Hybrid-Block Method for the Solution of Second Order Non-linear Differential Equations

Abstract

The Duffing equation is one of the most unique and special non-linear differential equations in light of its many real-world applications in areas ranging from physics to economics. This paper sets out to investigate and study some existing numerical methods proposed by different authors over the years and subsequently develop an alternative computational method that can be used to solve duffing oscillator equations. This new method was developed by adopting the power series as the basis function and integrating it within quarter-step intervals using the interpolation and collocation approach. The analysis of the new method was carried out and found to be zero-stable, consistent, and convergent. Four duffing problems were used to test the efficiency of the new method, and the results were found to be computationally reliable.

Keywords: Duffing Equations, Collocation, Block Method, Interpolation, Approximate Solution,

Significance Statement: The duffing equation is a dynamical system that displays chaotic behaviour such as the wave function. A good example is the wave function. In this paper, we present a new method for solving duffing second order differential equations based on numerical computational principles.

Introduction

Over recent years, nonlinear ordinary differential equations (NLODEs) have been used in modeling physical systems using different approaches. Various researchers have expressed interest in the efforts made to find the exact and/or nearest exact solutions.

Statistical methods in artificial intelligence have become prominent with the use of artificial neural networks (ANN**)** [1-3] and genetic algorithms to find solutions to nonlinear problems in engineering and the sciences. Examples of ANN-approaches in literature are [4, 5]. Other evolutionary computing approaches use algorithms like the differential transformed method [6], the improved Taylor matrix method [7]. [8] solved the van der pol oscillator by using a hybrid genetic algorithm, just to mention a few.

Other analytical and numerical techniques reported in the literature are the variation iteration method for solving the Duffy-Van der Pol equation [9], the linearization method for the solution of the Van der Pol and Duffing equations [10], and the decomposition method [11, 12, 13], with variations in DMs like the natural decomposition for solving the Duffy-Van der Pol equation and the restarted adomian decomposition method. [14] introduced a one-step sixth-order computational method for free damped and free undamped systems for second-order differential equations. The Homotopy Analysis Method (HAM) was also used for approximations of the Duffing oscillator with dual frequency excitations [15] and the coupling of the homotopy perturbation method and the variational approach for the solution of the nonlinear cubic-quintic duffing oscillator [16]. The hybrid method was used by [17], the modified differential transform method [18], and the trigonometrically fitted two-step Obrechkoff method [19]. The Duffing Oscillator phenomenon is remarkably used in classical applications in engineering, biology, and the sciences. It is named after Georg Duffing, a German who characterized oscillation and its chaotic properties in the early 1980s, which have since gained prominence due to the equation's exceptional ability to replicate its dynamics in the real world [20].The phenomenon is observed in the motion of a body subjected to nonlinear spring power, linear sticky damping, and periodic power. The Duffing oscillator is revealed in mechanical systems under the action of a periodic external force. Furthermore, Duffing oscillators have grown in relevance in magneto-elastic mechanical systems, fluid flow-induced vibration [21], large-amplitude oscillation of centrifugal governing systems [22], weak signal detection [23], etc.

All of the methods for solving second-order linear and nonlinear ODEs mentioned above are powerful and efficient in that they provide and are capable of higher-accuracy approximations and closed-form solutions if they exist; however, the rate of convergence can be slow; for example, the variational iteration method requires the evaluation of the Lagrangian multiplier, which frequently requires tedious algebraic calculations. Our new hybrid block method converges faster, needs less computing, and can be used to evaluate points that aren't on the grid.

In this paper, we will demonstrate a computational method for simulating Duffing oscillators of the form;

$$\mu''(t) + \rho\mu'(t) + \sigma\mu(t) + \tau\mu^3(t) = f(t) \tag{1}$$

with initial conditions, $\mu(0) = \psi,\ \mu'(0) = \omega$
where $\rho, \sigma, \tau, \psi, and\ \omega$ are real constants and $f(t)$ is a real-valued function. We shall assume that equation (1) satisfy the existence and uniqueness theorem stated below.

