_{L^\infty}^{2\theta-1}\int_{\mathbb{R}^2}|\nabla u(x,t)|^2\Delta u(x,t)dx+3||u(\cdot,t)||_{L^\infty}^{2\theta}||\Delta u(\cdot,t)||_{L^2}^2\\\le&C(||u(\cdot,t)||_{L^\infty})(||\nabla u(\cdot,t)||_{L^4}^2+||\Delta u(\cdot,t)||_{L^2}^2)\\\le&C(||u(\cdot,t)||_{L^\infty})[(C||\Delta\nabla u(\cdot,t)||_{L^2}^{\frac{1}{2}}||u(\cdot,t)||_{L^2}^{\frac{1}{2}})^2+(C||\Delta\nabla u(\cdot,t)||_{L^2}^{\frac{2}{3}}||u(\cdot,t)||_{L^2}^{\frac{1}{3}})^2]\\\le&C(||u(\cdot,t)||_{L^\infty})[C||\Delta\nabla u(\cdot,t)||_{L^2}^2+||u(\cdot,t)||_{L^2}^2+C||\Delta\nabla u(\cdot,t)||_{L^2}^2+||u(\cdot,t)||_{L^2}^2]\\\le&\frac{1}{4}||\nabla\Delta u(\cdot,t)||_{L^2}^2+C(||u_0||_{H^2}).\end{aligned}\tag{2.17}$$

Integrating (2.15) and using Gronwall inequality, we get (2.13).

Taking the Laplacian on both sides of (1.1), and then taking inner product with Δ^2u, we have by integration by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\Delta\nabla u(x,t)|^2dx+\int_{\mathbb{R}^2}|\Delta^2u(x,t)|^2dx=&\int_{\mathbb{R}^2}k\Delta[(1+\mu|u(x,t)|^{2\theta})u(x,t)]\Delta^2u(x,t)dx\\&-2\int_{\mathbb{R}^2}(\nabla u(x,t)\times\Delta\nabla u(x,t))\cdot\Delta^2u(x,t)dx\\\le&\frac{1}{4}||\Delta^2u(x,t)||_{L^2}^2+C(T,||u_0||_{H^3})(1+||\Delta\nabla u(x,t)||_{L^2}^2)+C(||u_0||_{H^3}),\end{aligned}\tag{2.18}$$

where we have used Hölder inequality, Gagliardu-Nirenberg's inequality and the embedding theorem of Sobolev spaces. Gronwall's inequality then implies the result (2.14). ☐

Lemma 2.3. *Let the dimension $d=3$ and assume that $u\in H^m(m\ge 2)$ with $||u_0||_{H^2}$ sufficiently small, then for the smooth solution of problem* (1.1)-(1.2) *one has*

$$||\Delta u(\cdot,t)||_{L^2}^2+\int_0^t||\Delta\nabla u(\cdot,s)||_{L^2}^2\le C(T,||u_0||_{H^2}),\qquad\forall T>0,t\in[0,T].\tag{2.19}$$

Furthermore, if $m \geq 3$, we have

$$||\Delta\nabla u(\cdot,s)||_{L^2}^2 + \int_0^t ||\Delta u(\cdot,s)||_{L^2}^2 ds \leq C(T, ||u_0||_{H^3}), \qquad \forall T > 0, t \in [0,T]. \tag{2.20}$$

Proof. Taking the Laplacian operator on both sides of (1.1), and then taking inner product of the resultant with Δu, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\Delta u|^2 dx + \int_{\mathbb{R}^3} |\Delta\nabla u|^2 dx + k\int_{\mathbb{R}^3} |\Delta u|^2 dx + k\int_{\mathbb{R}^3} \Delta(\mu|u|^{2\theta})u \cdot \Delta u dx = \int_{\mathbb{R}^3} (\nabla u \times \Delta\nabla u) \cdot \Delta u dx. \tag{2.21}$$

