Hybrid-Block Method for the Solution of Second Order Non-linear Differential Equations

Abstract
The Duffing equation is one of the unique and special non-linear differential equations in light of its many real-world applications in areas ranging from physics to economics. This paper sets out to investigate and study some existing numerical methods proposed by different authors over the years and subsequently develop an alternative computational method that can be used to solve duffing oscillators equations. This new method was developed by adopting the power series as the basis function and integrating it within quarter-step intervals using the interpolation and collocation approach. The analysis of the new method was carried out and found to be zero-stable, consistent and convergent. Four duffing problems were used to test the efficiency of the new method and the results were found to be computationally reliable.
Keywords: Duffing Equations, Collocation, block method, Interpolation, Approximate Solution,
Significance Statement: The duffing equation is a dynamical system that displays chaotic behaviour such as the wave function. A Good example is the wave function. In this research work, we provide a new method of solving duffing second order differential equations using the principles of numerical computational approach.

Introduction

Over recent years, nonlinear ordinary differential equations (NLODEs) have been used in modeling physical systems using different approaches. The efforts exerted in finding the exact and or the nearest exact solutions have received attention from different researchers.

Statistical methods in artificial intelligence have become prominent with the use of artificial neural networks (ANN**)** [1-3] and genetic algorithms to find solutions to nonlinear problems in engineering and the sciences. Examples of ANN-approaches in literature are [4, 5]. Other evolutionary computing approaches use algorithms like the differential transformed method [6], the improved Taylor matrix method [7]. [8] solved the van der pol oscillator by using a hybrid genetic algorithm, just to mention a few.

Other analytical and numerical techniques reported in literature are the variation iteration method for solving the Duffy-Van der Pol equation [9], the linearization method for the solution of the Van der Pol and Duffing equations [10], and the decomposition method [11, 12, 13], with variations in DMs like the natural decomposition for solving the Duffy-Van der Pol equation and the restarted adomian decomposition method. [14] introduced a one-step sixth-order computational method for free damped and free undamped systems for second-order differential equations. The Homotopy Analysis Method (HAM) was also used for approximations of the Duffing oscillator with dual frequency excitations [15] and the coupling of the homotopy perturbation method and the variational approach for the solution of the nonlinear cubic-quintic duffing oscillator [16]. The hybrid method was used by [17], the modified differential transform method [18], and the trigonometrically fitted two-step Obrechkoff method [19]. The Duffing Oscillator phenomenon is remarkably used in classical applications in engineering, biology, and the sciences. It is named after the German, Georg Duffing, who in the early 80s characterized oscillation and its chaotic properties, which has since gained prominence due to the equation's outstanding ability to replicate its dynamics in the real world [20]. The phenomenon is observed in the motion of a body subjected to nonlinear spring power, linear sticky damping, and periodic power. The Duffing oscillator is revealed in mechanical systems under the action of a periodic external force. Furthermore, Duffing oscillators have grown into relevance in magneto-elastic mechanical systems, fluid flow-induced vibration [21], large-size amplitude oscillation of centrifugal governing systems [22], and weak signal detection [23], etc.

In this paper, we will demonstrate a computational method for simulating Duffing oscillators of the form;

$$\mu''(t) + \rho\mu'(t) + \sigma\mu(t) + \tau\mu^3(t) = f(t) \qquad (1)$$

with initial conditions, $\mu(0) = \psi,\ \mu'(0) = \omega$
where $\rho, \sigma, \tau, \psi, and\ \omega$ are real constants and $f(t)$ is a real-valued function. We shall assume that equation (1) satisfy the existence and uniqueness theorem stated below.

Theorem 1 [30]

$$\text{Let } u^{(n)} = f(x, u, u', \dots, u^{(n-1)}), u^{(k)}(x_0) = c_k \qquad (2)$$

Where $k = 0,1, \dots, (n-1), u\ and\ f\ are\ scalars.$
$x_0 \le x \le x_0 + a, |s_j - c_j| \le b, j = 0,1, \dots, (n-1), (a > 0, b > 0.$ Suppose the function $f(x, s_0, s_1, \dots, s_{n-1})$: is defined in R and in addition

(i) f is non-negative and non-decreasing in each of $x,\ s_0, s_1, \dots, s_{n-1}\ in\ R$
(ii) $f(x, c_0, c_1, \dots, c_{n-1}) > 0, for\ x_0 \le x \le x_0 + a$, and
(iii) $c_k \ge 0, k = 0,1, \dots, n-1$.

