Original Research Article

Remarks on the Murray-von Neumann Equivalence of Projections

Abstract

In this paper, we characterize Murray-von Neumann equivalent projections. We also investigate and compare the relationship between the Murray von Neumann relation and other equivalence relations on the set $\mathcal{P}(B(\mathcal{H}))$ of orthogonal projections in the von Neumann algebra $B(\mathcal{H})$.

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1. Introduction

Let $\mathcal{H}$ denote a Hilbert space and $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$. If $T \in B(\mathcal{H})$, then T^* denotes the adjoint of T, while Ker (T), Ran (T), $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ stands for the kernel of T, range of T, closure of $\mathcal{M}$ and orthogonal complement of a closed subspace $\mathcal{M}$ of $\mathcal{H}$, respectively. We denote by $\sigma(T)$, $\| T \|$ the spectrum and the norm of T, respectively.

An operator $T \in B(\mathcal{H})$ is an orthogonal projection if $T = T^* = T^2$; an isometry if $T^*T = I$; unitary if $T^*T = TT^* = I$; symmetry if $T^* = T = T^{-1}$, i.e, if $T^*T = TT^* = T^2 = I$; normal if $T^*T = TT^*$; an involution if $T^2 = I$.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be invariant under $T \in B(\mathcal{H})$ if $T(M) \subseteq M$ and is said to reduce T if it is invariant under both T and T^*. Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{H})$ are said to be similar (denoted $A \sim B$) if there exists an invertible operator $N \in B(\mathcal{H})$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are unitarily equivalent (denoted by $A \simeq B$) if there exists a unitary operator $U \in B_+(\mathcal{H})$ (Banach algebra of all invertible operators in $B(\mathcal{H})$) such that $UA = BU$ (i.e. $A = U^*BU$, equivalently, $A = U^{-1}BU$). Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{H})$ are said to be metrically equivalent (denoted by $A \overset{\text{m.e}}{\sim} B$) if $\| Ax \| = \| Bx \|$, (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in \mathcal{H}$ or $A \overset{\text{m.}e}{\sim} B$ if $A^*A = B^*B$. (cf. [10]). Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{H})$ are said to be almost similar (denoted $A \overset{\text{a.s}}{\sim} B$) if there exists an invertible operator $N \in B(\mathcal{H})$ such that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$.

An operator $V \in B(\mathcal{H}, \mathcal{K})$ is called partial isometry if it is an isometry on Ker $(V)^\perp$. That is,

$$\| Vx \| = \begin{cases} \| x \|, x \in (\text{Ker } (V))^\perp \\ 0, x \in \text{Ran } (V) \end{cases}$$

In this case, $\mathcal{M} = \text{Ker } (V)^\perp$ is called the initial space of V and $\mathcal{N} = \text{Ran } (V)$ is called the final space of V.

Two projections P and Q in $B(\mathcal{H})$ are said to be Murray-von Neumann equivalent, denoted by $P \overset{\text{M-v-N}}{\sim} Q$, if there exists an operator $V \in B(\mathcal{H})$ such that $V^*V = P$ and $VV^* = Q$. The notion of Murray-von Neumann equivalence of projections was introduced by Berberian[1] and has since generated considerable interest to operator theorists (see [2], [9] and [11]).

A von Neumann algebra $\mathfrak{M}$ is a strongly closed C^*-subalgebra of $B(\mathcal{H})$. The commutant of a von Neumann algebra $\mathfrak{M}$ is the set $\mathfrak{M}' = \{T \in B(\mathcal{H}): TA = AT, \forall A \in \mathfrak{M}\}$. We denote by $\mathcal{P}(\mathfrak{M})$ the set of all orthogonal projections in $\mathfrak{M}$ and by $\mathcal{Z}(\mathfrak{M}) = \mathfrak{M} \cap \mathfrak{M}' = \{P \in \mathfrak{M}: PQ = QP, \text{ for all } Q \in \mathfrak{M}\}$ the center of $\mathfrak{M}$. If $\mathcal{Z}(\mathfrak{M}) = \{\alpha I: \alpha \in \mathbb{C}\}$, then $\mathfrak{M}$ is called a factor.

Given two projections P, Q, if $Q - P$ is a projection (i.e. $P \leq Q$ or equivalently, $Q - P \geq 0$), we call P a sub-projection of Q and write $P \preceq Q$, where " " denotes the Murray-von Neumann order on $\mathbb{P}(\mathfrak{M}) = \mathcal{P}(\mathfrak{M})/^{\text{N-v-N}}$. We write $P \preceq Q$ if and only if $P \overset{M-v-N}{\sim} Q_0$ for a certain sub-projection $Q_0 \leq Q$. Clearly $P \preceq P$ for any projection P. Note that for any $P, Q \in \mathcal{P}(\mathfrak{M}), P \preceq Q$ or $Q \preceq P$. Indeed, $P \preceq Q$ and $Q \preceq P$ implies that $P \overset{\text{M-v-N}}{\sim} Q$. Clearly, the order $\leq$ in $B(\mathcal{H})$ translates into the Murray-von Neumann order between orthogonal projections.

A projection $P \in \mathcal{P}(\mathfrak{M})$ is called a central projection if P commutes with every projection in $\mathcal{P}(\mathfrak{M})$. For each $P \in \mathcal{P}(\mathfrak{M}), C_P \in \mathcal{P}(\mathcal{Z}(\mathfrak{M}))$ stands for the central support or central carrier of P, where the central carrier C_A of an operator A in a von Neumann algebra $\mathfrak{M}$ is the projection $I - P$, where P is the union (that is, $P = \vee_\alpha P_\alpha$) of all central projections P_α in $\mathfrak{M}$ such that $PA = 0$. C_A can as well be defined as the intersection of all central projections Q such that $QA = A$. For any projection P, C_P is the smallest projection in the center $\mathcal{Z}(\mathfrak{M})$ containing P as a sub-projection (i.e., it is the smallest projection Z in the centre such that $P \leq Z$[11]). That is, every projection P in a von Neumann algebra has a central carrier since $\mathcal{Z}(\mathfrak{M})$ is itself a von Neumann algebra.