Theorem 1 [28]

$$\text{Let } u^{(n)} = f\left(x, u, u', \dots, u^{(n-1)}\right), u^{(k)}(x_0) = c_k, \tag{2}$$

where $k = 0,1, \dots, (n-1), x_0 \le x \le x_0 + a, |s_j - c_j| \le b, j = 0,1, \dots, (n-1), (a > 0, b > 0.$ Suppose the function $f(x, s_0, s_1, \dots, s_{n-1})$: is defined in R and in addition

(i) f is non-negative and non-decreasing in each of $x,\ s_0, s_1, \dots, s_{n-1}\ in\ R$
(ii) $f(x, c_0, c_1, \dots, c_{n-1}) > 0, for\ x_0 \le x \le x_0 + a,$ and
(iii) $c_k \ge 0, k = 0,1, \dots, n-1.$

Then, the initial value problem (2) has a unique solution in R. Consequently, this paper presents a new hybrid numerical method of ¼ step size for the solutions of duffing oscillators.

2 Mathematical Formulation:

We considered a power series approximate solution within the interval $\left[x_{n,}\ x_{n+\frac{1}{4}}\right]$ of the form

$$\mu(x) = \sum_{i=0}^{d+e-1} b_i x^i, \tag{3}$$

where d and e are the numbers of interpolation and collocation points respectively. The first and second derivatives of (3) give

$$\mu'(x) = \sum_{i=1}^{d+e-1} i b_i x^{i-1} \quad \& \quad \mu''(x) = \sum_{j=2}^{d+e-1} i(i-1) b_i x^{i-2} \tag{4}$$

Substituting $\mu''(x)$ of (4) into (1) gives

$$f\left(x, \mu, \mu'\right) = \sum_{i=2}^{d+e-1} i(i-1) b_i x^{i-2}. \tag{5}$$

Let the solution to (1) be sought on the partition $N: \alpha = x_0 < x_1 < x_2 < \dots < x_N = \beta$ with a constant step size (h) given as $h = x_{n+1/4} - x_n,\ n = 0,1,\dots,N$

Collocating (5) at $x_{n+e}, e = 0\left(\frac{1}{24}\right)\frac{1}{4}$. The polynomial in (5) passes through interpolation at points $x_{n+d}, d = \frac{1}{6}, \frac{5}{24}$, evaluating this gives $d + e$ system of the non-linear equation of the form

$$\mathrm{E}X = \Phi, \tag{6}$$

where

$$E = \begin{bmatrix} \varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_8 \end{bmatrix}^T,$$

$$\Phi = \begin{bmatrix} \mu_{\frac{1}{6}} & \mu_{n+5/24} & f_n & f_{n+\frac{1}{24}} & f_{n+\frac{1}{12}} & f_{n+\frac{1}{8}} & f_{n+\frac{1}{6}} & f_{n+\frac{5}{24}} & f_{n+\frac{1}{4}} \end{bmatrix}^T,$$

and

$$X = \begin{bmatrix}
1 & x_{n+\frac{1}{6}} & x^2_{n+\frac{1}{6}} & x^3_{n+\frac{1}{6}} & x^4_{n+\frac{1}{6}} & x^5_{n+\frac{1}{6}} & x^6_{n+\frac{1}{6}} & x^7_{n+\frac{1}{6}} & x^8_{n+\frac{1}{6}} \\
1 & x_{n+\frac{5}{24}} & x^2_{n+\frac{5}{24}} & x^3_{n+\frac{5}{24}} & x^4_{n+\frac{5}{24}} & x^5_{n+\frac{5}{24}} & x^6_{n+\frac{5}{24}} & x^7_{n+\frac{5}{24}} & x^8_{n+\frac{5}{24}} \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\
0 & 0 & 2 & 6x_{n+\frac{1}{24}} & 12x^2_{n+\frac{1}{24}} & 20x^3_{n+\frac{1}{24}} & 30x^4_{n+\frac{1}{24}} & 42x^5_{n+\frac{1}{24}} & 56x^6_{n+\frac{1}{24}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{12}} & 12x^2_{n+\frac{1}{12}} & 20x^3_{n+\frac{1}{12}} & 30x^4_{n+\frac{1}{12}} & 42x^5_{n+\frac{1}{12}} & 56x^6_{n+\frac{1}{12}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{8}} & 12x^2_{n+\frac{1}{8}} & 20x^3_{n+\frac{1}{8}} & 30x^4_{n+\frac{1}{8}} & 42x^5_{n+\frac{1}{8}} & 56x^6_{n+\frac{1}{8}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{6}} & 12x^2_{n+\frac{1}{6}} & 20x^3_{n+\frac{1}{6}} & 30x^4_{n+\frac{1}{6}} & 42x^5_{n+\frac{1}{6}} & 56x^6_{n+\frac{1}{6}} \\
0 & 0 & 2 & 6x_{n+\frac{5}{24}} & 12x^2_{n+\frac{5}{24}} & 20x^3_{n+\frac{5}{24}} & 30x^4_{n+\frac{5}{24}} & 42x^5_{n+\frac{5}{24}} & 56x^6_{n+\frac{5}{24}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12x^2_{n+\frac{1}{4}} & 20x^3_{n+\frac{1}{4}} & 30x^4_{n+\frac{1}{4}} & 42x^5_{n+\frac{1}{4}} & 56x^6_{n+\frac{1}{4}}
\end{bmatrix}.$$