Then, the initial value problem (2) has a unique solution in R

Consequently, this paper presents a new hybrid numerical method of ¼ step size for the solutions of duffing oscillators.

2 Mathematical Formulation:

We considered a power series approximate solution within the interval $\left[x_{n,}\ x_{n+\frac{1}{4}}\right]$ of the form

$$\mu(x) = \sum_{i=0}^{d+e-1} b_i x^i, \tag{3}$$

where d and e are the numbers of interpolation and collocation points respectively. The first and second derivatives of (3) give

$$\mu^{'}(x) = \sum_{i=1}^{d+e-1} i b_i x^{i-1} \ \& \ \mu^{''}(x) = \sum_{j=2}^{d+e-1} i(i-1) b_i x^{i-2} \tag{4}$$

Substituting $\mu^{''}(x)$ of (4) into (1) gives

$$f\left(x, \mu, \mu^{'}\right) = \sum_{i=2}^{d+e-1} i(i-1) b_i x^{i-2}. \tag{5}$$

Let the solution to (1) be sought on the partition $N: \alpha = x_0 < x_1 < x_2 < \dots < x_N = \beta$ with a constant step size (h) given as $h = x_{n+\frac{1}{4}} - x_n,\ n = 0,1,\dots,N$

Collocating (5) at $x_{n+e}, e = 0\left(\frac{1}{24}\right)\frac{1}{4}$. The polynomial in (5) passes through interpolation at points $x_{n+d}, d = \frac{1}{6}, \frac{5}{24}$, evaluating this gives $d + e$ system of the non-linear equation of the form

$$\mathrm{E}X = \Phi, \tag{6}$$

where

$$\mathrm{E} = \begin{bmatrix} \varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_8 \end{bmatrix}^T,$$

$$\Phi = \begin{bmatrix} \mu_{\frac{1}{6}} & \mu_{n+5/24} & f_n & f_{n+\frac{1}{24}} & f_{n+\frac{1}{12}} & f_{n+\frac{1}{8}} & f_{n+\frac{1}{6}} & f_{n+\frac{5}{24}} & f_{n+\frac{1}{4}} \end{bmatrix}^T,$$

and

$$
X=\begin{bmatrix}
1 & x_{n+\frac{1}{6}} & x^2_{n+\frac{1}{6}} & x^3_{n+\frac{1}{6}} & x^4_{n+\frac{1}{6}} & x^5_{n+\frac{1}{6}} & x^6_{n+\frac{1}{6}} & x^7_{n+\frac{1}{6}} & x^8_{n+\frac{1}{6}} \\
1 & x_{n+\frac{5}{24}} & x^2_{n+\frac{5}{24}} & x^3_{n+\frac{5}{24}} & x^4_{n+\frac{5}{24}} & x^5_{n+\frac{5}{24}} & x^6_{n+\frac{5}{24}} & x^7_{n+\frac{5}{24}} & x^8_{n+\frac{5}{24}} \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\
0 & 0 & 2 & 6x_{n+\frac{1}{24}} & 12x^2_{n+\frac{1}{24}} & 20x^3_{n+\frac{1}{24}} & 30x^4_{n+\frac{1}{24}} & 42x^5_{n+\frac{1}{24}} & 56x^6_{n+\frac{1}{24}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{12}} & 12x^2_{n+\frac{1}{12}} & 20x^3_{n+\frac{1}{12}} & 30x^4_{n+\frac{1}{12}} & 42x^5_{n+\frac{1}{12}} & 56x^6_{n+\frac{1}{12}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{8}} & 12x^2_{n+\frac{1}{8}} & 20x^3_{n+\frac{1}{8}} & 30x^4_{n+\frac{1}{8}} & 42x^5_{n+\frac{1}{8}} & 56x^6_{n+\frac{1}{8}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{6}} & 12x^2_{n+\frac{1}{6}} & 20x^3_{n+\frac{1}{6}} & 30x^4_{n+\frac{1}{6}} & 42x^5_{n+\frac{1}{6}} & 56x^6_{n+\frac{1}{6}} \\
0 & 0 & 2 & 6x_{n+\frac{5}{24}} & 12x^2_{n+\frac{5}{24}} & 20x^3_{n+\frac{5}{24}} & 30x^4_{n+\frac{5}{24}} & 42x^5_{n+\frac{5}{24}} & 56x^6_{n+\frac{5}{24}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12x^2_{n+\frac{1}{4}} & 20x^3_{n+\frac{1}{4}} & 30x^4_{n+\frac{1}{4}} & 42x^5_{n+\frac{1}{4}} & 56x^6_{n+\frac{1}{4}}
\end{bmatrix}.
$$