The von Neumann algebra plays a role in determining central carriers. For example, the central carrier of a projection P (different from 0 and I) relative to the algebra of all bounded operators $B(\mathcal{H})$ and relative to the von Neumann algebra generated by P and I. In the first case the central carrier is I and in the second it is P. It is well-known that for any operator A in a von Neumann algebra $\mathfrak{M}, C_A A = A$ (see [9] and [11])

If an operator $T \in B(\mathcal{H}, \mathcal{K})$ has closed range, the restriction $T|_{\mathrm{Ker}\,(T)}{}^{\perp} \longrightarrow$ Ran (T) is a boundedly invertible operator and the inverse defined on Ran (T) can be defined on all of $\mathcal{K}$ by letting $T|_{\mathrm{Ran}\,(T)^{\perp}} = T|_{\mathrm{Ker}\,(T^{*})} = 0$. The extension uniquely determined by T, and denoted by $T^{\dagger}$, is called the Moore-Penrose inverse or pseudo-inverse of T and it is the unique solution to the equations

$$TT^{\dagger}T = T, T^{\dagger}TT^{\dagger} = T^{\dagger}, TT^{\dagger} = \left(TT^{\dagger}\right)^{*}, T^{\dagger}T = \left(T^{\dagger}T\right)^{*}.$$

Clearly $T^{\dagger}$ exists if and only if Ran (T) is closed. In this case, $TT^{\dagger}$ and $T^{\dagger}T$ are the orthogonal projections onto Ran (T) and Ran (T^{*}), respectively and Ran $\left(T^{\dagger}\right)$ = Ran (T^{*})

Unlike the Moore-Penrose inverse, the Drazin inverse is defined on $B(\mathcal{H})$. The Drazin inverse of $T \in B(\mathcal{H})$ is the unique operator denoted by T^{D} satisfying

$$TT^{D} = T^{D}T, T^{D}TT^{D} = T^{D}, T^{k+1}T^{D} = T^{k},$$

where k = index (T), the smallest nonnegative integer k such that rank $\left(T^{k+1}\right)$ = Ran $\left(T^{k}\right)$. If index $(T) = 0$, then T is invertible and $T^{D} = T^{-1}$. The Drazin inverse was developed by Drazin in 1958[3] and it was proved that if $T, S \in B(\mathcal{H})$ with $TS = ST = 0$, then $(S + T)^{D} = S^{D} + T^{D}$. Thus $T \in B(\mathcal{H})$ is said to have a Drazin inverse or to be Drazin invertible if there exists $X \in B(\mathcal{H})$ such that

$$TX = XT, XTX = X, T^{k+1}X = T^{k}.$$

In this case $X = T^{D}$ is called the Drazin inverse of T. For every T there exists at most one such X.

To set the stage, we first state and prove some results which are useful in the proof of the main results.

2. Preliminary Results

From now on, if there is no danger of confusion, by a projection we mean an orthogonal projection.

Recall that $V \in B(\mathcal{H})$ is a partial isometry if there exists a subspace $\mathcal{M} \subseteq \mathcal{H}$ such that $\| Vx \| = \| x \|$, for all $x \in \mathcal{M} = Ker(V)^{\perp}$, (i.e., it is isometric on the orthogonal complement of its kernel) and $\| Vx \| = 0$ if $x \in \mathcal{M}^{\perp}$. This means that V^{*} is the Moore-Penrose inverse of V. That is, $VV^{*}V = V$. The class of partial isometries was first studied by Halmos and McLaughlin [7] and they have shown that every partial isometry $V \in B(\mathcal{H})$ has a canonical representation as $V = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}$ on $\mathcal{H}$ = Ker $(V) \oplus$ Ker $(V)^{\perp}$, with $B^{*}B + D^{*}D = I$. This result was improved by Halmos ([6], §127), where it is stated that every partial isometry V is bounded and if $V \neq 0$, then $\| V \| = 1$. Clearly a partial isometry is a contraction and therefore its spectrum is necessarily a subset of the closed unit disc in $\mathbb{C}$. A non-empty compact subset Ω of $\mathbb{C}$ is the spectrum of a partial isometry V if and only if either $\Omega \subseteq \partial\mathbb{D}$, the unit circle/circumference (i.e. it does not contain the origin: V is invertible and hence unitary) or $\Omega \subseteq \overline{\mathbb{D}}$, with $0 \in \Omega$ (i.e. V is not invertible) ([6], §133), where $\mathbb{D}$ denotes the open unit disc of the plane. For more properties the reader may consult ([6], Chapter 15).

Proposition 2.1 ([5], Theorem 3). Let V be an operator on a Hilbert space $\mathcal{H}$. Then the following statements are mutually equivalent.

(a). V is a partial isometry.

(b). V^* is a partial isometry.

(c). $VV^*V = V$

(d). $V^*VV^* = V^*$.

(e). V^*V and VV^* are projection operators.

(f). $V^* = V^\dagger = V^D$.

Proof. See ([5], Theorem 3 and [4], Theorem 2.3).

Remark. The projections V^*V and VV^* in Proposition 2.1 are called the initial and final projections of V, respectively. The class of partial isometries is wider than the class of isometries. It contains isometries, co-isometries and also projection operators. The set of all partial isometries on a Hilbert space $\mathcal{H}$ forms a semigroup. By a semigroup of operators on $\mathcal{H}$ we simply mean a set $\mathcal{S}$ closed under multiplication; it is said to be self-adjoint if $\mathcal{S} = \mathcal{S}^* := \{V^* : V \in \mathcal{S}\}$. Thus, the concept of self-adjoint semigroups of partial isometries is a direct and natural generalization of that of groups of unitary operators or semigroup orthogonal projections, which is abelian.

Theorem 2.2. Let $T \in B(\mathcal{H})$ such that $T^2 = T$. Then T is a partial isometry if and only if $T = T^*$. Furthermore, if T is both idempotent and a partial isometry, then $T = T^* = T^\dagger = T^D$, where $T^\dagger$ and T^D denote the Penrose and Drazin inverse of T, respectively.

Proof. Suppose $T^2 = T$. If $T = T^*$, then $(TT^*)^2 = TT^*$. This means that $T^*TT^* = T^*$ and therefore by Proposition 2.1(d), T is a partial isometry. Conversely, if $TT^*T = T$, then $\| T \| \leq 1$ and since $T^2 = T, T$ is necessarily self-adjoint. In this case, $T = T^* = T^\dagger = T^D$.

Proposition 2.3. If V is a partial isometry then the following statements are equivalent.

(a). the non-zero eigenvalues of V lie on the unit circle.

(b). $V = U^*TU$, for some unitary operator U and a triangular operator matrix $T = [t_{ij}]$ with $|t_{ii}| = 1$ or 0 , for each i.

Example. The operator $V = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$ acting on $\mathcal{H} = \mathbb{C}^2$ satisfies the properties of Proposition 2.3.

Theorem 2.4 ([8], Proposition 5.87). If $T: H \longrightarrow K$ is a partial isometry, then $T = VP$ where V: Ker $(T)^\perp \longrightarrow \mathcal{K}$ is an isometry and $P: \mathcal{H} \longrightarrow \mathcal{H}$ is the orthogonal projection onto Ker $(T)^\perp$. Conversely, let $\mathcal{M}$ be any subspace of $\mathcal{H}$. If $V: \mathcal{M} \longrightarrow \mathcal{H}$ is an isometry and

$P: \mathcal{H} \longrightarrow \mathcal{H}$ is the orthogonal projection onto $\mathcal{M}$, then $T = VP: \mathcal{H} \longrightarrow \mathcal{K}$ is a partial isometry.

Proposition 2.5. A partial isometry $V \in B(\mathcal{H})$ is an isometry if and only if Ker $(V)^{\perp} = \mathcal{H}$. That is, if Ker $(V) = \{0\}$.

