Original Research Article

Global Clustering Coefficient of the Join and Corona of Graphs

Abstract

The global clustering coefficient is one of the most useful indices in complex network analysis. It is another metric that somehow measures how close a graph from being a complete graph. In this paper we present some expressions for the global clustering coefficient of the join $G \vee H$ and corona $G \circ H$ of arbitrary simple and undirected graphs G and H. As corollaries to these results, we will show that for the path P_m, cycle C_m, fan F_m, and wheel W_m, both $Cc(P_m \vee P_m)$ and $Cc(C_m \vee C_m)$ approach to 0 as m increases without bound, while $Cc(F_m)$, $Cc(W_m)$, $Cc(P_m \circ P_m)$, and $Cc(C_m \circ C_m)$ all approach to 2/3.

Keywords: clustering coefficient; join; vertex corona.

2020 Mathematics Subject Classification: 05C09, 05C38, 05C76

1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. Consider $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ to be the open neighborhood of a vertex $v \in V(G)$, $\deg_G v$ the degree of v, and $t_G(v)$ the number of distinct triangles in G which are incident to v. The *local clustering coefficient* of vertex v in G, denoted by $Cc_v(G)$, is a measure that can be defined as

$$Cc_v(G) = \begin{cases} 0, & \text{if } \deg_G v \le 1, \\ \frac{t_G(v)}{\binom{\deg_G v}{2}}, & \text{if } \deg_G v \ge 2. \end{cases} \tag{1.1}$$

This equation can be traced as a unifying version between its treatment in [13] and [10]. On the other hand, the *global clustering coefficient* $Cc(G)$ of a graph G with order n is a measure that indicates the overall clustering of G, obtained by averaging the local clustering coefficients of all the vertices in G. That is,

$$Cc(G) = \frac{1}{n} \sum_{\substack{v \in V(G) \\ \deg_G v \ge 2}} Cc_v(G) = \frac{1}{n} \sum_{\substack{v \in V(G) \\ \deg_G v \ge 2}} \frac{2t_G(v)}{\deg_G v(\deg_G v - 1)}. \tag{1.2}$$

This measure was introduced in the field of social network analysis by Duncan J. Watts and Steven Strogatz [14] in 1998 to determine, among other things, whether a graph is a "small-world network". Since then, several related studies in various contexts have also emerged. In [3] the authors gave some expressions and bounds for the global clustering coefficient of the tensor product of graphs, while in [4], the authors gave asymptomatic values for the global clustering coefficients of the cartesian and tensor product of complete graphs with order m, where $m \to \infty$. A related study on finding the number of distinct triangles in the join and corona was done in [5], while a triangle-counting algorithm for large networks appeared in [12].

In this paper we present some expressions for the global clustering coefficient of the join $G \vee H$ and corona $G \circ H$ of arbitrary graphs G and H. As corollaries to these results, we will show that for the path P_m, cycle C_m, fan F_m, and wheel W_m, as m increases without bound: both $Cc(P_m \vee P_m)$ and $Cc(C_m \vee C_m)$ approach to 0; while $Cc(F_m)$, $Cc(W_m)$, $Cc(P_m \circ P_m)$, and $Cc(C_m \circ C_m)$ all approach to 2/3. Please be it noted that the scope of this work falls within the general motivation of investigating graphs under some binary operations and expressing some of their parameterized values in terms of some relevant invariants of the constituent graphs such as the ones done in [1, 3, 4, 5, 6, 7, 8, 11]. For basic graph theory terminologies not specifically described nor defined in this paper, please refer to either [2] or [9].

2 Join of Graphs

The *join* of two graphs G and H, denoted by $G \vee H$, has the vertex set $V(G \vee H) = V(G) \dot{\cup} V(H)$ and edge set $E(G \vee H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G), v \in V(H)\}$, with the symbol $\dot{\cup}$ denoting disjoint union. Recall that the *fan graph* F_n of order $n+1$, where $n \geq 2$, is given by $F_n = K_1 \vee P_n$ while the *wheel graph* W_n of order $n+1$, where $n \geq 3$, is the graph given by $W_n = K_1 \vee C_n$.

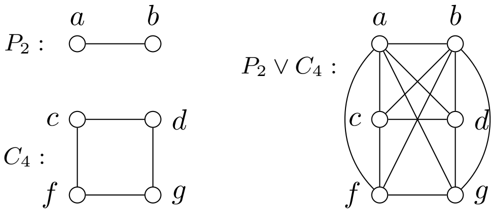

Figure 1: Graphs P_2, C_4, and $P_2 \vee C_4$

Lemma 2.1. *Let G and H be graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$, and $|E(H)| = m_2$. If $v \in V(G \vee H)$, then the number of triangles in $G \vee H$ incident to vertex v is given by the formula*

$$t_{G \vee H}(v) = \begin{cases} t_G(v) + n_2 \cdot \deg_G(v) + m_2 & \text{if } v \in V(G), \\ t_H(v) + n_1 \cdot \deg_H(v) + m_1 & \text{if } v \in V(H). \end{cases}$$

Proof: Note that if $v \in V(G \vee H)$, then by the definition of the join of graphs, either $v \in V(G)$ or $v \in V(H)$. If G and H are both null graphs, then the desired formula easily holds, with both sides of the formula reduced to zero. So suppose G and H are not simultanously null graphs, which means that either G or H has an edge. Let $uw \in E(\langle N_{G \vee H}(v) \rangle)$. Then the vertices u, w, and v are pairwise adjacent in $G \vee H$.