We applied Gaussian elimination with the backward method to Solve (6) for $b_i s$. We then substituted our $b_i s$ into equation (2) to get the continuous hybrid linear multistep method of the form

$$
\mu(x)=\alpha_{\frac{1}{6}}(t)\mu_{n+\frac{1}{6}}+\alpha_{\frac{5}{25}}(t)\mu_{n+\frac{5}{25}}+h^2\left(\sum_{i=0}^{1}\beta_i(t)f_{n+i}+\beta_j(t)f_{n+j}\right),\quad j=\frac{1}{24},\frac{1}{12},\frac{1}{8},\frac{1}{6},\frac{5}{24},\frac{1}{4} \tag{7}
$$

The coefficients of μ_{n+i}, f_{n+i} and f_{n+j} give

$$
\alpha_{\frac{1}{6}}=5-24t,\qquad \alpha_{\frac{5}{24}}=24t-4,
$$

$$
\beta_{\frac{1}{24}}=\frac{-1}{2903\,040}\begin{pmatrix}82\,556\,485\,632t^8-91\,729\,428\,480t^7+41\,469\,345\,792t^6-9698\,476\,032t^5\\+1212\,309\,504t^4-69\,672\,960t^3+176\,214t-5305\end{pmatrix}
$$

$$
\beta_0=\frac{1}{7\,418\,240}\begin{pmatrix}82\,556\,485\,632t^8-96\,315\,899\,904t^7+46\,820\,229\,120t^6+12\,290\,310\,144t^5\\+1885\,814\,784t^4-170\,698\,752t^3+8709\,120t^2-221\,166t+2045\end{pmatrix}
$$

$$
\beta_{\frac{1}{12}}=\frac{1}{5806\,080}\begin{pmatrix}412\,782\,428\,160t^8-435\,714\,785\,280t^7+183\,267\,753\,984t^6-38\,543\,081\,472t^5\\+4075\,868\,160t^4-74\,182\,400t^3-112\,182t+19\,465\end{pmatrix}
$$

$$
\beta_{\frac{1}{8}}=\frac{-1}{4354\,560}\begin{pmatrix}412\,782\,428\,160t^8-412\,782\,428\,160t^7+161\,864\,220\,672t^6-31\,102\,009\,344t^5\\+2949\,488\,640t^4-116\,121\,600t^3-264\,942t-23\,165\end{pmatrix}
$$

$$
\beta_{\frac{1}{6}}=\frac{1}{5806\,080}\begin{pmatrix}412\,782\,428\,160t^8-389\,850\,071\,040t^7+143\,136\,129\,024t^6-25\,667\,518\,464t^5\\+2299\,207\,680t^4-87\,091\,200t^3-158\,790t+35\,425\end{pmatrix}
$$

$$
\beta_{\frac{5}{24}}=\frac{-1}{2903\,040}\begin{pmatrix}82\,556\,485\,632t^8-73\,383\,542\,784t^7+25\,416\,695\,808t^6-4347\,592\,704t^5\\+376\,233\,984t^4-13\,934\,592t^3+20\,886t-1945\end{pmatrix}
$$

$$\beta_{\frac{1}{4}} = \frac{1}{2903\,040}\begin{pmatrix} 82\,556\,485\,632t^8 - 73\,383\,542\,784t^7 + 25\,416\,695\,808t^6 - 4347\,592\,704t^5 \\ +376\,233\,984t^4 - 13\,934\,592t^3 + 20\,886t - 1945 \end{pmatrix}$$

where $t = \frac{x - x_n}{h}$, $\mu_{n+i} = \mu(x_n + ih)$ and $f_{n+j} = f((x_n ih), \mu(x_n + ih), y'(x_n + ih))$.

We then solved (7) for the independent solution at the hybrid points to get our continuous block scheme

$$\mu(x) = \sum_{i=0}^{1} \frac{(ih)^{(m)}}{m!}\mu_n^{(m)} + h^2\left(\sum_{i=0}^{1}\sigma_i(x) f_{n+i} + \sigma_j f_{n+j}\right),\ j = \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4} \quad (8)$$