For the term on the right, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^3} (\nabla u \times \Delta\nabla u) \cdot \Delta u dx\right| &\leq 2\|\nabla u\|_{L^6}\|\Delta u\|_{L^3}\|\nabla\Delta u\|_{L^2} \\ &\leq C'\|u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}^2 \leq \frac{1}{4}||\Delta\nabla u(\cdot,t)||_{L^2}^2, \end{aligned} \tag{2.22}$$

thanks to (2.7), provided that $C'\|u\|_{L^\infty} \leq CC'\|u_0\|_{H^2} \leq 1/4$, i.e., provided that $\|u_0\|_{H^2} \leq 1/(4CC')$. Here the constant C is the constant in (2.7). Thanks to (2.17), we have for the last term on the left that

$$\left|k\int_{\mathbb{R}^3} \Delta(\mu|u|^{2\theta})u dx\right| \leq C(||u_0||_{H^2}) + \frac{1}{4}||\Delta\nabla u(\cdot,t)||_{L^2}^2. \tag{2.23}$$

Then integrating over time, we obtain(2.19).

If we take the inner product of the resultant with $\Delta^2 u$, we then have by integration by parts

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\Delta\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta^2 u|^2 dx = \int_{\mathbb{R}^3} k\Delta[(1+\mu|u|^{2\theta})u]\Delta^2 u dx - 2\int_{\mathbb{R}^3} (\nabla u \times \Delta\nabla u) \cdot \Delta^2 u dx. \tag{2.24}$$

Similarly, we get

$$\left|-2\int_{\mathbb{R}^3} (\nabla u \times \Delta\nabla u) \cdot \Delta^2 u dx\right| \leq \frac{1}{4}||\Delta^2 u(\cdot,t)||_{L^2}^2 + C(T, ||u_0||_{H^3})(1 + ||\Delta\nabla u||_{L^2}^2), \tag{2.25}$$

and

$$\left|\int_{\mathbb{R}^3} \Delta(|u|^{2\theta}u)\Delta^2 u dx\right| \leq \frac{1}{4}||\Delta^2 u||_{L^2}^2 + C(||u_0||_{H^3}). \tag{2.26}$$

Combining (2.24)-(2.26), and using Gronwall's inequality, we have (2.20). ☐

In the following, we consider higher order regularities of the LLB equation.

Lemma 2.4. *Let $m \geq 4$, $d = 2, 3$ and $u_0 \in H^m$. For any $T > 0$, the global smooth solution of* (1.1)-(1.2) *satisfies the following estimates*

$$||\Delta^2 u(t)||_{L^2}^2 + \int_0^t ||\Delta^2\nabla u(\cdot,s)||_{L^2}^2 ds \leq C(T, ||u_0||_{H^4}), \qquad \forall T > 0, \ t \in [0,T]. \tag{2.27}$$

Proof. Taking Δ on both sides of the LLB equation and then taking inner product with $\Delta^3 u$, we get by integration by parts that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\Delta^2 u||_{L^2}^2 + ||\Delta^2\nabla u||_{L^2}^2 = &-2\sum_{j,k=1}^{d}\int_{\mathbb{R}^d} (\partial_{x_j}\partial_{x_k} u \times \Delta\partial_{x_j} u) \cdot \Delta^2\partial_{x_k} u dx + k\sum_{j=1}^{d}\int_{\mathbb{R}^d} \Delta\partial_{x_j} u \cdot \Delta^2\partial_{x_j} u dx \\ &-2\sum_{j,k=1}^{d}\int_{\mathbb{R}^d} (\partial_{x_j} u \times \Delta\partial_{x_j}\partial_{x_k} u) \cdot \Delta^2\partial_{x_k} u dx - \sum_{j,k=1}^{d}\int_{\mathbb{R}^d} (\partial_{x_j} u \times \Delta^2 u) \cdot \Delta^2\partial_{x_j} u dx - k\mu\int_{\mathbb{R}^d} \Delta(|u|^{2\theta}u) \cdot \Delta^3 u dx. \end{aligned} \tag{2.28}$$