Assume first that $v \in V(G)$. By the nature of the join of graphs, vertices u and w can both belong to $V(G)$ or both to $V(H)$, or one could be in G while the other is in H. If u and w are both in $V(G)$, then u and w must be in $N_G(v)$. This implies then that $uw \in E(\langle N_G(v)\rangle)$. Now, if u and w are both in $V(H)$, then $uw \in E(H)$. Lastly for this case, if $u \in V(G)$ and $w \in V(H)$, then $u \in N_G(v)$ is a must. Conversely, if either $uw \in E(\langle N_G(v)\rangle)$, $uw \in E(H)$, or $u \in V(G)$ and $w \in V(H)$, then necessarily $uw \in E(\langle N_{G\vee H}(v)\rangle)$. Therefore, if $v \in V(G)$, we have

$$E(\langle N_{G\vee H}(v)\rangle) = E(\langle N_G(v)\rangle) \ \dot{\cup}\ E(H) \ \dot{\cup}\ \Big(N_G(v) \times V(H)\Big).$$

Taking the cardinality of the expression above, we obtain

$$\begin{aligned} t_{G\vee H}(v) &= |E(\langle N_{G\vee H}(v)\rangle)| \\ &= |E(\langle N_G(v)\rangle)| \ + \ |E(H)| \ + \ |N_G(v)| \cdot |V(H)| \\ &= t_G(v) + m_2 + \deg_G(v) \cdot n_2. \end{aligned}$$

On the other hand, if $v \in V(H)$, we can apply the same type of argument above to produce a similar conclusion that $t_{G\vee H}(v) = t_H(v) + m_1 + \deg_H(v) \cdot n_1$. We then combine the two cases together to easily complete the proof. $\square$

In Figure 2 below, the number of highlighted edges emphasized the number of triangles in $G \vee H$ incident to vertex $v \in V(G)$, with the number of blue edges, number of red edges, and number of green edges correspond to the first term, second term, and third term of the right-hand side of the formula for $t_{G\vee H}(v)$, respectively. A similar formation of triangles holds for the vertices of H since the join of graphs is commutative.

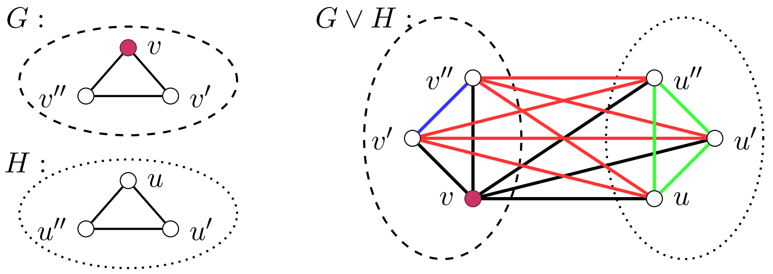

Figure 2: The join $G \vee H$ of two arbitrary graphs G and H, in which vertices v, v', v'' are pairwise adjacent in G and vertices u, u', u'' are pairwise adjacent in H

Our next result gives an exact expression for the local clustering coefficient of the join of two nontrivial connected graphs.

Theorem 2.2. *Let G and H be nontrivial connected graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. The local clustering coefficient $Cc_v(G \vee H)$ of vertex $v \in V(G \vee H)$ in the join $G \vee H$ is given by*

$$Cc_v(G \vee H) = \begin{cases} \check{f}_G(u,n_2) \cdot Cc_v(G) \ + \ \check{g}_G(u,n_2,m_2) & \textit{if } v \in V(G) \\ \check{f}_H(v,n_1) \cdot Cc_v(H) \ + \ \check{g}_H(v,n_1,m_1) & \textit{if } v \in V(H) \end{cases},$$

where $\check{f}_G(u,n_2) = \frac{\deg_G(u)(\deg_G(u)-1)}{(\deg_G(u)+n_2)(\deg_G(u)+n_2-1)}$, $\quad \check{g}_G(u,n_2,m_2) = \frac{n_2\deg_G(u)+m_2}{\binom{\deg_G(u)+n_2}{2}}$,
$\check{f}_H(v,n_1) = \frac{\deg_H(v)(\deg_H(v)-1)}{(\deg_H(v)+n_1)(\deg_H(v)+n_1-1)}$, $\quad$ *and* $\quad \check{g}_H(v,n_1,m_1) = \frac{n_1\deg_H(v)+m_1}{\binom{\deg_H(v)+n_1}{2}}$.