The coefficient of f_{n+i} and f_{n+j} give

$$\alpha_0 = \frac{165888}{35}t^8 - \frac{27\,648}{5}t^7 + 2688t^6 - \frac{3528}{5}t^5 + \frac{1624}{15}t^4 - \frac{49}{5}t^3 + \frac{1}{2}t^2$$

$$\alpha_{\frac{1}{24}} = -\frac{995\,328}{35}t^8 + \frac{221\,184}{7}t^7 - \frac{71\,424}{5}t^6 + \frac{16\,704}{5}t^5 - \frac{2088}{5}t^4 + 24t^3$$

$$\alpha_{\frac{1}{12}} = \frac{497\,664}{7}t^8 - \frac{525\,312}{7}t^7 + \frac{157\,824}{5}t^6 - \frac{33\,192}{5}t^5 + 702t^4 - 30t^3$$

$$\alpha_{\frac{1}{8}} = -\frac{663\,552}{7}t^8 + \frac{663\,552}{7}t^7 - \frac{185\,856}{5}t^6 + \frac{35\,712}{5}t^5 - \frac{2032}{3}t^4 + \frac{80}{3}t^3$$

$$\alpha_{\frac{1}{6}} = \frac{497\,664}{7}t^8 - \frac{470\,016}{7}t^7 + \frac{123\,264}{5}t^6 - \frac{22\,104}{5}t^5 + 396t^4 - 15t^3$$

$$\alpha_{\frac{5}{24}} = -\frac{995\,328}{35}t^8 + \frac{884\,736}{35}t^7 - \frac{43\,776}{5}t^6 + \frac{7488}{5}t^5 - \frac{648}{5}t^4 + \frac{24}{5}t^3$$

$$\alpha_{\frac{1}{4}} = \frac{165\,888}{35}t^8 - \frac{27\,648}{7}t^7 + \frac{6528}{5}t^6 - 216t^5 + \frac{274}{15}t^4 - \frac{2}{3}t^3$$

Evaluating (8) at $t = \frac{1}{24}\left(\frac{1}{24}\right)\frac{1}{4}$ gives us our discrete block which takes the form of

$$\mathbf{A}^{(0)}U_m^{(i)} = \sum_{i=0}^{1} h^i p_i \mu_n^{(i)} + h^2 q_{\mathbf{i}} f(\mu_n) + h^2 r_{\mathbf{i}} \mathbf{f}(U_m),\ i = 0,1, \quad (9)$$

where

$$U_m = \begin{bmatrix} \mu_{n+\frac{1}{24}} & \mu_{n+\frac{1}{12}} & \mu_{n+\frac{1}{8}} & \mu_{n+\frac{1}{6}} & \mu_{n+\frac{5}{24}} & \mu_{n+\frac{1}{4}} \end{bmatrix}^T,$$

$$\mathbf{f}(U_m) = \begin{bmatrix} f_{n+\frac{1}{24}} & f_{n+\frac{1}{8}} & f_{n+\frac{1}{6}} & f_{n+\frac{1}{8}} & \mu_{n+\frac{5}{24}} & \mu_{n+\frac{1}{4}} \end{bmatrix}^T,$$

$$\mu_n^{(i)} = \begin{bmatrix} \mu_{n-1}^{(i)} & \mu_{n-2}^{(i)} & \mu_{n-3}^{(i)} & \mu_{n-4}^{(i)} & \mu_{n-5}^{(i)} & \mu_n^{(i)} \end{bmatrix}^T,$$

$$\mathbf{f}(\mu_n) = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_{n-4} & f_{n-5} & f_n \end{bmatrix}^T,$$

where $A^{(0)} = 6 \times 6$ Identity matrix.

when $i = 0$:

$$p_0 = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} 0&0&0&0&0&\frac{1}{24} \\ 0&0&0&0&0&\frac{1}{12} \\ 0&0&0&0&0&\frac{1}{6} \\ 0&0&0&0&0&\frac{1}{8} \\ 0&0&0&0&0&\frac{5}{24} \\ 0&0&0&0&0&\frac{1}{4} \end{bmatrix},$$

$$e_0 = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{bmatrix} e_1 = \begin{bmatrix} 0&0&0&0&0&\frac{1}{24} \\ 0&0&0&0&0&\frac{1}{12} \\ 0&0&0&0&0&\frac{1}{8} \\ 0&0&0&0&0&\frac{1}{6} \\ 0&0&0&0&0&\frac{5}{24} \\ 0&0&0&0&0&\frac{1}{4} \end{bmatrix}, \quad q_0 = \begin{bmatrix} 0&0&0&0&0&\frac{28\,549}{1088\,640} \\ 0&0&0&0&0&\frac{1027}{17\,010} \\ 0&0&0&0&0&\frac{253}{2688} \\ 0&0&0&0&0&\frac{1088}{8505} \\ 0&0&0&0&0&\frac{32225}{217\,728} \\ 0&0&0&0&0&\frac{41}{210} \end{bmatrix}$$