We applied Gaussian elimination with the backward method to Solve (6) for $b_i s$. We then substituted our $b_i s$ into equation (2) to get the continuous hybrid linear multistep method of the form

$$\mu(x) = \alpha_{\frac{1}{6}}(t)\mu_{n+\frac{1}{6}} + \alpha_{\frac{5}{25}}(t)\mu_{n+\frac{5}{25}} + h^2\left(\sum_{i=0}^{1}\beta_i(t) f_{n+i} + \beta_j(t) f_{n+j}\right), \quad j = \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4} \tag{7}$$

The coefficients of μ_{n+i}, f_{n+i} and f_{n+j} give

$$\alpha_{\frac{1}{6}} = 5 - 24t, \qquad \alpha_{\frac{5}{24}} = 24t - 4,$$

$$\beta_{\frac{1}{24}} = \frac{-1}{2903\,040}\left(\begin{array}{l} 82\,556\,485\,632t^8 - 91\,729\,428\,480t^7 + 41\,469\,345\,792t^6 - 9698\,476\,032t^5 \\ +1212\,309\,504t^4 - 69\,672\,960t^3 + 176\,214t - 5305 \end{array}\right) \qquad \beta_0 = \frac{1}{7\,418\,240}\left(\begin{array}{l} 8 \\ + \end{array}\right.$$

$$\beta_{\frac{1}{12}} = \frac{1}{5806\,080}\left(\begin{array}{l} 412\,782\,428\,160t^8 - 435\,714\,785\,280t^7 + 183\,267\,753\,984t^6 - 38\,543\,081\,472t^5 \\ +4075\,868\,160t^4 - 74\,182\,400t^3 - 112\,182t + 19\,465 \end{array}\right)$$

$$\beta_{\frac{1}{8}} = \frac{-1}{4354\,560}\left(\begin{array}{l} 412\,782\,428\,160t^8 - 412\,782\,428\,160t^7 + 161\,864\,220\,672t^6 - 31\,102\,009\,344t^5 \\ +2949\,488\,640t^4 - 116\,121\,600t^3 - 264\,942t - 23\,165 \end{array}\right)$$

$$\beta_{\frac{1}{6}} = \frac{1}{5806\,080}\left(\begin{array}{l} 412\,782\,428\,160t^8 - 389\,850\,071\,040t^7 + 143\,136\,129\,024t^6 - 25\,667\,518\,464t^5 \\ +2299\,207\,680t^4 - 87\,091\,200t^3 - 158\,790t + 35\,425 \end{array}\right)$$

$$\beta_{\frac{5}{24}} = \frac{-1}{2903\,040}\left(\begin{array}{l} 82\,556\,485\,632t^8 - 73\,383\,542\,784t^7 + 25\,416\,695\,808t^6 - 4347\,592\,704t^5 \\ +376\,233\,984t^4 - 13\,934\,592t^3 + 20\,886t - 1945 \end{array}\right)$$

$$\beta_{\frac{1}{4}} = \frac{1}{2903\,040}\begin{pmatrix} 82\,556\,485\,632t^8 - 73\,383\,542\,784t^7 + 25\,416\,695\,808t^6 - 4347\,592\,704t^5 \\ +376\,233\,984t^4 - 13\,934\,592t^3 + 20\,886t - 1945 \end{pmatrix}$$

where $t = \dfrac{x - x_n}{h}$, $\mu_{n+i} = \mu(x_n + ih)$ and $f_{n+j} = f\left((x_n ih), \mu(x_n + ih), y'(x_n + ih)\right)$.