Proof: Using Equation 1.1, Lemma 2.1, and the fact that

$$\deg_{G\vee H}(v) = \begin{cases} \deg_G(v) + n_2 & \text{if } v \in V(G) \\ \deg_H(v) + n_1 & \text{if } v \in V(H) \end{cases},$$

we have

$$Cc_v(G \vee H) = \frac{t_{G\vee H}(v)}{\binom{\deg_{G\vee H}(v)}{2}} = \begin{cases} \frac{t_G(v)+n_2\cdot\deg_G(v)+m_2}{\binom{\deg_G(v)+n_2}{2}} & \text{if } \ v \in V(G), \\ \frac{t_H(v)+n_1\cdot\deg_H(v)+m_1}{\binom{\deg_H(v)+n_1}{2}} & \text{if } \ v \in V(H). \end{cases}$$

Now if we consider first the case in which $v \in V(G)$, we obtain

$$\begin{aligned} Cc_v(G \vee H) &= \frac{t_G(v) + n_2 \cdot \deg_G(v) + m_2}{\binom{\deg_G(v)+n_2}{2}} \\ &= \frac{t_G(v)}{\binom{\deg_G(v)+n_2}{2}} + \frac{n_2 \cdot \deg_G(v) + m_2}{\binom{\deg_G(v)+n_2}{2}} \\ &= Cc_v(G) \cdot \frac{\binom{\deg_G(v)}{2}}{\binom{\deg_G(v)+n_2}{2}} + \frac{n_2 \cdot \deg_G(v) + m_2}{\binom{\deg_G(v)+n_2}{2}} \\ &= Cc_v(G) \cdot \frac{\deg_G(v)(\deg_G(v)-1)}{(\deg_G(v)+n_2)(\deg_G(v)+n_2-1)} + \frac{n_2 \cdot \deg_G(v) + m_2}{\binom{\deg_G(v)+n_2}{2}}. \end{aligned}$$

A similar simplification applies to the second case, $v \in V(H)$, which yields

$$\begin{aligned} Cc_v(G \vee H) &= \frac{t_H(v) + n_1 \cdot \deg_H(v) + m_1}{\binom{\deg_H(v)+n_1}{2}} \\ &= Cc_v(H) \cdot \frac{\deg_H(v)(\deg_H(v)-1)}{(\deg_H(v)+n_1)(\deg_H(v)+n_1-1)} + \frac{n_1 \cdot \deg_H(v) + m_1}{\binom{\deg_H(v)+n_1}{2}}, \end{aligned}$$

and the claimed formula holds. □

The next result, which is for $Cc(G \vee H)$, is a consequence of Theorem 2.2.

Corollary 2.3. *Let G and H be nontrivial connected graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. The global clustering coefficient $Cc(G \vee H)$ of the join $G \vee H$ is given by*

$$\begin{aligned} Cc(G \vee H) = \frac{1}{n_1+n_2} \Bigg(&\sum_{u \in V(G)} \left[\check{f}_G(u, n_2) \cdot Cc_u(G) \ + \ \check{g}_G(u, n_2, m_2)\right] \\ &+ \sum_{v \in V(H)} \left[\check{f}_H(v, n_1) \cdot Cc_v(H) \ + \ \check{g}_H(v, n_1, m_1)\right] \Bigg), \end{aligned}$$

$$\textit{where } \check{f}_G(u,n_2) = \frac{\deg_G(u)(\deg_G(u)-1)}{(\deg_G(u)+n_2)(\deg_G(u)+n_2-1)}, \quad \check{g}_G(u,n_2,m_2) = \frac{n_2\deg_G(u)+m_2}{\binom{\deg_G(u)+n_2}{2}},$$
$$\check{f}_H(v,n_1) = \frac{\deg_H(v)(\deg_H(v)-1)}{(\deg_H(v)+n_1)(\deg_H(v)+n_1-1)}, \quad \textit{and} \quad \check{g}_H(v,n_1,m_1) = \frac{n_1\deg_H(v)+m_1}{\binom{\deg_H(v)+n_1}{2}}.$$

In particular, if G is a nontrivial connected graph, then

$$Cc(G \vee G) = \frac{1}{n_1} \sum_{u \in V(G)} \left[\check{f}_G(u, n_1) \cdot Cc_u(G) \ + \ \check{g}_G(u, n_1, m_1)\right].$$