$$r_0 = \begin{bmatrix} \frac{275}{5184} & -\frac{5717}{120960} & \frac{10621}{272160} & -\frac{7703}{362880} & \frac{403}{60480} & \frac{28\,549}{1088\,640} \\ \frac{194}{945} & -\frac{8}{81} & \frac{788}{8505} & -\frac{97}{1890} & \frac{46}{2835} & \frac{1027}{17\,010} \\ \frac{165}{448} & -\frac{267}{4480} & \frac{5}{32} & -\frac{363}{4480} & \frac{57}{2240} & \frac{253}{2688} \\ \frac{1504}{2835} & -\frac{8}{945} & \frac{2624}{8505} & -\frac{8}{81} & \frac{32}{945} & \frac{1088}{8505} \\ \frac{8375}{12096} & \frac{3125}{72576} & \frac{25625}{54432} & -\frac{625}{24192} & \frac{275}{5184} & -\frac{1375}{217\,728} \\ \frac{6}{7} & \frac{3}{35} & \frac{68}{105} & \frac{3}{70} & \frac{6}{35} & 0 \end{bmatrix}$$

$i = 1$

$$e_1 = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{bmatrix} e_1 = \begin{bmatrix} 0&0&0&0&0&\frac{19087}{181440} \\ 0&0&0&0&0&\frac{1139}{11340} \\ 0&0&0&0&0&\frac{137}{1344} \\ 0&0&0&0&0&\frac{286}{2835} \\ 0&0&0&0&0&\frac{3715}{36288} \\ 0&0&0&0&0&\frac{41}{420} \end{bmatrix}$$

$$r_1 = \begin{bmatrix} \frac{2713}{7560} & -\frac{15487}{60480} & \frac{586}{2835} & -\frac{6737}{60480} & \frac{263}{7560} & -\frac{863}{181440} \\ \frac{94}{189} & \frac{11}{3780} & \frac{332}{2835} & -\frac{269}{3780} & \frac{22}{945} & \frac{37}{11\,340} \\ \frac{27}{56} & \frac{387}{2240} & \frac{34}{105} & -\frac{243}{2240} & \frac{9}{280} & -\frac{29}{6720} \\ \frac{464}{945} & \frac{128}{945} & \frac{1504}{2835} & \frac{58}{945} & \frac{16}{945} & -\frac{8}{2835} \\ \frac{725}{1512} & \frac{2125}{12096} & \frac{250}{567} & \frac{3875}{12096} & \frac{235}{1512} & -\frac{1375}{36288} \\ \frac{18}{35} & \frac{9}{140} & \frac{68}{105} & \frac{9}{140} & \frac{18}{35} & \frac{41}{420} \end{bmatrix}$$

3 Analysis of the New Method
3.1 Order of the block [26]

Let the linear operator $\{\mu(x):h\}$ associated with the discrete block method (10) be defined as

$$\{y(x):h\} = \mathbf{A}^{(0)}U_m^{(i)} - \sum_{i=0}^{1} h^i p_i \mu_n^{(i)} - h^2\left(q_{\mathbf{i}} f(\mu_n) + h^2 r_{\mathbf{i}} \mathbf{f}(U_m)\right). \tag{9}$$

Expanding (10) in the Taylor series and comparing the coefficients of h gives $L\{\mu(x):h\} = C_0\mu(x) + C_1\mu'(x) + \ldots + C_s h^s \mu^s(x) + C_{s+1}h^{s+2}\mu^{s+2}(x) + C_{s+2}h^{s+2}\mu^{s+2}(x) + \ldots$ where the constant coefficients $C_s, s = 0,1,2,\ldots$

Where

$$C_0 = \sum_i^k \alpha i, C_1 = \sum_i^k (i\alpha_i - \beta_i), \quad \ldots C_0 = \sum_i^k \left[\frac{1}{\delta!} i^\delta \alpha_i - \frac{1}{(\delta-1)!} i^{\delta-1}\beta_i\right], \delta = 2,3\ldots$$

The linear operator L associated with the derived block formula is of order s *if* $C_0 = C_1 = \ldots = C_s = C_{s+1} = 0$ *and* $C_{s+2} \neq 0$. C_{s+2} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{s+2}h^{s+2}y^{s+2}(x) + O\left(h^{s+3}\right)$.