Proof. Let $VV^*V = V$. If V is an isometry, then $V^*V = I$. Therefore Ker $(V) = \{0\}$ and hence Ker $(V)^{\perp} = \mathcal{H}$. Conversely, suppose that Ker $(V)^{\perp} = \mathcal{H}$. Then Ker $(V) = \{0\}$, and since $VV^*V = V$ we have that $V^*V = I$. This establishes the claim.

Example. The operator $V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} = \mathbb{R}^2$ is a partial isometry which is not an isometry.

Proposition 2.6. Every orthogonal projection $P \in B(\mathcal{H})$ is a partial isometry.

Proof. . If $P^* = P = P^2$, then $P^*P = P$ and therefore $PP^*P = P^2 = P$.

For any $T \in B(\mathcal{H})$, the maps $T \longrightarrow TT^*$ and $T \longrightarrow T^*T$ are continuous In particular, if V is a partial isometry, then the mapst $V \longrightarrow P_{Ran}(V) = VV^*$ and $V \longrightarrow P_{Ker}(V)^{\perp} = V^*V$ are continuous.

Proposition 2.7. Let $A, B \in B(\mathcal{H})$ such that $AB = BA \neq 0$. Then AB is invertible if and only if A and B are invertible.

Theorem 2.8 ([5], Theorem 1§3.7.1). The following conditions on an operator T are equivalent:

(i). T is a partial isometry and quasinormal.

(ii). T is the direct sum of an isometry and zero.

Theorem 2.9 ([5], Theorem 2§3.7.1). Let T be an operator on a Hilbert space $\mathcal{H}$. Then (i). T is a normal partial isometry if and only if T is the direct sum of a unitary operator and zero.

(ii). T is a subnormal partial isometry if and only if T is the direct sum of an isometry and zero.

Theorem 2.10. Let V be a partial isometry such that $P = V^*V$ and $Q = VV^*$. Then

(i). Ker (P) = Ker (V), $\overline{\text{Ran}\ (Q)} = \overline{\text{Ran}\ (V)}$.

(ii). Ker (Q) = Ker (V^*), $\overline{\text{Ran}\ (P)} = \overline{\text{Ran}\ (V^*)}$.

(iii). if V is self-adjoint. then Ker (P) = Ker (Q) and $\overline{\text{Ran}\ (P)} = \overline{\text{Ran}\ (Q)}$.

Proof. The proof of (i) and (ii) follows from the definition. To show (iii), we note that since $V = V^*$, by (i) and (ii), we have Ker (Q) = Ker (V^*) = Ker (V) = Ker (P) and $\overline{\text{Ran}\ (P)} = \overline{\text{Ran}\ (V^*)} = \overline{\text{Ran}\ (V)} = \overline{\text{Ran}\ (Q)}$.

Proposition 2.11. If $T \in B(\mathcal{H})$ is such that $\| I - T \| < 1$, then T is invertible.

3. Main Results

Recall that two projections P and Q in $B(\mathcal{H})$ are said to be Murray-von Neumann equivalent, denoted by $P^{\mathrm{M-v-N}} \sim Q$ if there exists an operator $V \in B(\mathcal{H})$ such that $V^*V = P$ and $VV^* = Q$.

Remark. Clearly, the operator V implementing the Murray-von Neumann equivalence of any two orthogonal projections is automatically a partial isometry. That is, it satisfies $VV^*V = V$. Thus, two orthogonal projections are Murray-von Neumann equivalent exactly when there is a partial isometry with one projection as the initial projection and the other as the final projection. For any partial isometry V, its initial projection P is the smallest projection (with respect to the partial ordering $\preceq$ on $\mathcal{P}(\mathfrak{M})$) such that $VP = V$ and its final projection Q is the smallest projection such that $QV = V$.

Theorem 3.1. Let V be a partial isometry and let P and Q be Murray-von Neumann equivalent projections with respect to V. Then $V = VP = QV$.

Proof. Suppose $P = V^*V$ and $Q = VV^*$ for a partial isometry V. Then from the definition and the above remark, we have $V = VV^*V = VP = QV = QVP$.

Theorem 3.1 also says that if P and Q be Murray-von Neumann equivalent projections with respect to V, then $V^* = PV^* = V^*Q$. A consequence of this result is that $P = V^*QV$. Using Theorem 3.1, we have that two projections P and Q are Murray-von Neumann equivalent if there exists a partial isometry V such that $P = V^*QV$

Proposition 3.2. The Murray-von Neumann relation is an equivalence relation on the family $\mathcal{P}(B(\mathcal{H}))$ of projections in $B(\mathcal{H})$.

Proof. Suppose that P and Q are projections such that $P = V^*V$ and $Q = VV^*$, for some partial isometry V. Reflexivity follows easily from Proposition 2.6, because a projection is also a partial isometry. Symmetry follows from Proposition 2.1 since V^* is a partial isometry (with the initial and final spaces interchanged) whenever V is. Now suppose P, Q and R are projections and that $P \overset{\mathrm{M-v-N}}{\sim} Q$ and $Q \overset{\mathrm{M-v-N}}{\sim} R$. Then there exists partial isometries V and W such that $P = V^*V, Q = VV^* = W^*W$ and $R = WW^*$. Now let $Z = WV$. Then using the proof of Theorem 3.1, Z is also a partial isometry and that $Z^*Z = V^*W^*WV = V^*QV = V^*V = P$, and $ZZ^* = WVV^*W^* = WQW^* = WW^* = R$ which proves transitivity.

For any partial isometry $V \in B(\mathcal{H})$, its initial space is Ran (V^*V) and its final space is Ran (VV^*). Clearly V is understood to be a unitary operator in $B\big(\text{Ran } (V^*V), \text{Ran } (VV^*)\big)$, in the sense that $V^*V = I_{\text{Ran } (V^*V)}$ and $VV^* = I_{\text{Ran } (VV^*)}$.

For a given partial isometry $V \in B(\mathcal{H})$, either Ran (V^*V) = Ran (VV^*), Ran $(V^*V) \subset$ Ran (VV^*) or Ran $(VV^*) \subset$ Ran (V^*V). If Ran (V^*V) = Ran (VV^*), then V is a unitary in $B\big(\text{Ran } (A^*A)\big)$. If Ran $(V^*V) \subset$ Ran (VV^*), then the partial isometry V^* satisfies Ran $((V^*)^*V)$ = Ran $(VV^*) \supset$ Ran (V^*V). This is equivalent to saying that the initial space of V^* coincides with the final space of V and the initial space of V coincides with the final space of V^*. In other words, the final space of V^* is properly contained in the initial space of V^*. Similarly, if Ran $(VV^*) \subset$ Ran (V^*V), then the initial space of V^* is properly contained in the final space of V^*.

3.1. Murray-von Neumann Relation and other Operator Relations.

The following result gives a condition under which Murray-von Neumann equivalence implies unitary equivalence and hence similarity of projection operators.