Proof: Using Equation 1.2, Theorem 2.2, and the fact that $|V(G \vee H)| = n_1 + n_2$, we obtain

$$\begin{aligned}
Cc(G \vee H) &= \frac{1}{n_1+n_2} \sum_{w \in V(G \vee H)} Cc_w(G \vee H) \\
&= \frac{1}{n_1+n_2} \Bigg(\sum_{u \in V(G)} \left[Cc_u(G) \cdot \frac{\deg_G(u)(\deg_G(u)-1)}{(\deg_G(u)+n_2)(\deg_G(u)+n_2-1)} + \frac{n_2 \cdot \deg_G(u) + m_2}{\binom{\deg_G(u)+n_2}{2}} \right] \\
&\quad + \sum_{v \in V(H)} \left[Cc_v(H) \cdot \frac{\deg_H(v)(\deg_H(v)-1)}{(\deg_H(v)+n_1)(\deg_H(v)+n_1-1)} + \frac{n_1 \cdot \deg_H(v) + m_1}{\binom{\deg_H(v)+n_1}{2}} \right] \Bigg).
\end{aligned}$$

The claimed general formula follows. Moreover, if $G = H$, then

$$\begin{aligned}
Cc(G \vee G) &= \frac{2}{n_1+n_1} \sum_{u \in V(G)} \left[Cc_u(G) \cdot \frac{\deg_G(u)(\deg_G(u)-1)}{(\deg_G(u)+n_1)(\deg_G(u)+n_1-1)} + \frac{n_1 \cdot \deg_G(u) + m_1}{\binom{\deg_G(u)+n_1}{2}} \right] \\
&= \frac{1}{n_1} \sum_{u \in V(G)} \left[Cc_u(G) \cdot \frac{\deg_G(u)(\deg_G(u)-1)}{(\deg_G(u)+n_1)(\deg_G(u)+n_1-1)} + \frac{n_1 \cdot \deg_G(u) + m_1}{\binom{\deg_G(u)+n_1}{2}} \right].
\end{aligned}$$

This completes the proof. □

Corollary 2.4. *Let G and H be connected graphs with orders n_1 and n_2, respectively. If G and H are regular graphs with regularities $d_G \geq 2$ and $d_H \geq 2$, respectively, then*

$$Cc(G \vee H) = \frac{n_1(\check{f}_G \cdot Cc(G) + \check{g}_G) + n_2(\check{f}_H \cdot Cc(H) + \check{g}_H)}{n_1+n_2},$$

where $\check{f}_G = \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)}$, $\check{g}_G = \frac{n_2 d_G + m_2}{\binom{d_G+n_2}{2}}$, $\check{f}_H = \frac{d_H(d_H-1)}{(d_H+n_1)(d_H+n_1-1)}$, *and* $\check{g}_H = \frac{n_1 d_H + m_1}{\binom{d_H+n_1}{2}}$.

In particular, if G is a connected regular graph with regularity $d_G \geq 2$, then

$$Cc(G \vee G) = \check{f}_G \cdot Cc(G) + \check{g}_G.$$

Proof: Using Corollary 2.3, we have

$$\begin{aligned}
Cc(G \vee H) &= \frac{1}{n_1+n_2} \Bigg(\sum_{u \in V(G)} \left[Cc_u(G) \cdot \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)} + \frac{n_2 d_G + m_2}{\binom{d_G+n_2}{2}} \right] \\
&\quad + \sum_{v \in V(H)} \left[Cc_v(H) \cdot \frac{d_H(d_H-1)}{(d_H+n_1)(d_H+n_1-1)} + \frac{n_1 d_H + m_1}{\binom{d_H+n_1}{2}} \right] \Bigg) \\
&= \frac{1}{n_1+n_2} \Bigg(\frac{n_1(n_2 d_G + m_2)}{\binom{d_G+n_2}{2}} + \sum_{u \in V(G)} \left[Cc_u(G) \cdot \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)} \right. \\
&\quad \left. + \frac{n_2(n_1 d_H + m_1)}{\binom{d_H+n_1}{2}} \right] + \sum_{v \in V(H)} \left[Cc_v(H) \cdot \frac{d_H(d_H-1)}{(d_H+n_1)(d_H+n_1-1)} \right] \Bigg) \\
&= \frac{1}{n_1+n_2} \Bigg(\frac{n_1(n_2 d_G + m_2)}{\binom{d_G+n_2}{2}} + \frac{n_1 d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)} \cdot \frac{1}{n_1} \cdot \sum_{u \in V(G)} Cc_u(G) \\
&\quad + \frac{n_2(n_1 d_H + m_1)}{\binom{d_H+n_1}{2}} + \frac{n_2 d_H(d_H-1)}{(d_H+n_1)(d_H+n_1-1)} \cdot \frac{1}{n_2} \cdot \sum_{v \in V(H)} Cc_v(H) \Bigg)
\end{aligned}$$

$$= \frac{1}{n_1+n_2}\left(n_1\left(\frac{n_2 d_G + m_2}{\binom{d_G+n_2}{2}} + \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)}\cdot\frac{1}{n_1}\cdot Cc(G)\right)\right.$$
$$\left. + n_2\left(\frac{n_1 d_H + m_1}{\binom{d_H+n_1}{2}} + \frac{d_H(d_H-1)}{(d_H+n_1)(d_H+n_1-1)}\cdot\frac{1}{n_2}\cdot Cc(H)\right)\right)$$

The claimed formula follows. The case where $G = H$ can be seen to be an immediate consequence of the derived formula. ☐

The next corollary can be useful for the clustering coefficient of the fan graph, wheel graph, and corona of graphs.