Comparing the coefficient efficient of h gives $C_0 = C_1 = \ldots = C_8 = 0$ and implies that the order is the derived method is 7 and the error constant is given by

$$C_8 = [2.5164 \times 10^{-15}, \; 6.222 \times 10^{-15}, \; 9.7336 \times 10^{-15}, -5.8282 \times 10^{-11}, \; 3.3777 \times 10^{-9}, 1.5991 \times 10^{-14}]$$

3.2 Zero stability of the method

Definition: *A block method is said to be zero stable if as* $h \to 0$, *the roots* $r_j, = 1(1)k$ *of the first characteristic polynomial* $\rho(r) = 0$ *that is* $\rho(r) = \det\left[\sum A^{(0)}R^{k-1}\right] = 0$ *satisfying* $|R| \leq 1$, must have multiplicity equal to unity [26]

For our method

$$\rho(R) = \left| R\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\rho(R) = R^5(R-1) = 0, R = 0,0,0,0,0,1$$

Hence the derived method is zero stable.

3.3. Stability Interval [25]

Definition: *The derived block method is said to be absolutely stable within a given interval if for a given* h, *all roots* z_s *of the characteristic polynomial*

$\pi(z,h) = \rho(z) + h^2\sigma(z) = 0$, *satisfies* $|z_s| < 1, s = 1,2,\ldots,n$. *where* $h = \lambda^2 h^2$ *and* $\lambda = \frac{\partial f}{\partial y}$. *We adopted the boundary locus method to determine the stability interval of our block.*

Substituting $y'' = -\lambda^2 y$ into the block (8) and substituting $e^{i\theta} = \cos\theta + i\sin\theta$, gives

$$\frac{2-2\cos\frac{1}{4}\theta}{\frac{5123}{679360348\,051492483\,591386076\,649881600}-\frac{157}{256\,183968017\,402574624\,179569303\,832494080}\cos\frac{1}{4}\theta}$$

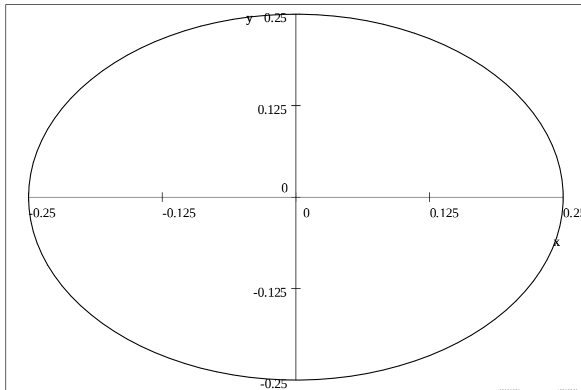


Fig. 1: Showing the stability region of the derived block method.

3.4 Consistency of the Method

Consistency ensures that the magnitude of the local truncated error at each stage of computation is regulated. According to [15], [26], for a linear multistep method to be consistent, the order of the method $s, s \geq 1$. Hence the derived method is consistent since it has uniform order $s = 7 \geq 1$.

3.5. Convergence of the derived method

According to [26], a linear multistep method is convergent if it is stable and consistent. Therefore, the derived method is convergent.

4 Implementation of the New Method

In this section, we present the implementation of the new method.
We shall use "EDM" to mean error in our new derived method.

Example 1: Consider the damped duffing equation,

$$\mu''(t) + \mu'(t) + \mu(t) + \mu^3(t) = cos^3(t) - sin\ (t)$$

whose initial conditions are:

$$\mu(0) = 1, \mu'(0) = 0$$

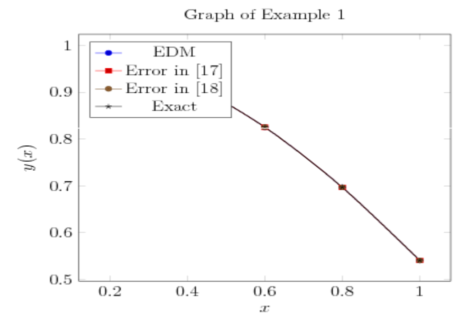
The exact solution is given by

$$\mu(t) = cos\ (t)$$

Source [6], [27]

Table 1: Showing the comparison of the absolute errors in [6], [27] and the derived method

t	*Error in [6]*	*Error in [27]*	*EDM*	*Time/s*
0.2	6×10^{-10}	9.32×10^{-12}	2.22×10^{-16}	*0.3349*
0.4	4×10^{-10}	4.25×10^{-11}	6.66×10^{-16}	*0.4397*
0.6	1.7×10^{-09}	8.63×10^{-11}	1.22×10^{-15}	0.5577
0.8	3.99×10^{-08}	1.29×10^{-10}	4.33×10^{-15}	*0.6535*
1.0	3.599×10^{-7}	1.61×10^{-10}	8.22×10^{-15}	*0.7487*