Theorem 3.3. Let P and Q be projections such that $P^{M-v-N} \sim Q$ with an implementing partial isometry V. If V is invertible then P and Q are similar projections.

Proof. Suppose $P = V^*V$ and $Q = VV^*$. Invertibility of V implies V is unitary and hence by Theorem 3.1$V^* = PV^{-1} = V^{-1}Q$. Thus $Q = VV^* = VPV^{-1} = VPV^*$

Theorem 3.3 also says if V in invertible, then $P = V^*V$ and $Q = VV^*$ are also invertible. This also means that $P = Q = I$, since the only invertible projection is the identity operator. In addition, we conclude that V is unitary, since every invertible partial isometry is unitary.

Corollary 3.4. Let P and Q be invertible projections. If $P \overset{M-v-N}{\sim} Q$ with an implementing partial isometry V then $P = Q$.

Corollary 3.4 says that for invertible projections in a Hilbert space $\mathcal{H}$, the notions of unitary equivalence, similarity, quasisimilarity, metric equivalence and Murray von Neumann equivalence coincide with equality. The statement is also valid if we assume that V is a normal partial isometry. Note that a normal partial isometry need not be unitary. The operator $V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a normal partial isometry which is a direct sum of a unitary and zero but it is not unitary.

Corollary 3.5. Let P and Q be projections such that $P = V^*V$ and $Q = VV^*$ for some partial isometry V. If V is normal then $P = Q$.

Proof. This follows from $0 = V^*V - VV^* = P - Q$.

Remark. We note that for any orthogonal projections $P, Q \in B(\mathcal{H})$, if $P = V^*V$ and $Q = VV^*$ for some partial isometry V, the condition of V being either normal, unitary, or invertible and the condition of invertibility of both P and Q all coincide. If any of these conditions is satisfied, then $P = Q$.

Proposition 3.6. If P and Q are unitarily equivalent projections then they are Murray-von Neumann equivalent.

Proof. Suppose that $P = UQU^*$ for some unitary operator U. Since a unitary operator is a partial isometry, the result follows from the proof of Theorem 3.1.

We remark that Murray-von Neumann equivalence does not in general imply unitary equivalence or metric equivalence of projection operators. Let S be a non-unitary partial isometry, for instance, the unilateral shift on ℓ^2. Then $P = S^*S$ and $SS^* = Q$ are projections. Clearly $P \overset{M-v-N}{\sim} Q$ but P and Q are not unitarily equivalent. A simple calculation also shows that these projection operators are not metrically equivalent. In finite dimensions, it is clear that Murray-von Neumann equivalence implies similarity. However, this is not true in infinite dimensions. To see this, let $P = S^*S$ and $SS^* = Q$, where S is the unilateral shift operator on ℓ^2. This example also shows that Murray-von Neumann equivalence does not in general imply metric equivalence of projection operators.

If $A_1, A_2, \ldots, A_n$ are elements in $B(\mathcal{H})$, then diag $(A_1, A_2, \ldots, A_n)$ denotes the $n \times n$ matrix whose main diagonal consists of the elements $A_1, A_2, \ldots, A_n$. The following result gives a condition when Murray-von Neumann equivalence implies unitary equivalence of operators.

Proposition 3.7. Let P and Q be projections. If $P \overset{M-v-N}{\sim} Q$ then diag $(P, 0)$ is unitarily equivalent to diag $(Q, 0)$.

Proof. Suppose there is a partial isometry V such that $P = V^*V$ and $Q = VV^*$. Then by Theorem 3.1, we have that $V = VP = QV = QVP$. Using this fact, the operators $U = \begin{pmatrix} V & I-Q \\ I-P & V^* \end{pmatrix}$ and $W = \begin{pmatrix} Q & I-Q \\ I-Q & Q \end{pmatrix}$ are unitary and hence WU is also unitary. Clearly

$$WU\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}U^*W^* = W\begin{pmatrix} UPU^* & 0 \\ 0 & 0 \end{pmatrix}W^* = W\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}W^* = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}.$$

The following two results are a consequence of Proposition 3.7. Corollary 3.8. Let P and Q be projections. If $P \overset{M-v-N}{\sim} Q$ then $\sigma(P)/\{0\} = \sigma(Q)/\{0\}$

This result can be improved as follows.

Corollary 3.9. Let P and Q be Murray-von Neumann equivalent projections. If P and Q are invertible, then $\sigma(P) = \sigma(Q)$.

Proof. There are several ways to prove this result. One of them is to use the fact that since P and Q are invertible, 0 is not contained in the spectra of P and Q. The rest of the proof follows upon application of Corollary 3.8. The other is to use Corollary 3.4. That is, $P = Q$.

Theorem 3.10. Two projections P and Q acting on a Hilbert space $\mathcal{H}$ are unitarily equivalent if and only if dim (Ran (P)) = dim (Ran (Q)) and dim (Ker (P)) = dim (Ker (Q))

Proof. Suppose $P = UQU^*$, where U is a unitary operator. Then $PU = UQ$. Since U is an isomorphism (indeed, an automorphism), it preserves the dimension of the Hilbert space. Therefore dim (Ran (P)) = dim (Ran (Q)). The rest of the proof follows from the self-adjointness of P and Q and the fact that Ker (A^*) = Ran $(A)^{\perp}$, for any $A \in B(\mathcal{H})$

Theorem 3.10 says that two projections on a Hilbert space $\mathcal{H}$ are unitarily equivalent if and only if they have the same rank and nullity. The following weaker condition is necessary and sufficient for Murray-von Neumann equivalence.

Theorem 3.11. Two projections P are Q on a Hilbert space $\mathcal{H}$ are Murray-von Neumann equivalent if and only if dim (Ran (P)) = dim (Ran (Q)).

This says that two projections are Murray-von Neumann equivalent if and only if they have the same rank.

Example. Consider the projections $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ acting on the Hilbert space $\mathcal{H} = \mathbb{R}^3$. A simple calculation shows that P and Q are Murray-von Neumann equivalent, with the equivalence being implemented by the partial isometry $V = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. A simple calculation also shows that P and R are Murray-von Neumann equivalent, with the equivalence being implemented by the partial isometry $W = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Also Q and R are Murray-von Neumann equivalent, with the equivalence being implemented by the partial isometry $Z = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By Theorem 3.11, S is not Murray-von Neumann equivalent to either P, Q or R.

A simple calculation also shows that P, Q and R are pairwise similar and also pairwise almost similar. In particular, $X = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 3 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ implements the similarity and also the almost similarity between P and Q. A closer look also reveals that P, Q and R are pairwise unitarily equivalent. The operator S is Murray-von Neumann equivalent to, for instance, the operator $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, with the equivalence being implemented by the partial isometry $Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Remark. We note that the partial isometry implementing the Murray-von Neumann equivalence of two projections need not be unique. For instance, in the example above, the partial isometry $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ also implements the Murray-von Neumann equivalence between P and R. The next result gives the relation between the partial isometries implementing a Murray-von Neumann equivalence of two projections P and Q.