Theorem 2.5. *Let G be a nontrivial connected graph with order n and size m. If K_1 is the trivial graph, then*

$$Cc_v(G \vee K_1) = \begin{cases} Cc_v(G)\cdot\dfrac{\deg_G(v)-1}{\deg_G(v)+1} + \frac{2}{\deg_G(v)+1} & \textit{if } v \in V(G), \\ \dfrac{2m}{n(n-1)} & \textit{if } v \in V(K_1). \end{cases}$$

Proof: From Lemma 2.1, we have

$$t_{G\vee K_1}(v) = \begin{cases} t_G(v) + \deg_G(v) & \text{if } v \in V(G) \\ m & \text{if } v \in V(K_1). \end{cases}$$

Using Equation (1.1) and the fact that

$$\deg_{G\vee K_1}(v) = \begin{cases} \deg_G(v) + 1 & \text{if } v \in V(G) \\ n & \text{if } v \in V(K_1), \end{cases}$$

we obtain

$$Cc_v(G \vee K_1) = \frac{t_{G\vee K_1}(v)}{\binom{\deg_{G\vee K_1}(v)}{2}} = \begin{cases} \frac{t_G(v)+\deg_G(v)}{\binom{\deg_G(v)+1}{n}} & \text{if } v \in V(G) \\ \frac{m}{\binom{n}{2}} = \frac{2m}{n(n-1)} & \text{if } v \in V(K_1). \end{cases}$$

Further simplification to the case wherein $v \in V(G)$ yields

$$\begin{aligned} \frac{t_G(v)+\deg_G(v)}{\binom{\deg_G(v)+1}{2}} &= \frac{Cc_v(G)\binom{\deg_G(v)}{2}}{\binom{\deg_G(v)+1}{2}} + \frac{\deg_G(v)}{\binom{\deg_G(v)+1}{2}} \\ &= Cc_v(G)\frac{\deg_G(v)-1}{\deg_G(v)+1} + \frac{2}{\deg_G(v)+1}. \end{aligned}$$

The proof is now complete. ☐

The figure below shows the fan graph F_5 and wheel graph W_5 where each vertex is labeled by their respective local clustering coefficients. Observe that for the fan graph F_m, $m-3$ vertices have $2/3$ as their local clustering coefficient whereas for the wheel graph W_m, $m-1$ vertices have $2/3$ as their local clustering coefficient.

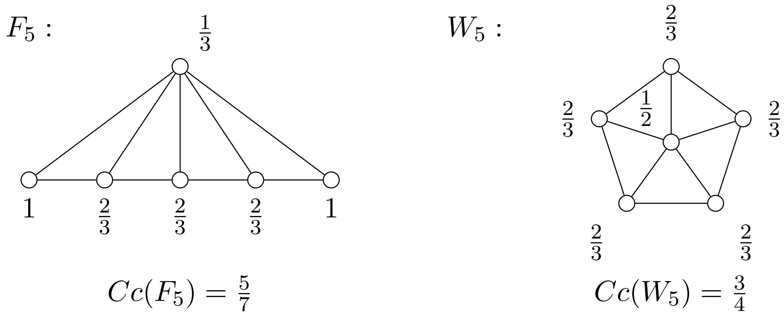

Figure 3: The fan graph F_5 and wheel graph W_5 where each vertex is labeled by their respective local clustering coefficients

Corollary 2.6. *For the fan graph* F_m *and wheel graph* W_m*,*

$$Cc(F_m) = \frac{2m^2 + 2m + 6}{3m^2 + 3m} \quad \text{and} \quad Cc(W_m) = \frac{2m^2 + 4}{3m^2 - 3}.$$

Moreover, as m *increases without bound,* $Cc(F_m) \to 2/3$ *and* $Cc(W_m) \to 2/3$.