Example 2:

Consider the damped Duffing equation,

$$\mu''(t) + 2\mu'(t) + \mu(t) + 8\mu^3(t) = e^{-3t}$$

With the initial conditions,

$$\mu(0) = \frac{1}{2}, \mu'(0) = -\frac{1}{2}$$

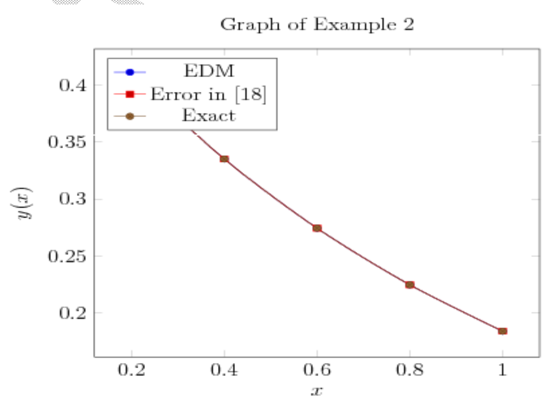
The exact solution is given by

$$\mu(t) = \frac{1}{2}e^{-t}$$

Source [27]

Table 2: Showing the comparison of the absolute errors in [27] and the derived method

t	*Exact Solution*	*Approximate Solution*	*EDM*	*Error in [27]*	*Time/s*
0.1	*0.4524187090179798*	*0.4524187090179803*	*4.996004e-16*	*1.4872-08*	*0.1124*
0.2	*0.4093653765389910*	*0.4093653765389926*	*1.609823e-15*	*1.286e-07*	*0.1701*
0.3	*0.3704091103408589*	*0.3704091103408619*	*2.997602e-15*	*1.464e-07*	*0.1871*
0.4	*0.3351600230178196*	*0.3351600230178241*	*4.440892e-15*	*1.393e-07*	*0.1927*
0.5	*0.3032653298563167*	*0.3032653298563224*	*5.717649e-15*	*1.845e-07*	*0.1984*
0.6	*0.2744058180470131*	*0.2744058180470199*	*6.772360e-15*	*2.422e.07*	*0.2035*
0.7	*0.2482926518957047*	*0.2482926518957123*	*7.577272e-15*	*2.468e-07*	*0.2113*
0.8	*0.2246644820586107*	*0.2246644820586189*	*8.132384e-15*	*2.127e-07*	*0.2172*
0.9	*0.2032848298702995*	*0.2032848298703079*	*8.437695e-15*	*1.987e-07*	*0.2231*
1.0	*0.1839397205857211*	*0.1839397205857296*	*8.493206e-15*	*2.071e-07*	*0.2300*



Example 3:

Consider the undamped Duffing Equation,

$$\mu''(t) + \mu(t) + \mu^3(t) = (cost + \varepsilon sin10t)^3 - 99\varepsilon sin10t$$

With initial conditions,

$$\mu(0) = 1, \mu'(0) = 10\varepsilon$$

where $\varepsilon = 10^{-10}$.

The exact solution is given by,

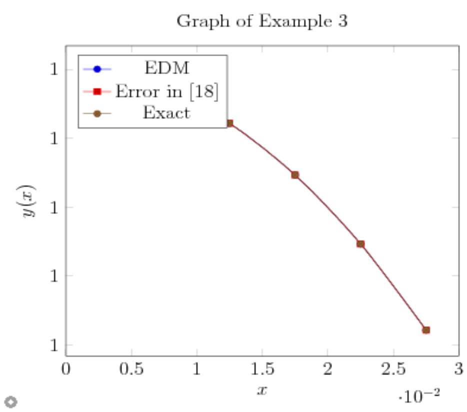
$$\mu(t) = cost + \varepsilon sin10t.$$

This equation describes a periodic motion of low frequency with a small perturbation of high frequency.

Source [27]