Theorem 3.12. Suppose P and Q are Murray-von Neumann equivalent projections with distinct partial isometries V and W implementing the equivalence. Then $W^*W = V^*V = P$ and $WW^* = VV^* = Q$.

Theorem 3.12 says that if partial isometries W and V implement the Murray-von Neumann equivalence of P and Q, then $W \overset{\text{m.e}}{\sim} V$ and $W^* \overset{\text{m.e}}{\sim} V^*$. That is, W, V and their adjoints W^*, V^* are pairwise metrically equivalent operators. For a deeper and comprehensive theory about metric equivalence operators, see [10].

Theorem 3.13. If $P^{M-v-N} \sim Q$ then $\| P \| = \| Q \|$.

Proof. This follows from $\| P \| = \|V^*V\| = \|VV^*\| = \| Q \|$.

The converse of Theorem 3.13 is not true in general. The orthogonal projections $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have equal norm but P and Q are not Murray – von Neumann equivalent.

Theorem 3.14. Two orthogonal projections P and Q are unitarily equivalent if and only if they are Murray-von Neumann equivalent and $I - P$ and $I - Q$ are Murray-von Neumann equivalent.

Proof. Suppose $Q = UPU^*$, for some unitary operator U. Put $V = UP$ and $W = U(I - P)$. Then $V^*V = PU^*UP = P^2 = P, VV^* = UP^2U^* = UPU^* = Q$ and similarly $W^*W = I - P, WW^* = I - Q$. Thus P and Q are Murray-von Neumann equivalent and $I - P$ and $I - Q$ are Murray-von Neumann equivalent. Conversely suppose P and Q are Murray-von Neumann equivalent and $I - P$ and $I - Q$ are Murray-von Neumann equivalent. Then there exists partial isometries V: Ran $(P) \longrightarrow$ Ran (Q) and W: Ran $(I - P) \longrightarrow$ Ran $(I - Q)$ satisfying the above conditions. Now, let $Z = V + W$. Direct calculation shows that Z is unitary with $Z^* = U^*$ and therefore $ZPZ^* = UPU^* = Q$, which proves the claim.

Proposition 3.6, Theorem 3.10 and Theorem 3.14 say that unitary equivalence of projections is stronger than Murray-von Neumann equivalence of projections. However, in finite dimensions they are equivalent. That is, they are equal modulo passing to matrix algebras.

Given a contraction $T \in B(\mathcal{H})$, both $(I - T^*T)$ and $(I - TT^*)$ are positive operators and hence have unique square roots. We define $D_T = (I - T^*T)^{\frac{1}{2}}$ and $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ and call them the defect operators of T. The closures of their ranges $\mathcal{D}_T = \overline{D_T(\mathcal{H})}$ and $\mathcal{D}_{T^*} = \overline{D_{T^*}(\mathcal{H})}$ are called the defect spaces of T. The respective dimensions(ranks) d_T and d_{T^*} are called the defect indices or numbers of T. Equivalently, the defect numbers of a contraction operator $T \in B(\mathcal{H})$, can be defined as d_T = dim (Ran $(T)^\perp$) = dim ($\mathcal{H} \ominus \overline{\text{Ran } (T)}$) and d_{T^*} = dim (Ran $(T^*)^\perp$) = dim $\left(\mathcal{H} \ominus \overline{\text{Ran } (T^*)}\right)$. We note also that d_T can be characterized in terms of the adjoint operator T^*, namely, d_T = dim $\left(\text{Ker } (T^*)\right)$. In applications, the equation $Tx = y$, where T is invertible, is solvable if and only if y is orthogonal to $\mathcal{H} \ominus$ Ran (T). So d_T is a measure of the number of "orthogonality" conditions ensuring the solvability of the equation $Tx = y$. Recall that Ran $(T) = T(\mathcal{H}) = \{y \in \mathcal{H} : y =$ Tx, some $x \in \mathcal{H}\}$.

Since partial isometries are contractions, the defect numbers for projections and partial isometries are well-defined.

Proposition 3.15. Let P and Q be orthogonal projections and suppose that $P = V^*V$ and $Q = VV^*$, for some partial isometry V. Then $D_V = D_P$ and $D_{V^*} = D_Q$

Proof. By definition we note that $D_V = (I - V^*V)^{\frac{1}{2}} = (I - P)^{\frac{1}{2}} = (I - P^*P)^{\frac{1}{2}} = D_P$ and $D_{V^*} = (I - VV^*)^{\frac{1}{2}} = (I - Q)^{\frac{1}{2}} = (I - Q^*Q)^{\frac{1}{2}} = D_Q$

The following result shows that Murray-von Neumann equivalent projections have equal defect numbers.

Proposition 3.16. Let P and Q be projections acting on a finite dimensional Hilbert space $\mathcal{H}$. Then $d_P = d_Q$ if and only if P and Q are Murray-von Neumann equivalent.

Recall that an operator S is a symmetry if $S^* = S = S^{-1}$ (i.e. $S^*S = SS^* = S^2 = I$).

Proposition 3.17. If $S \in B(\mathcal{H})$ is a symmetry and $P \in B(\mathcal{H})$ is a projection, then $V = SP$ is a partial isometry.

Proof. The result follows from $VV^*V = SPP^*S^*SP = SP = V$.

Theorem 3.18. If $P \overset{M-v-N}{\sim} Q$ and $PQ = 0$, then there exists a symmetry S such that $SPS = Q$. Proof. Suppose there exists a partial isometry V such that $P = V^*V$ and $Q = VV^*$. By Theorem 3.1, $Q = VPV^*$. Let $S = V + V^* + I - (P + Q)$. Clearly $S^* = S$. Since $PQ = QP = 0$, by Theorem 3.1, we also have $VPQ = VVV^* = 0$, $V^2 = V^{*2} = 0$ and $V^*VV = 0$. A simple computation using this fact shows that $S^*S = SS^* = I$. Therefore $S^* = S = S^{-1}$. Another calculation shows that $SP = QS$. This establishes the claim.

Theorem 3.18 gives conditions under which Murray-von Neumann equivalence implies unitary equivalence.

We define the distance between two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of the Hilbert space $\mathcal{H}$ by $d(\mathcal{M}, \mathcal{N}) = \| P - Q \|$, where P and Q are the orthogonal projections with ranges $\mathcal{M}$ and $\mathcal{N}$, respectively. This function defines a metric on the set of all closed subsets of $\mathcal{H}$. We can define $\| P - Q \|$ as follows

$$\begin{aligned} \| P - Q \| &= \sup \{\| (P - Q)x \| : x \in \mathcal{H}, \| x \| \leq 1\} \\ &\geq \sup \{\| (P - Q)x \| : x \in \mathcal{M}^{\perp}, \| x \| \leq 1\} \\ &= \sup \{\| Qx \| : x \in \mathcal{M}^{\perp}, \| x \| \leq 1\} = \alpha \end{aligned}$$

and

$$\begin{aligned} \| P - Q \| &= \sup \{\| (P - Q)x \| : x \in \mathcal{H}, \| x \| \leq 1\} \\ &\geq \sup\{\| (P - Q)x \| : x \in \mathcal{N}^{\perp}, \| x \| \leq 1\} \\ &= \sup \{\| Px \| : x \in \mathcal{N}^{\perp}, \| x \| \leq 1\} = \beta. \end{aligned}$$

Theorem 3.20. Let P and Q be orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then

$$\| P - Q \| = \mathrm{m}\ \{\alpha, \beta\},$$

where α and β are as defined in Remark 3.19.