We then solved (7) for the independent solution at the hybrid points to get our continuous block scheme

$$\mu(x) = \sum_{i=0}^{1} \frac{(ih)^{(m)}}{m!}\mu_n^{(m)} + h^2\left(\sum_{i=0}^{1}\sigma_i(x) f_{n+i} + \sigma_j f_{n+j}\right),\ j = \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4} \qquad (8)$$

The coefficient of f_{n+i} and f_{n+j} give

$$\alpha_0 = \frac{165888}{35}t^8 - \frac{27\,648}{5}t^7 + 2688t^6 - \frac{3528}{5}t^5 + \frac{1624}{15}t^4 - \frac{49}{5}t^3 + \frac{1}{2}t^2$$

$$\alpha_{\frac{1}{24}} = -\frac{995\,328}{35}t^8 + \frac{221\,184}{7}t^7 - \frac{71\,424}{5}t^6 + \frac{16\,704}{5}t^5 - \frac{2088}{5}t^4 + 24t^3$$

$$\alpha_{\frac{1}{12}} = \frac{497\,664}{7}t^8 - \frac{525\,312}{7}t^7 + \frac{157\,824}{5}t^6 - \frac{33\,192}{5}t^5 + 702t^4 - 30t^3$$

$$\alpha_{\frac{1}{8}} = -\frac{663\,552}{7}t^8 + \frac{663\,552}{7}t^7 - \frac{185\,856}{5}t^6 + \frac{35\,712}{5}t^5 - \frac{2032}{3}t^4 + \frac{80}{3}t^3$$

$$\alpha_{\frac{1}{6}} = \frac{497\,664}{7}t^8 - \frac{470\,016}{7}t^7 + \frac{123\,264}{5}t^6 - \frac{22\,104}{5}t^5 + 396t^4 - 15t^3$$

$$\alpha_{\frac{5}{24}} = -\frac{995\,328}{35}t^8 + \frac{884\,736}{35}t^7 - \frac{43\,776}{5}t^6 + \frac{7488}{5}t^5 - \frac{648}{5}t^4 + \frac{24}{5}t^3$$

$$\alpha_{\frac{1}{4}} = \frac{165\,888}{35}t^8 - \frac{27\,648}{7}t^7 + \frac{6528}{5}t^6 - 216t^5 + \frac{274}{15}t^4 - \frac{2}{3}t^3$$

Evaluating (8) at $t = \frac{1}{24}\left(\frac{1}{24}\right)\frac{1}{4}$ gives us our discrete block which takes the form of

$$\mathbf{A}^{(0)}U_m^{(i)} = \sum_{i=0}^{1} h^i p_i \mu_n^{(i)} + h^2 q_{\mathbf{i}} f(\mu_n) + h^2 r_{\mathbf{i}}\mathbf{f}(U_m),\ i = 0,1, \qquad (9)$$

where

$$U_m = \begin{bmatrix} \mu_{n+\frac{1}{24}} & \mu_{n+\frac{1}{12}} & \mu_{n+\frac{1}{8}} & \mu_{n+\frac{1}{6}} & \mu_{n+\frac{5}{24}} & \mu_{n+\frac{1}{4}} \end{bmatrix}^T,$$

$$\mathbf{f(}U_m\mathbf{)} = \begin{bmatrix} f_{n+\frac{1}{24}} & f_{n+\frac{1}{8}} & f_{n+\frac{1}{6}} & f_{n+\frac{1}{8}} & \mu_{n+\frac{5}{24}} & \mu_{n+\frac{1}{4}} \end{bmatrix}^T,$$

$$\mu_n^{(i)} = \begin{bmatrix} \mu_{n-1}^{(i)} & \mu_{n-2}^{(i)} & \mu_{n-3}^{(i)} & \mu_{n-4}^{(i)} & \mu_{n-5}^{(i)} & \mu_n^{(i)} \end{bmatrix}^T,$$

$$\mathbf{f(}\mu_n\mathbf{)} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_{n-4} & f_{n-5} & f_n \end{bmatrix}^T,$$

where $A^{(0)} = 6\times 6$ Identity matrix.

when $i = 0$:

$$p_0=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},\quad p_1=\begin{bmatrix}0&0&0&0&0&\frac{1}{24}\\0&0&0&0&0&\frac{1}{12}\\0&0&0&0&0&\frac{1}{6}\\0&0&0&0&0&\frac{1}{8}\\0&0&0&0&0&\frac{5}{24}\\0&0&0&0&0&\frac{1}{4}\end{bmatrix},$$