For the first term, by using Hölder inequality, Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned}
\left|2\sum_{j,k=1}^{d}\int_{\mathbb{R}^3}(\partial_{x_j}\partial_{x_k}u\times\Delta\partial_{x_j}u)\cdot\Delta^2\partial_{x_k}udx\right| &\leq 2\sum_{j,k=1}^{d}||\partial_{x_j}\partial_{x_k}u||_{L^4}||\Delta\partial_{x_j}u)||_{L^4}||\Delta^2\partial_{x_k}u||_{L^2}\\
&\leq C(||\Delta\nabla u||_{L^2}^{\frac{d+8}{12}}||u||_{L^2}^{\frac{4-d}{12}})(||\Delta^2u||_{L^2}^{\frac{d+12}{16}}||u||_{L^2}^{\frac{4-d}{16}})||\Delta^2\nabla u||_{L^2} \qquad (2.29)\\
&\leq C||\Delta\nabla u||_{L^2}^{\frac{8(8+d)}{3(4-d)}}||u||_{L^2}^{\frac{14}{3}}+||\Delta^2u||_{L^2}^2+\frac{1}{6}||\Delta^2\nabla u||_{L^2}^2\\
&\leq C(T,||u_0||_{H^3})+||\Delta^2u||_{L^2}^2+\frac{1}{6}||\Delta^2\nabla u||_{L^2}^2,
\end{aligned}$$

and similarly

$$\begin{aligned}
2\left|\sum_{j,k=1}^{d}\int_{\mathbb{R}^3}(\partial_{x_j}u\times\Delta\partial_{x_j}\partial_{x_k}u)\cdot\Delta^2\partial_{x_k}udx\right| \leq& 2|\sum_{j,k=1}^{d}||\partial_{x_j}u||_{L^\infty}||\Delta\partial_{x_j}\partial_{x_k}u||_{L^2}||\Delta^2\partial_{x_k}u||_{L^2}\\
\leq& C||\Delta^2u||_{L^2}^2+\frac{1}{6}||\Delta^2\nabla u||_{L^2}^2.
\end{aligned} \qquad (2.30)$$

For the last term, we have

$$\left|k\mu\int_{\mathbb{R}^3}\Delta(|u|^{2\theta}u)\cdot\Delta^3udx\right|\leq C(T,||u_0||_{H^3})+\frac{1}{6}||\Delta^2\nabla u||_{L^2}^2. \qquad (2.31)$$

Combining (2.29)-(2.31) we get (2.27) by Gronwall's inequality. ☐

Lemma 2.5. *Let* $d=2,3$, $u_0\in H^m(m\geq 2)$. *If Furthermore,* $||u_0||_{H^2}$ *is sufficiently small when* $d=3$, *then the smooth solution of the LLB equation (1.1)-(1.2) satisfies*

$$||\partial_t^j\partial_x^\alpha u(\cdot,t)||_{L^2}^2\leq C(T;||u_0||_{H^m}),\qquad \forall T>0,\quad t\in[0,T],\quad \textit{and} \qquad (2.32)$$

$$\int_0^t||\partial_t^h\partial_x^\beta u(\cdot,s)||_{L^2}^2ds\leq C(T;||u_0||_{H^m}),\qquad \forall T>0,\ t\in[0,T], \qquad (2.33)$$

where $2j+|\alpha|\leq m$, $2k+|\beta|\ \leq m+1$.

Proof. The proof is standard by mathematical induction, since we have already proven the key estimates in the previous lemmas. For the time derivative estimates, we should only note that we can trade time derivatives with spatial derivatives by using the LLB equation (1.1). The details are omitted by clarity. ☐

Next, we consider the uniqueness of the global smooth solutions. We have the following lemma.

Lemma 2.6. *Let* $d=2,3$, *and* u *and* v *be two smooth solutions of problem (1.1)–(1.2), with the same initial data* $u_0=v_0\in H^\infty(\mathbb{R}^d)$, *then* $u(t,\cdot)\equiv v(t,\cdot)$ *for any* $t\geq 0$.