Table 3: Showing the comparison of the absolute errors in [27] and the derived method for Example 3

t	*Exact Solution*	*Approximate Solution*	*EDM*	*Error in [27]*	*Time/s*
0.0025	*0.9999968750041274*	*0.9999968750041274*	*0.000000e+00*	*0.000000e+00*	*0.0265*
0.0050	*0.9999875000310395*	*0.9999875000310395*	*0.000000e+00*	*1.110223e-016*	*0.0287*
0.0075	*0.9999718751393287*	*0.9999718751393286*	*1.110223e-16*	*8.881784e-016*	*0.0297*
0.0100	*0.9999500004266486*	*0.9999500004266486*	*0.000000e+00*	*7.771561e-016*	*0.0305*
0.0125	*0.9999218760297148*	*0.9999218760297148*	*0.000000e+00*	*4.440892e-016*	*0.0315*
0.0150	*0.9998875021243030*	*0.9998875021243031*	*1.110223e-16*	*9.992007e-016*	*0.0323*
0.0175	*0.9998468789252486*	*0.9998468789252487*	*1.110223e-16*	*1.665335e-015*	*0.0332*
0.0200	*0.9998000066864446*	*0.9998000066864449*	*2.220446e-16*	*2.775558e-015*	*0.0340*
0.0225	*0.9997468857008414*	*0.9997468857008415*	*1.110223e-16*	*5.440093e-015*	*0.0349*
0.0250	*0.9996875163004431*	*0.9996875163004431*	*0.000000e+00*	*7.216450e-015*	*0.0363*
0.0275	*0.9996218988563066*	*0.9996218988563066*	*0.000000e+00*	*9.436896e-015*	*0.0373*



Example 4:
Consider the undamped Duffing oscillator,

$$\mu''(t) + 3(\mu) - 2\mu^3(t) = cos(t)sin(2t)$$

with the initial condition,

$$\mu(0) = 0, \mu'(0) = 1$$

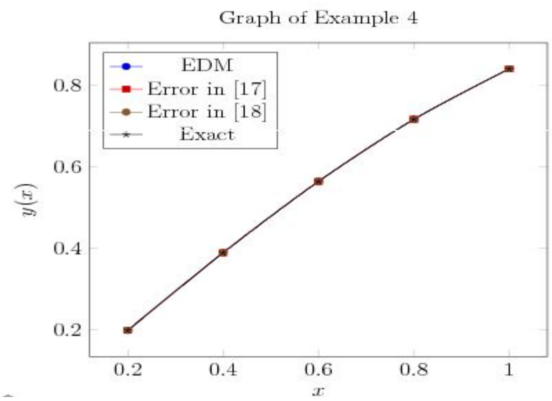
The exact solution is given by,

$$\mu(t) = sin(t)$$

Source:[7, 27]

Table 4: Showing the comparison of the absolute errors in [27, 28] and the derived method for Example 4

t	*Exact Solution*	*Approximate Solution*	*EDM*	*Error in [7]*	*Error in [27]*	*Time/s*
0.1	*0.0998334166468282*	*0.0998334166468282*	*0.000000e+00*	*3.603424e-07*	*3.024248e-13*	*0.0325*
0.2	*0.1986693307950612*	*0.1986693307950611*	*1.110223e-16*	*1.020596e-05*	*4.584944e-13*	*0.0410*
0.3	*0.2955202066613396*	*0.2955202066613391*	*4.440892e-16*	*2.357701e-05*	*7.316370e-14*	*0.0479*
0.4	*0.3894183423086506*	*0.3894183423086494*	*1.165734e-15*	*9.788940e-07*	*1.692257e-12*	*0.0541*
0.5	*0.4794255386042031*	*0.4794255386042009*	*2.220446e-15*	*1.601644e-05*	*4.596878e-12*	*0.0579*
0.6	*0.5646424733950355*	*0.5646424733950319*	*3.552714e-15*	*3.106965e-05*	*8.754997e-12*	*0.0613*
0.7	*0.6442176872376912*	*0.6442176872376860*	*5.218048e-15*	*8.505959e-06*	*1.390665e-11*	*0.0645*
0.8	*0.7173560908995230*	*0.7173560908995161*	*6.883383e-15*	*2.193132e-05*	*1.959244e-11*	*0.0698*
0.9	*0.7833269096274836*	*0.7833269096274751*	*8.548717e-15*	*3.183986e-05*	*2.519718e-11*	*0.0731*
1.0	*0.8414709848078967*	*0.8414709848078867*	*9.992007e-15*	*3.225774e-05*	*2.999911e-11*	*0.0766*



5. Discussion of Results

We have considered 4 numerical examples for the implementation of the new hybrid block method. The results in Tables 1-4 showed clearly that the new hybrid block method performed comparatively with the existing methods. **The graphs of the four examples demonstrate the convergence of our numerical method.**

6. Conclusion

In this paper, we have presented a new hybrid block method of order 7 for the solution of duffing oscillators. The derivation of the approximate solution is given in section 2. Section 3 contains the analysis of the new method, while section 4 presents the implementation. Finally, we present the discussion of results and conclusion in sections 5 and 6 respectively. The results of the new method showed its superiority in accuracy, stability and efficiency.

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