,

$$r_0=\begin{bmatrix}\frac{275}{5184}&-\frac{5717}{120960}&\frac{10621}{272160}&-\frac{7703}{362880}&\frac{403}{60480}&\frac{28\,549}{1088\,640}\\ \frac{194}{945}&-\frac{8}{81}&\frac{788}{8505}&-\frac{97}{1890}&\frac{46}{2835}&\frac{1027}{17\,010}\\ \frac{165}{448}&-\frac{267}{4480}&\frac{5}{32}&-\frac{363}{4480}&\frac{57}{2240}&\frac{253}{2688}\\ \frac{1504}{2835}&-\frac{8}{945}&\frac{2624}{8505}&-\frac{8}{81}&\frac{32}{945}&\frac{1088}{8505}\\ \frac{8375}{12096}&\frac{3125}{72576}&\frac{25625}{54432}&-\frac{625}{24192}&\frac{275}{5184}&-\frac{1375}{217\,728}\\ \frac{6}{7}&\frac{3}{35}&\frac{68}{105}&\frac{3}{70}&\frac{6}{35}&0\end{bmatrix}$$

$i=1$

$$e_1=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\quad e_1=\begin{bmatrix}0&0&0&0&0&\frac{19087}{181440}\\0&0&0&0&0&\frac{1139}{11340}\\0&0&0&0&0&\frac{137}{1344}\\0&0&0&0&0&\frac{286}{2835}\\0&0&0&0&0&\frac{3715}{36288}\\0&0&0&0&0&\frac{41}{420}\end{bmatrix}$$

$$r_1=\begin{bmatrix}\frac{2713}{7560}&-\frac{15487}{60480}&\frac{586}{2835}&-\frac{6737}{60480}&\frac{263}{7560}&-\frac{863}{181440}\\ \frac{94}{189}&\frac{11}{3780}&\frac{332}{2835}&-\frac{269}{3780}&\frac{22}{945}&\frac{37}{11\,340}\\ \frac{27}{56}&\frac{387}{2240}&\frac{34}{105}&-\frac{243}{2240}&\frac{9}{280}&-\frac{29}{6720}\\ \frac{464}{945}&\frac{128}{945}&\frac{1504}{2835}&\frac{58}{945}&\frac{16}{945}&-\frac{8}{2835}\\ \frac{725}{1512}&\frac{2125}{12096}&\frac{250}{567}&\frac{3875}{12096}&\frac{235}{1512}&-\frac{1375}{36288}\\ \frac{18}{35}&\frac{9}{140}&\frac{68}{105}&\frac{9}{140}&\frac{18}{35}&\frac{41}{420}\end{bmatrix}$$

3 Analysis of the New Method
3.1 Order of the block [26]

Let the linear operator $\{\mu(x):h\}$ associated with the discrete block method (10) be defined as

$$\{y(x):h\}=\mathbf{A}^{(0)}U_m^{(i)}-\sum_{i=0}^{1}h^i p_i\mu_n^{(i)}-h^2\left(q_{\mathbf{i}}f(\mu_n)+h^2 r_{\mathbf{i}}\mathbf{f}(U_m)\right). \quad (9)$$

Expanding (10) in the Taylor series and comparing the coefficients of h gives

$L\{\mu(x):h\} = C_0\mu(x) + C_1\mu'(x) + \ldots + C_s h^s \mu^s(x) + C_{s+1} h^{s+2}\mu^{s+2}(x) + C_{s+2} h^{s+2}\mu^{s+2}(x) + \ldots$ where the constant coefficients $C_s, s = 0,1,2,\ldots$
Where

$$C_0 = \sum_{i}^{k} \alpha i, C_1 = \sum_{i}^{k} (i\alpha_i - \beta_i), \quad \ldots C_0 = \sum_{i}^{k} \left[\frac{1}{\delta!} i^{\delta}\alpha_i - \frac{1}{(\delta-1)!} i^{\delta-1}\beta_i\right], \delta = 2,3\ldots$$

The linear operator L associated with the derived block formula is of order s *if* $C_0 = C_1 = \ldots = C_s = C_{s+1} = 0$ *and* $C_{s+2} \neq 0.$ C_{s+2} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{s+2} h^{s+2} y^{s+2}(x) + O\left(h^{s+3}\right).$

Comparing the coefficient efficient of h gives $C_0 = C_1 = \ldots = C_8 = 0$ and implies that the order is the derived method is 6 and the error constant is given by

$$C_8 = [2.5164 \times 10^{-15}, \; 6.222 \times 10^{-15}, \; 9.7336 \times 10^{-15}, -5.8282 \times 10^{-11}, \; 3.3777 \times 10^{-9}, 1.5991 \times 10^{-14}]$$

3.2 Zero stability of the method

Definition: *A block method is said to be zero stable if as* $h \to 0,$ *the roots* $r_j, = 1(1)k$ *of the first characteristic polynomial* $\rho(r) = 0$ *that is* $\rho(r) = \det\left[\sum A^{(0)} R^{k-1}\right] = 0$ *satisfying* $|R| \le 1,$ must have multiplicity equal to unity [26]

For our method

$$\rho(R) = \left[R\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] = 0$$

$$\rho(R) = R^5(R-1) = 0, R = 0,0,0,0,0,1$$

Hence the derived method is zero stable.