Proof. Set $\omega=u-v$, then

$$\omega_t=\Delta\omega+\omega\times\Delta u+v\times\Delta\omega-k\omega-k(|u|^{2\theta}u-|v|^{2\theta}v). \qquad (2.34)$$

Taking the inner product with ω on both sides leads to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|\omega|^2dx=&-\int_{\mathbb{R}^d}|\nabla\omega|^2dx+\int_{\mathbb{R}^d}(v\times\Delta\omega)\cdot\omega dx\\
&-k\int_{\mathbb{R}^d}|\omega|^2dx-k\int_{\mathbb{R}^d}(|u|^{2\theta}u-|v|^{2\theta}v)\cdot\omega dx.
\end{aligned} \qquad (2.35)$$

The second term can be estimated as

$$\left|\int_{\mathbb{R}^d}(v\times\Delta\omega)\cdot\omega dx\right|=\left|\int_{\mathbb{R}^d}(\nabla v\times\nabla\omega)\cdot\omega dx\right|\leq 2||\nabla v||_{L^\infty}^2||\omega||_{L^2}^2+\frac{1}{2}||\nabla\omega||_{L^2}^2. \qquad (2.36)$$

For the last term, we have

$$\begin{aligned}\int_{\mathbb{R}^d}(|u|^{2\theta}u-|v|^{2\theta}v)\cdot\omega dx\leq&\int_{\mathbb{R}^d}(|u|+|v|)^{2\theta}\cdot|\omega|^2dx\\ \leq&(||u||_{L^\infty}+||v||_{L^\infty})^{2\theta}\int_{\mathbb{R}^d}|\omega|^2dx\leq C\int_{\mathbb{R}^d}|\omega|^2dx\end{aligned}\tag{2.37}$$

where C depends on the $||\cdot||_{L^\infty}$ of the solutions and hence can be controlled by the initial data $\|u_0\|_{H^2}$ thanks to (2.7). So we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}|\omega|^2dx\leq C\int_{\mathbb{R}^d}|\omega|^2dx.\tag{2.38}$$

Gronwall's inequality and the fact that $\omega(x,0)\equiv 0$ then imply the uniqueness. ☐

Proof of Theorem 1.1 and 1.2. The proof is standard since we have shown the desired estimates of the smooth solutions. For the time derivative estimates in (1.3) and (1.4), we need only to trade the spatial derivatives for the time derivatives. Uniqueness is proved in Lemma 2.6 since the solution is sufficiently regular. ☐

References

[1] U. Atxitia, D. Hinzke and U. Nowak, Fundamentals and applications of the Landau-Lifshitz-Bloch equation, *Journal of Physics D: Applied Physics*, 2017, 50(3): 033003.

[2] U. Atxitia, P. Nieves and O. Chubykalo-Fesenko, Landau-Lifshitz-Bloch equation for ferrimagnetic materials, *Physical Review B*, 2012, 86(10): 104414.

[3] A. Berti and C. Giorgi, Derivation of the Landau-Lifshitz-Bloch equation from continuum thermodynamics, *Physica B: Condensed Matter*, 2016, 500: 142–153.

[4] Z. Y. Chen, Z. R. Yan, M. H. Qin, J.-M. Liu, Landau-Lifshitz-Bloch equation for domain wall motion in antiferromagnets, *Physical Review B* 2019, 99(21): 214436.

[5] R. F. L. Evans, D. Hinzke, U. Atxitia, U. Nowak, R. W. Chantrell and O. Chubykalo-Fesenko, Stochastic form of the Landau-Lifshitz-Bloch equation, *Physical Review B*, 2012, 85(1): 014433.

[6] B. Guo and S. Ding, Landau-Lifshitz equations, *World Scientific*, 2008.

[7] B. Guo and Y. Han, Global smooth solution of Hydrodynamical equation for the Heisenberg paremagnet, *Math. Meth. Appl. Sci.*, 2004, 27(2): 181-191.

[8] K. Le, Weak solutions of the Landau-Lifshitz-Bloch equation, *J. Differential Equations*, 2016, 261(12), 6699-6717.

[9] Q. Li, B. Guo and M. Zeng, Smooth solutions of the Landau-Lifshitz-Bloch equation, *J. Appl. Anal. Comput.*, 2021, 11(6), 2713-2721.

[10] A. Lunardi, *Semigroups and optimal regularity in parabolic problems*, Springer Science and Business Media, 2012.

[11] P. Nieves, D. Serantes and O. Chubykalo-Fesenko, The Landau-Lifshitz-Bloch equation for quantum spin, *Springer Proceedings in Physics*, 2015: 140-142.

[12] X. Pu and L. Yang, Global smooth solutions for the Landau-Lifshitz-Bloch equation with helicity term, *Appl. Math. Lett.* **133** (2022), Paper No. 108215, 7 pp.