Proof. see ([12], Theorem 4.33).

With this definition, and for any two orthogonal projections P and Q on a Hilbert space $\mathcal{H}$, it is easy to see that $0 \leq \| P - Q \| \leq 1$. Thus $0 \leq \max\{\alpha, \beta\} \leq 1$.

Theorem 3.21 ([6], Problem 130). If U and V are partial isometries such that $\| U - V \| < 1$ then dim (Ran (U)) = dim (Ran (V))

Corollary 3.22 ([6], Problem 57). If P and Q are projections on a Hilbert space $\mathcal{H}$ such that $\| P - Q \| < 1$ then dim (Ran (P)) = dim (Ran (Q)) and dim (Ran $(I - P)$) = dim (Ran $(I - Q)$)

Proof. First, we note that $(I - P)\mathcal{H}$ = Ran $(P)^{\perp}$ and $(I - Q)\mathcal{H}$ = Ran $(Q)^{\perp}$. Thus $(I - P)(I - Q)\mathcal{H}$ = Ran $(P)^{\perp} \cap$ Ran $(Q)^{\perp}$. Put $Z = PQ + (I - P)(I - Q)$. Then a simple computation shows that $ZZ^* = Z^*Z = I - (P - Q)^2$. This means that Z is normal and since $\| P - Q \| < 1$, we deduce that $I - (P - Q)^2$ is invertible (using the fact that $\| T \| < 1$ implies that $I - T$ is invertible, for any bounded linear operator T) and by Proposition 2.7, Z is invertible. Since Z(Ran (Q)) $\subseteq$ Ran (P) and Z(Ran $(Q)^{\perp}$) $\subseteq$ Ran $(P)^{\perp}$, the invertibility of Z implies that Z(Ran (Q)) = Ran (P) and Z(Ran $(Q)^{\perp}$) = Ran $(P)^{\perp}$. This is equivalent to saying that Ran $(P) \equiv$ Ran (Q) and Ran $(P)^{\perp} \equiv$ Ran $(Q)^{\perp}$, and hence have equal dimensions, respectively.

Corollary 3.23. If P and Q are projections such that $\| P - Q \| < 1$ then there exists a unitary operator U such that $P = UQU^*$.

Proof. The conclusion of Corollary 3.22 guarantees existence of such a unitary operator U. From Corollary 3.22, $Z = PQ + (I - P)(I - Q)$ is normal and invertible and if $Z = U|Z|$ is the polar decomposition of Z, we have that $|Z|$ is invertible and U is unitary. Since Z is normal, the operators Z, $|Z|$ and U all commute. Also

$$P|Z|^2 = P(I - (P - Q)^2) = PQP = (I - (P - Q)^2)P = |Z|^2 P$$

we deduce that $P|Z| = |Z|P$. Thus

$$|Z|PU = P|Z|U = PU|Z| = PZ = PQ = ZQ = |Z|UQ.$$

Therefore $PU = UQ$, or equivalently, $P = UQU^*$.

Example. The projections $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are such that

$\| P - Q \| = \frac{\sqrt{2}}{2} < 1$. A simple calculation shows that $P = UQU^*$ with $U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ being unitary. Another calculation shows that $P^{\mathrm{M-v-N}} \sim Q$, with the partial isometry $V = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ implementing the equivalence.

Remark. The converse to Corollary 3.23 is not true in general. The following result gives a condition under which the converse holds.

Corollary 3.24. Let P and Q be projections on a Hilbert space $\mathcal{H}$. Then $\| P - Q \| < 1$ if and only if $PQ \neq QP$, $\| PQ \| < 1$ and $Q = XPX^{-1}$, for some invertible operator X.

Theorem 3.25. If $P, Q \in B(\mathcal{H})$ are orthogonal projections and if there exists an invertible operator X such that $\|Q - XPX^{-1}\| < \frac{1}{2}$ then $P \overset{M-v-N}{\sim} Q$. Proof. Let $P_0 = XPX^{-1}$ and $Z = P_0Q + (I - P_0)(I - Q)$. Clearly, $P_0^2 = P_0$ and

$$\begin{aligned} \| Z - I \| &= \|(P_0Q + (I - P_0)(I - Q)) - I\| \\ &= \|(P_0Q + (I - P_0)(I - Q)) - (Q + (I - Q))\| \\ &\leq \|(P_0 - I)Q\| + \|((I - P_0) - I)(I - Q)\| \\ &= \|(P_0 - Q)Q\| + \|[(I - P_0) - (I - Q)](I - Q)\| \\ &= \|(P_0 - Q)Q\| + \|(Q - P_0)(I - Q)\| \\ &< 1 \end{aligned}$$

By Proposition 2.11, the operator Z is invertible. Thus, the partial isometry U in the polar decomposition of $Z = U|Z|$ is unitary and $U = Z|Z|^{-1}$. Clearly, $P_0 = UQU^*$ by Corollary 3.23. Therefore, $Q = U^*XPX^{-1}U = (U^*X)P(U^*X)^{-1}$, where $T = U^*X$ is invertible and therefore the partial isometry W in the polar decomposition of $T = U^*X = W|T|$ is unitary. A simple calculation shows that $Q = WPW^*$. Therefore P and Q are unitarily equivalent, and hence Murray-von Neumann equivalent, by Proposition 3.6.

Theorem 3.25 shows that similar orthogonal projection operators are Murray-von Neumann equivalent. But it also says that there may be orthogonal projections which are not similar but are Murray-von Neumann equivalent. This evidently happens if

$0 < \|Q - XPX^{-1}\| < \frac{1}{2}$.

Theorem 3.26. Similar orthogonal projections P and Q acting on a Hilbert space $\mathcal{H}$ are Murray-von Neumann equivalent.

Proof. The proof follows immediately from the fact that similar normal operators are unitarily equivalent. The rest of the proof follows from Proposition 3.6. This result also follows from Theorem 3.25.

We conclude that for orthogonal projections acting on a Hilbert space $\mathcal{H}$:

Metric equivalence $\Longrightarrow$ Unitary equivalence $\Longrightarrow$ Similar $\Longrightarrow$ Murray-von N. equivalent

Are there orthogonal projections in a Hilbert space $\mathcal{H}$, which are not similar but are Murray-von Neumann equivalent?