3.3. Stability Interval [25]

Definition: *The derived block method is said to be absolutely stable within a given interval if for a givenh all roots* z_s *of the characteristic polynomial*
$\pi(z,h) = \rho(z) + h^2\sigma(z) = 0.$ *Satisfying* $|z_s| < 1, s = 1,2,\ldots,n,$ *where* $h = \lambda^2 h^2$ *and* $\lambda = \frac{\delta f}{\delta y}.$
We adopted the boundary locus method to determine the stability interval of our block.
Substituting$y'' = -\lambda^2 y$ into the block (8) and substituting$e^{i\theta} = cos\theta + isin\theta$, gives

$$\frac{2-2cos\frac{1}{4}\theta}{\frac{5123}{679360348\,051492483\,591386076\,649881600} - \frac{157}{256\,183968017\,402574624\,179569303\,832494080}cos\frac{1}{4}\theta}.$$

Hence, the method is absolutely stable as shown in Figure 1 below.

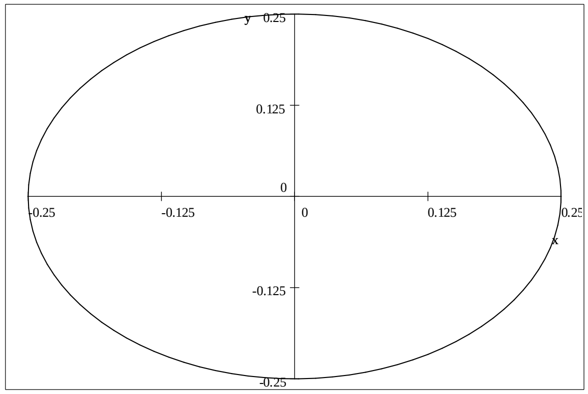


Fig. 1: Showing the stability region of the derived block method.

3.4 Consistency of the Method

Consistency ensures that the magnitude of the local truncated error at each stage of computation is regulated. According to [15], [26], for a linear multistep method to be consistent, the order of the method $s, s \geq 1$. Hence the derived method is consistent since it has uniform order $s = 7 \geq 1$.

3.5. Convergence of the derived method

According to [26], a linear multistep method is convergent if it is stable and consistent. Therefore, the derived method is convergent.

4 Implementation of the New Method

In this section, we present the implementation of the new method.We shall use "EDM" to mean error in our new derived method.

Example 1: Consider the damped duffing equation,

$$\mu''(t) + \mu'(t) + \mu(t) + \mu^3(t) = cos^3(t) - sin\,(t)$$

whose initial conditions are:

$$\mu(0) = 1, \mu'(0) = 0$$

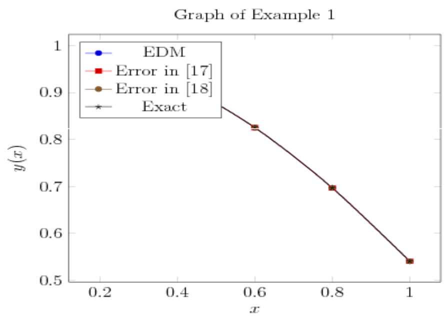
The exact solution is given by

$$\mu(t) = cos\,(t)$$

Source [6], [27]

Table 1: Showing the comparison of the absolute errors in [6], [27] and the derived method

t	*Error in [6]*	*Error in [27]*	*EDM*	*Time/s*
0.2	6×10^{-10}	9.32×10^{-12}	2.22×10^{-16}	*0.3349*
0.4	4×10^{-10}	4.25×10^{-11}	6.66×10^{-16}	*0.4397*
0.6	1.7×10^{-09}	8.63×10^{-11}	1.22×10^{-15}	0.5577
0.8	3.99×10^{-08}	1.29×10^{-10}	4.33×10^{-15}	*0.6535*
1.0	3.599×10^{-7}	1.61×10^{-10}	8.22×10^{-15}	*0.7487*



Example 2:

Consider the damped Duffing equation,

$$\mu''(t) + 2\mu'(t) + \mu(t) + 8\mu^3(t) = e^{-3t}$$

With the initial conditions,

$$\mu(0) = \frac{1}{2}, \mu'(0) = -\frac{1}{2}$$

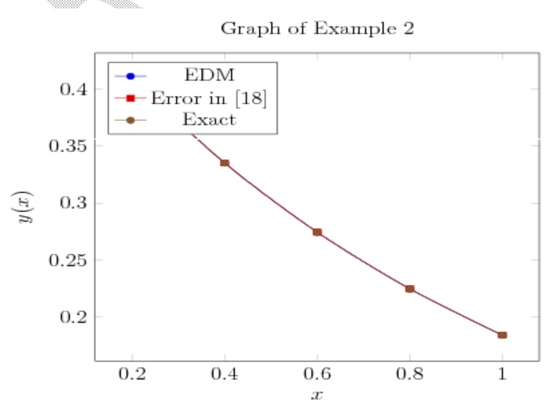
The exact solution is given by

$$\mu(t) = \frac{1}{2}e^{-t}$$

Source [27]

Table 2: Showing the comparison of the absolute errors in [27] and the derived method

t	*Exact Solution*	*Approximate Solution*	*EDM*	*Error in [27]*	*Time/s*
0.1	0.4524187090179798	0.4524187090179803	4.996004e-16	1.4872-08	0.1124
0.2	0.4093653765389910	0.4093653765389926	1.609823e-15	1.286e-07	0.1701
0.3	0.3704091103408589	0.3704091103408619	2.997602e-15	1.464e-07	0.1871
0.4	0.3351600230178196	0.3351600230178241	4.440892e-15	1.393e-07	0.1927
0.5	0.3032653298563167	0.3032653298563224	5.717649e-15	1.845e-07	0.1984
0.6	0.2744058180470131	0.2744058180470199	6.772360e-15	2.422e.07	0.2035
0.7	0.2482926518957047	0.2482926518957123	7.577272e-15	2.468e-07	0.2113
0.8	0.2246644820586107	0.2246644820586189	8.132384e-15	2.127e-07	0.2172
0.9	0.2032848298702995	0.2032848298703079	8.437695e-15	1.987e-07	0.2231
1.0	0.1839397205857211	0.1839397205857296	8.493206e-15	2.071e-07	0.2300



Example 3:

Consider the undamped Duffing Equation,

$$\mu''(t) + \mu(t) + \mu^3(t) = (cost + \varepsilon sin10t)^3 - 99\varepsilon sin10t$$

With initial conditions,

$$\mu(0) = 1, \mu'(0) = 10\varepsilon$$

where $\varepsilon = 10^{-10}$.

The exact solution is given by,

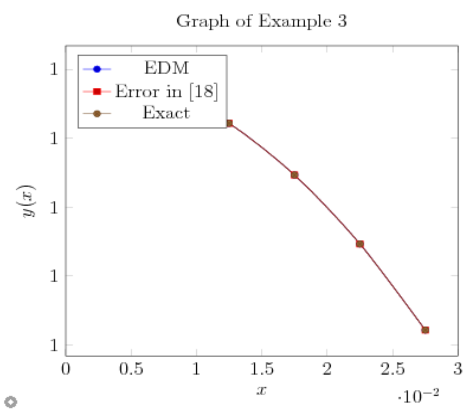
$$\mu(t) = cost + \varepsilon sin10t.$$

This equation describes a periodic motion of low frequency with a small perturbation of high frequency.

Source [27]

Table 3: Showing the comparison of the absolute errors in [27] and the derived method for Example 3

t	*Exact Solution*	*Approximate Solution*	*EDM*	*Error in [27]*	*Time/s*
0.0025	*0.9999968750041274*	*0.9999968750041274*	*0.000000e+00*	*0.000000e+00*	*0.0265*
0.0050	*0.9999875000310395*	*0.9999875000310395*	*0.000000e+00*	*1.110223e-016*	*0.0287*
0.0075	*0.9999718751393287*	*0.9999718751393286*	*1.110223e-16*	*8.881784e-016*	*0.0297*
0.0100	*0.9999500004266486*	*0.9999500004266486*	*0.000000e+00*	*7.771561e-016*	*0.0305*
0.0125	*0.9999218760297148*	*0.9999218760297148*	*0.000000e+00*	*4.440892e-016*	*0.0315*
0.0150	*0.9998875021243030*	*0.9998875021243031*	*1.110223e-16*	*9.992007e-016*	*0.0323*
0.0175	*0.9998468789252486*	*0.9998468789252487*	*1.110223e-16*	*1.665335e-015*	*0.0332*
0.0200	*0.9998000066864446*	*0.9998000066864449*	*2.220446e-16*	*2.775558e-015*	*0.0340*
0.0225	*0.9997468857008414*	*0.9997468857008415*	*1.110223e-16*	*5.440093e-015*	*0.0349*
0.0250	*0.9996875163004431*	*0.9996875163004431*	*0.000000e+00*	*7.216450e-015*	*0.0363*
0.0275	*0.9996218988563066*	*0.9996218988563066*	*0.000000e+00*	*9.436896e-015*	*0.0373*