Proof: The proof is immediate from Theorem 2.5 and Corollary 2.3. □

The figure below shows the graphs $P_5 \vee P_7$ and $C_5 \vee C_7$. Observe that for the graph $P_5 \vee P_7$, $Cc_{u_1}(P_5 \vee P_7) = Cc_{u_5}(P_5 \vee P_7) = 13/28$ and $Cc_{u_i}(P_5 \vee P_7) = 5/9$, for every $i \in \{2, 3, 4\}$, while $Cc_{v_1}(P_5 \vee P_7) = Cc_{v_7}(P_5 \vee P_7) = 9/15$ and $Cc_{v_j}(P_5 \vee P_7) = 2/3$, for every $j \in \{2, 3, 4, 5, 6\}$. As for the graph $C_5 \vee C_7$, $Cc_{u_i}(C_5 \vee C_7) = 7/12$ for every $u_i \in V(C_5)$, while $Cc_{v_j}(C_5 \vee C_7) = 5/7$ for every $v_j \in V(C_7)$.

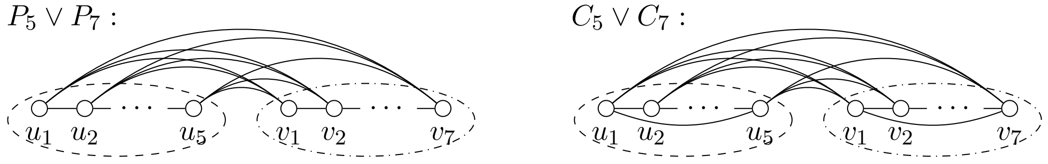

Figure 4: The graphs $P_5 \vee P_7$ and $C_5 \vee C_7$

Corollary 2.7. *For the join* $P_m \vee P_m$ *and* $C_m \vee C_m$*,* $Cc(P_m \vee P_m) \to 0$ *and* $Cc(C_m \vee C_m) \to 0$ *as* m *increases without bound.*

Proof: Using the particular case for the value of $Cc(G \vee G)$ in Corrollary 2.3 and the particular case for regular graphs in Corollary 2.4 , we have

$$Cc(P_m \vee P_m) = \frac{6m^3 + 4m^2 + 4m - 4}{m^4 + 3m^3 + 2m} \qquad \text{and} \qquad Cc(C_m \vee C_m) = \frac{3m}{\binom{m+2}{2}},$$

respectively. Observe that as m increases without bound, both $Cc(P_m \vee P_m)$ and $Cc(C_m \vee C_m)$ approach to zero. □

3 Corona of Graphs

The *corona* of a graph G by a graph H, denoted by $G \circ H$, is the graph obtained by taking one copy of G together with $|V(G)|$ copies of H and then joining the i-th vertex of G to every vertex in the i-th copy of H, this particular copy of H will be denoted by H^i. We may also denote the vertices v in H^i as the ordered pair (i, v) or simply $v \in V(H^i)$ for some vertex $i \in V(G)$.

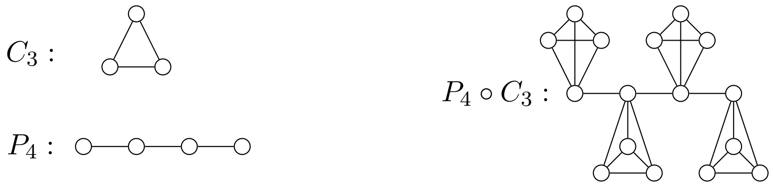

Figure 5: Graphs P_4, C_3, and $P_4 \circ C_3$

Lemma 3.1. *Let G and H be graphs with $|E(H)| = m$. If $v \in V(G \circ H)$, then the number of triangles in $G \circ H$ incident to vertex v is given by the formula*

$$t_{G\circ H}(v) = \begin{cases} t_G(v) + m & \text{if } v \in V(G), \\ t_H(v) + \deg_H(v) & \text{if } v \in V(H^i) \text{ for some } i \in V(G). \end{cases}$$

Proof: Let $uw \in E(\langle N_{G\circ H}(v)\rangle)$. Note that by the definition of the corona of graphs, either $v \in V(G)$ or $v \in V(H^i)$ for some $i \in V(G)$. We now proceed by cases.

Case 1: Assume that $v \in V(G)$. By the definition of the corona of graphs, either both vertices u,$w \in V(G)$, or both u,$w \in V(H^v)$, but not one vertex at each. This is due to the fact that every vertex in H^v is only adjacent to one vertex v of G (aside from its neighbors within H^v). From our preliminary assumption that $uw \in E(\langle N_{G\circ H}(v)\rangle)$, if u,$w \in V(G)$, then $uw \in E(\langle N_G(v)\rangle)$. On the other hand, if $u, w \in V(H^v)$, then $uw \in E(H^v)$. Of course, if either $uw \in E(\langle N_G(v)\rangle)$ or $uw \in E(H^v)$, then necessarily $uw \in E(\langle N_{G\circ H}(v)\rangle)$ Thus, for the case that $v \in V(G)$, $E(\langle N_{G\circ H}(v)\rangle) = E(\langle N_G(v)\rangle) \ \dot{\cup}\ E(H^v)$.
Taking the cardinality of the expression above gives us

$$t_{G\circ H}(v) = |E(\langle N_{G\circ H}(v)\rangle)| = |E(\langle N_G(v)\rangle)| \ + \ |E(H^v)| = t_G(v) + m.$$

Case 2: Assume that $v \in V(H^i)$ for some $i \in V(G)$. Observe that every vertex or edge involved in determining the local clustering coefficient of v in the corona $G \circ H$ is within the subgraph $i \vee H^i$. Hence we can apply Lemma 2.1, which gives us $t_{G\circ H}(v) = t_H(v) + \deg_H(v)$.