Corollary 3.27. Let P and Q be orthogonal projections such that $P \neq I$ and $Q \neq I$. If P and Q are Murray-von Neumann equivalent then they are unitarily equivalent.

Proof. The result follows from Theorem 3.25.

Corollary 3.28. If P and Q are orthogonal projections acting on a Hilbert space $\mathcal{H}$ then the following are equivalent.

(a). Ran $(P) \subseteq$ Ran (Q)

(b). $QP = P$

(c). $PQ = P$

(d). $\| Px \| \leq \| Qx \|$ for all $x \in \mathcal{H}$

Proof. (a) $\Longrightarrow$ (b). Suppose Ran $(P) \subseteq$ Ran (Q). Then Px $\in$ Ran (Q) for all $x \in \mathcal{H}$. Thus $QPx = Px$ for all $x \in \mathcal{H}$. This is equivalent to $QP = P$.

(b) $\Longrightarrow$ (c). Since P and Q are self-adjoint, if $QP = P$, then we also have $P = P^* = (QP)^* = P^*Q^* = PQ$

(c) $\Longrightarrow$ (d). Suppose $PQ = P$. Then $\| Px \| = \| PQx \| \leq \| P \| \| Qx \| \leq \| Qx \|$ for all $x \in \mathcal{H}$.

(d) $\Longrightarrow$ (a). Suppose $\| Px \| \leq \| Qx \|$ for all $x \in \mathcal{H}$, but Ran $(P) \nsubseteq$ Ran (Q). Then there exists $x \in$ Ran (P) such that $x \notin$ Ran (Q). Let $x = y + z$, with $y \in$ Ran (Q) and $z \in$ Ran $(Q)^{\perp}$. By the Pythagorean Theorem, $\| Px \|^2 = \| y \|^2 + \| z \|^2 > \| y \|^2 = \| Qx \|^2$, a contradiction. Thus Ran $(P) \subseteq$ Ran (Q)

If P and Q are orthogonal projections such that $PQ = QP = P$, then $Q - P$ is a projection and $\| Q - P \| = 1$. If $Q - P$ is a non-trivial projection, then P and Q need not be unitarily equivalent.

Let $P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $PQ = QP = P$ and $\| Q - P \| = 1$. Clearly these projections are not unitarily equivalent. This shows that the strict less than condition in Corollary 3.23 cannot be dropped.

Corollary 3.30. If P and Q are orthogonal projections acting on a Hilbert space $\mathcal{H}$ with Ran $(P) =$ Ran (Q), then $P = Q$.

Proof. From Corollary 3.27, $P = PQ = QP = Q$.

Theorem 3.31. Let P and Q be orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then $P-Q$ is a projection if and only if Ran $(Q) \subseteq$ Ran (P).

Theorem 3.32. Let $P, Q \in B(\mathcal{H})$ be orthogonal projections. Then $P-Q$ and $Q-P$ are orthogonal projections projections if and only if Ran (P) = Ran (Q).

Proof. The result follows from Corollaries 3.27, 3.30 and Theorem 3.31.

Theorem 3.33. Let $\mathcal{H}$ be a finite dimensional Hilbert space and $T \in B(\mathcal{H})$. Let $\mathcal{M}$ and $\mathcal{N}$ be reducing subspaces for T, with dim $(\mathcal{M})$ = dim $(\mathcal{N})$ and suppose that $T_1 = T|_{\mathcal{M}} = PT|_{\mathcal{M}}$ and $T_2 = T|_{\mathcal{N}} = QT|_{\mathcal{N}}$, with $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, where $P = P_{\mathcal{M}}$ and $Q = P_{\mathcal{N}}$ are in $\mathcal{P}(\mathfrak{M})$. Then T_1 and T_2 are unitarily equivalent if and only if P and Q are Murray-von Neumann equivalent.

3.2. Orbit of Murray-von Neumann equivalent Projections

Recall that for a projection $P \in \mathfrak{M}$, we denote by C_P the central carrier of P. That is, C_P is the smallest projection Z in the center of a von Neumann algebra $\mathfrak{M}$ such that $P \leq Z$. That is

$$C_P = \text{smallest } \{Z \in \mathcal{Z}(\mathfrak{M}): P \leq Z\} = \text{smallest } \{Z \in \mathfrak{M}: ZQ = QZ, \forall Q \in \mathfrak{M}, P \leq Z\}$$

Theorem 3.34. Let P and Q be projections. If $P \leq Q$, then $P = C_P Q$.

In Example 3.29, $P \leq Q$. A simple calculation shows that $P = C_P Q$, with $C_P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem 3.35. Let $P, Q \in B(\mathcal{H})$ be orthogonal projections. If $P \leq Q$, then $C_P \overset{M-v-N}{\sim} Q$

Recall that for any two projections $P, Q \in B(\mathcal{H})$, we let " $\preceq$ " denote the Murrayvon Neumann order on $\mathcal{P}(\mathfrak{M})/^{\overset{M-v-\mathbb{N}}{\sim}}$, the set of Murray-von Neumann equivalent projections. We write $P \preceq Q$ if and only if $P \overset{M-\text{v}-\text{N}}{\sim} Q_0$ for a certain sub-projection $Q_0 \leq Q$

Proposition 3.36. $\preceq$ induces a partial order on $\mathcal{P}(\mathfrak{M})/^{\overset{M-v-N}{\sim}}$, the set of Murray-von Neumann equivalent projections.

Clearly for any two operators $P, Q \in B(\mathcal{H})$, either $P \preceq Q$ or $Q \preceq P$. This means that any two projections in $B(\mathcal{H})$ are comparable. That is the partial ordering $\preceq$ is a linear ordering.

Theorem 3.37 (Murray-von Neumann Schröder- Bernstein). Let P and Q be orthogonal projections on a Hilbert space $\mathcal{H}$. If $P \preceq Q$ and $Q \preceq P$ then $P \overset{M-v-N}{\sim} Q$.

Theorem 3.38. Let P and Q be orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then

(a). $P \preceq Q$ if and only if tr $(P) \leq$ tr (Q).

(b). $P \overset{M-v-N}{\sim} Q$ if and only if tr (P) = tr (Q).

Theorem 3.39. If P is a central projection in $\mathcal{P}(B(\mathcal{H}))$, then $I - P$ is a central projection in $\mathcal{P}(B(\mathcal{H}))$.

Proof. The proof that $I - P$ is a projection is trivial. For all $Q \in \mathcal{P}(B(\mathcal{H}))$, we have that $Q(I - P) = Q - QP = Q - PQ = (I - P)Q$. This establishes the claim.