Example 4:
Consider the undamped Duffing oscillator,

$$\mu''(t) + 3(\mu) - 2\mu^3(t) = cos(t)sin(2t)$$

with the initial condition,

$$\mu(0) = 0, \mu'(0) = 1$$

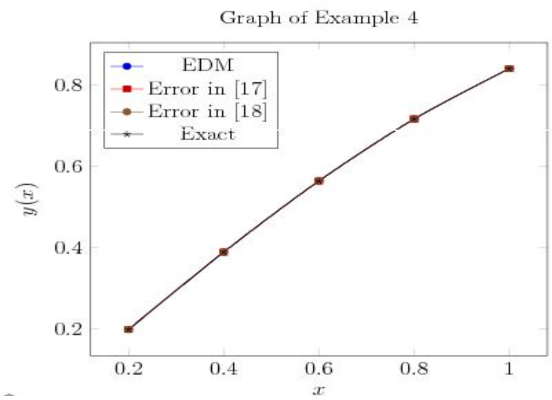
The exact solution is given by,

$$\mu(t) = sin(t)$$

Source:[7, 27]

Table 4: Showing the comparison of the absolute errors in [27, 28] and the derived method for Example 4

t	*Exact Solution*	*Approximate Solution*	*EDM*	*Error in [7]*	*Error in [27]*	*Time/s*
0.1	*0.0998334166468282*	*0.0998334166468282*	*0.000000e+00*	*3.603424e-07*	*3.024248e-13*	*0.0325*
0.2	*0.1986693307950612*	*0.1986693307950611*	*1.110223e-16*	*1.020596e-05*	*4.584944e-13*	*0.0410*
0.3	*0.2955202066613396*	*0.2955202066613391*	*4.440892e-16*	*2.357701e-05*	*7.316370e-14*	*0.0479*
0.4	*0.3894183423086506*	*0.3894183423086494*	*1.165734e-15*	*9.788940e-07*	*1.692257e-12*	*0.0541*
0.5	*0.4794255386042031*	*0.4794255386042009*	*2.220446e-15*	*1.601644e-05*	*4.596878e-12*	*0.0579*
0.6	*0.5646424733950355*	*0.5646424733950319*	*3.552714e-15*	*3.106965e-05*	*8.754997e-12*	*0.0613*
0.7	*0.6442176872376912*	*0.6442176872376860*	*5.218048e-15*	*8.505959e-06*	*1.390665e-11*	*0.0645*
0.8	*0.7173560908995230*	*0.7173560908995161*	*6.883383e-15*	*2.193132e-05*	*1.959244e-11*	*0.0698*
0.9	*0.7833269096274836*	*0.7833269096274751*	*8.548717e-15*	*3.183986e-05*	*2.519718e-11*	*0.0731*
1.0	*0.8414709848078967*	*0.8414709848078867*	*9.992007e-15*	*3.225774e-05*	*2.999911e-11*	*0.0766*



5. Discussion of Results

For the intent of application of the novel hybrid block technique, we have analyzed and studied four numerical examples. The findings shown in Tables 1 through 4 made it abundantly evident that the newly developed hybrid block approach functioned brilliantly in contrast to the methods described in [6,7] and [27]. The comparison was established since both authors handled the same problems, and the order of the method in [27] is equal to the order in our new method. The difference in error between our new method and the methods proposed by [6,7] and [27] is also less according to the implementation of our new method. The convergence of our numerical method may be seen graphically in each of the four examples' graphs.

6. Conclusion

For the purpose of finding a solution to duffing oscillator equations, we have presented a brand new hybrid block method of order 6 in this article. The section 2 explanation will walk you through how to derive the approximate solution. The investigation of the novel method is described in Section 3, and the application of it can be found in Section 4. Finally, in sections 5 and 6, respectively, we give our explanation of the findings and our conclusion to the reader. The findings of the newly developed method demonstrated that it is superior in terms of accuracy, consistency, and effectiveness.

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