The claimed formula now follows. □

In Figure 6 below, the highlighted vertices and edges emphasized the number of triangles in $G \circ H$ incident to vertex $u \in V(G)$ and $v \in V(H^i)$, for some $i \in V(G)$. For $u \in V(G)$ the number of red edges and blue edges correspond to the first term and second term of the right-hand side of the formula for $t_{G\circ H}(u)$, respectively, whereas for $v \in V(H^i)$, for some $i \in V(G)$, the number of orange edges and green edges correspond to the first term and second term of the right-hand side of the formula for $t_{G\circ H}(v)$, respectively.

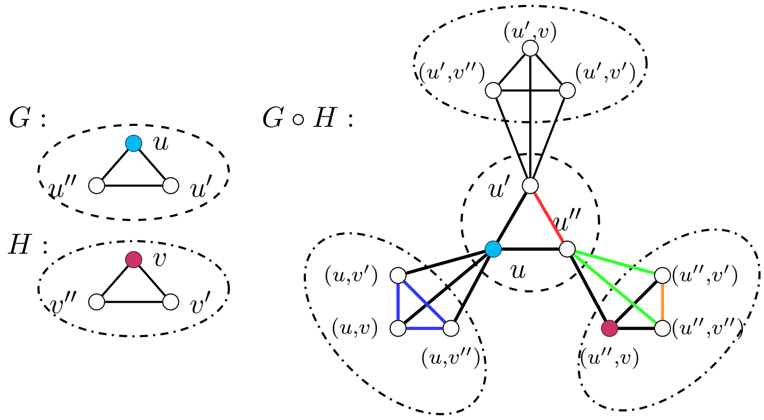

Figure 6: The corona $G \circ H$ of two arbitrary graphs G and H, in which vertices u, u', u'' are pairwise adjacent in G and vertices v, v', v'' are pairwise adjacent in H

Theorem 3.2. *Let G and H be graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. Then the local clustering coefficient of vertex $v \in V(G \circ H)$ in the corona $G \circ H$ is given by the formula*

$$Cc_v(G \circ H) = \begin{cases} \mathring{f}(v, n_2) \cdot Cc_v(G) \ + \ \mathring{g}(v, n_2, m_2) & \textit{if } v \in V(G) \\ Cc_v(H) \cdot \frac{\deg_H(v)-1}{\deg_H(v)+1} + \frac{2}{\deg_H(v)+1} & \textit{if } v \in V(H^i) \textit{ for some } i \in V(G), \end{cases}$$

where we have $\mathring{f}(v, n_2) = \frac{\deg_G(v)(\deg_G(v)-1)}{(\deg_G(v)+n_2)(\deg_G(v)+n_2-1)}$ *and* $\mathring{g}(v, n_2, m_2) = \frac{2m_2}{(\deg_G(v)+n_2)(\deg_G(v)+n_2-1)}$.

Proof: Using Equation 1.1, Lemma 3.1, and the fact that

$$\deg_{G \circ H}(v) = \begin{cases} \deg_G(v) + n & \text{if } \ v \in V(G), \\ \deg_H(v) + 1 & \text{if } \ v \in V(H^i) \text{ for some } i \in V(G), \end{cases}$$

we have

$$Cc_v(G \circ H) = \frac{t_{G \circ H}(v)}{\binom{\deg_{G \circ H}(v)}{2}} = \begin{cases} \frac{t_G(v)+m_2}{\binom{\deg_G(v)+n_2}{2}} & \text{if } \ v \in V(G), \\ \frac{t_H(v)+\deg_H(v)}{\binom{\deg_H(v)+1}{2}} & \text{if } \ v \in V(Hi) \text{ for some } i \in V(G). \end{cases}$$

Now if we consider the case in which $v \in V(G)$, we obtain

$$\begin{aligned} \frac{t_G(v) + m_2}{\binom{\deg_G(v)+n_2}{2}} &= \frac{t_G(v)}{\binom{\deg_G(v)+n_2}{2}} + \frac{m_2}{\binom{\deg_G(v)+n_2}{2}} \\ &= Cc_v(G) \cdot \frac{\binom{\deg_G(v)}{2}}{\binom{\deg_G(v)+n_2}{2}} + \frac{m_2}{\binom{\deg_G(v)+n_2}{2}} \\ &= Cc_v(G) \cdot \frac{\deg_G(v)(\deg_G(v) - 1)}{(\deg_G(v) + n_2)(\deg_G(v) + n_2 - 1)} + \frac{m_2}{\binom{\deg_G(v)+n_2}{2}}. \end{aligned}$$