Let $P \in B(\mathcal{H})$ be an orthogonal projection. Let $\text{Orb}_{Mv}(P)$ be the set or orbit of projections which are Murray-von Neumann equivalent to P. That is, $\text{Orb}_{Mv}(P) = \{Q \in \mathcal{P}(B(\mathcal{H})): Q \overset{\text{M-v-N}}{\sim} P\}$. If Q and R are both in $\text{Orb}_{Mv}(P)$, then $Q \overset{\text{M-v-N}}{\sim} P$ and $R \overset{\text{M-v-N}}{\sim} P$ and by symmetry, $R \overset{\text{M-v-N}}{\sim} Q$. This leads to the following result. Theorem 3.40. Let $P \in \mathcal{P}(B(\mathcal{H}))$. Then any two elements of $\text{Orb}_{Mv}(P)$ are Murray-von Neumann equivalent.

Corollary 3.41. Let $P \in \mathcal{P}(B(\mathcal{H}))$. If $Q \in \text{Orb}_{Mv}(P)$ and $R \overset{M-v-N}{\sim} Q$, then $R \in \text{Orb}_{Mv}(P)$

Corollary 3.41 says that any element of $\mathcal{P}(B(\mathcal{H}))$ that is Murray-von Neumann equivalent to an element of $\text{Orb}_{Mv}(P)$ is itself an element of $\text{Orb}_{Mv}(P)$. Consequently, for any two elements $P, Q \in \mathcal{P}(B(\mathcal{H}))$, the sets $\text{Orb}_{Mv}(P)$ and $\text{Orb}_{Mv}(Q)$ are either identical or disjoint.

Corollary 3.42. Let $P, Q \in \mathcal{P}(B(\mathcal{H}))$. If $P \overset{M-v-N}{\sim} Q$, then $\text{Orb}_{Mv}(P) = \text{Orb}_{Mv}(Q)$

Ø. Corollary 3.43. Let $P, Q \in \mathcal{P}(B(\mathcal{H}))$. If $P^{M-v-N}\chi Q$, then $\text{Orb}_{Mv}(P) \cap Orb_{Mv}(Q) = \emptyset$

The sets in the collection $\{\text{Orb}_{Mv}(P): P \in \mathcal{P}(B(\mathcal{H}))\}$ are the Murray-von Neumann equivalence classes or sets of $\mathcal{P}(B(\mathcal{H}))$ under the relation $\overset{\text{M-v-N}}{\sim}$. Thus $\mathcal{P}(B(\mathcal{H}))$ is the disjoint union of the Murray-von Neumann equivalence classes. No such Murrayvon Neumann equivalence class is empty, since $P \in \text{Orb}_{Mv}(P)$. The collection of equivalence classes under the equivalence relation $\overset{\text{M-v-N}}{\sim}$ is the quotient of $\mathcal{P}(B(\mathcal{H}))$ with respect to $\overset{\text{M-v-N}}{\sim}$, and we denote it by $\mathbb{P}(B(\mathcal{H})) = \mathcal{P}(B(\mathcal{H}))_{/\text{M-v-}\mathbb{N}}$.

If $\mathfrak{M} = B(\mathcal{H})$, then $\preceq$ defines a linear ordering on $\mathbb{P}(B(\mathcal{H})) = \mathcal{P}(B(\mathcal{H}))/^{M-v-\text{N}}_{\sim}$ which is order isomorphic to $\{0,1,2, \dots, n\}$ if $\dim(\mathcal{H}) = n$ or $\omega + 1$ if $\dim(\mathcal{H}) = \aleph_0$.

Example. If $\mathcal{H} = \mathbb{R}^2$, then $\mathbb{P}(B(\mathcal{H})) = \left\{\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\right\}$.

3.3. Orthogonal projections and Associated Involution Operators

Every idempotent operator P acting on a Hilbert space $\mathcal{H}$ has an associated involution defined by $T_P = 2P - I$.

The following assertions follow easily from definition.

Proposition 3.44. If P is an idempotent operator, then the associated operator $T_P = 2P - I$ is an involution.

Proposition 3.45. If P and Q are unitarily equivalent projections, then their associated involutions T_P and T_Q are unitarily equivalent.

Proposition 3.46. Two involutions are similar if and only if their associated idempotents are similar.

Proposition 3.47. If P is an orthogonal projection (self-adjoint idempotent), then the associated involution $T = I - 2P$ is unitary. Theorem 3.48. If P and Q are Murray-von Neumann equivalent projections, then their associated involutions T_P and T_Q are unitarily equivalent.

Discussion

In this paper we have managed to show, for the first time, that two orthogonal projections P and Q acting on a Hilbert space $\mathcal{H}$ are Murray-von Neumann equivalent if and only if there exists a partial isometry $V \in B(\mathcal{H})$ such that $P = V^*QV$.

For two orthogonal projections $P, Q \in B(\mathcal{H})$ with ranges $\mathcal{M}$ and $\mathcal{N}$, respectively, the question of invertibility of the operator $P - Q$ is of great interest as it is connected with the question of when $\mathcal{H}$ is decomposable as a direct sum $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, with the existence of an idempotent operator X satisfying the equations

$$PX = X, XP = P, Q(I - X) = I - X, (I - X)Q = Q.$$

In von Neumann algebras the notion of Murray-von Neumann equivalence is fundamental in the classification and structure theory. Projection operators are useful in vast areas of physics, especially in quantum theory, multi-body system dynamics, Markov Chains, singular difference and differential equations, numerical analysis and group theory. In ergodic theory, the orthogonal projections P and Q could be taken to correspond to measurable subsets of a measure space in which a group is acting.

References

1 S.K. Berberian, Equivalence of projections, Proc. Amer. Math. Soc. 33, No.2 (1972), 485-490.

2 Ximena Catepillán, Marek Ptak, and Wacław Szymanski, Multiple canonical decompositions of families of operators and a model of quasinormal families, Proceedings of the American Mathematical Society 121, No.4 (1994), 1165-1172.

3 M.P. Drazin, Pseudoinverse in associative rings and semigroups, Amer. Math. Monthly **65** (1958), 506-514.

4 P.A. Fuhrmann, Linear systems and operators in Hilbert space, McGraw-Hill, New York, 1981.

5 T. Furuta, Invitation to linear operators:From matrices to bounded linear operators on a hilbert space, Taylor Francis, London, 2001.

6 P.R. Halmos, A Hilbert space problem book, Springer-Verlag, New York, 1982.

7 P.R. Halmos and J.E. McLaughlin, Partial isometries, Pacific J. Math. 13 (1962), 585-596.

8 C.S. Kubrusly, Elements of operator theory, Birkhäuser, Basel, Boston, 2001.

9 P. Niemiec, Unitary equivalence and decompositions of finite systems of closed densely defined operators in Hilbert spaces, Dissertationes Math. (Rozprawy Mat.), 2012.

10 B.M. Nzimbi, G.P. Pokhariyal, and S.K. Moindi, A note on metric equivalence of some operators, Far East J. of Math. Sci. (FJMS) 75, No.2 (2013), 301-318.

11 A. Wegert, Steering projections in von Neumann algebras, Opuscula Math. 35, No.2 (2015), 251 – 271

12 J. Weidmann, Linear operators in Hilbert spaces, Springer-Verlag, New York, 1980.