As for the case that $v \in V(H^i)$ for some $i \in V(G)$, since the neighborhood of vertex v in this case has a similar structure of the neighborhood with vertex $v \in V(H)$ in $H \vee K_1$, we can use the first case of Theorem 2.5, which then gives us

$$\frac{t_H(v)+\deg_H(v)}{\binom{\deg_H(v)+1}{2}} = Cc_v(H)\cdot\frac{\deg_H(v)-1}{\deg_H(v)+1}+\frac{2}{\deg_H(v)+1}.$$

The claimed formula now follows. □

The next result, which is for $Cc(G\circ H)$, is a consequence of Theorem 3.2.

Corollary 3.3. *Let G and H be graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. Then the global clustering coefficient $Cc(G\circ H)$ of the corona $G\circ H$ is given by the formula*

$$\begin{aligned} Cc(G\circ H) = \frac{1}{n_1+n_1n_2}\Bigg(&\sum_{u\in V(G)}[\mathring{f}(u,n_2)\cdot Cc_u(G)+\mathring{g}(u)] \\ &+ n_1\sum_{v\in V(H)}\left[Cc_v(H)\cdot\frac{\deg_H(v)-1}{\deg_H(v)+1}+\frac{2}{\deg_H(v)+1}\right]\Bigg), \end{aligned}$$

where we have $\mathring{f}(u,n_2) = \frac{\deg_G u(\deg_G(u)-1)}{(\deg_G(u)+n_2)(\deg_G(u)+n_2-1)}$ *and* $\mathring{g}(u,n_2,m_2) = \frac{2m_2}{(\deg_G(u)+n_2)(\deg_G(u)+n_2-1)}$.

Proof: Using Equation 1.2, Theorem 3.2, and the fact that $|V(G\circ H)| = n_1+n_1n_2$ from the definition of the corona of graphs, we obtain

$$\begin{aligned} Cc(G\circ H) &= \frac{1}{n_1+n_1n_2}\sum_{w\in V(G\circ H)} Cc_w(G\circ H) \\ &= \frac{1}{n_1+n_1n_2}\Bigg(\sum_{u\in V(G)}\left[\mathring{f}(u,n_2)\cdot Cc_u(G)+\mathring{g}(u,n_2,m_2)\right] \\ &\quad+\sum_{i\in V(G)}\sum_{v\in V(H^i)}\left[Cc_v(H)\cdot\frac{\deg_H(v)-1}{\deg_H(v)+1}+\frac{2}{\deg_H(v)+1}\right]\Bigg) \\ &= \frac{1}{n_1+n_1n_2}\Bigg(\sum_{u\in V(G)}\left[\mathring{f}(u,n_2)\cdot Cc_u(G)+\mathring{g}(u,n_2,m_2)\right] \\ &\quad+n_1\sum_{v\in V(H)}\left[Cc_v(H)\cdot\frac{\deg_H(v)-1}{\deg_H(v)+1}+\frac{2}{\deg_H(v)+1}\right]\Bigg). \end{aligned}$$

where $\mathring{f}(u,n_2) = \frac{\deg_G u(\deg_G(u)-1)}{(\deg_G(u)+n_2)(\deg_G(u)+n_2-1)}$ and $\mathring{g}(u,n_2,m_2) = \frac{2m_2}{(\deg_G(u)+n_2)(\deg_G(u)+n_2-1)}$. □

Corollary 3.4. *Let G and H be graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. If G and H are regular graphs with regularities $d_G \geq 2$ and $d_H \geq 2$, respectively, then*

$$Cc(G\circ H) = \frac{1}{n_2+1}\left(\mathring{f}\cdot Cc(G)+\mathring{g}+n_2\left(Cc(H)\cdot\frac{d_H-1}{d_H+1}+\frac{2}{d_H+1}\right)\right),$$

where $\mathring{f} = \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)}$ *and* $\mathring{g} = \frac{2m_2}{(d_G+n_2)(d_G+n_2-1)}$.

Proof: Using Corollary 3.3, we have

$$Cc(G\circ H) = \frac{1}{n_1+n_1n_2}\Bigg(\sum_{u\in V(G)}\Bigg[\frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)}\cdot Cc_u(G)$$

$$
\begin{aligned}
&\quad + \frac{2m_2}{(d_G+n_2)(d_G+n_2-1)}\Bigg] + n_1 \sum_{v\in V(H)} \left[Cc_v(H)\cdot \frac{d_H-1}{d_H+1} + \frac{2}{d_H+1}\right]\Bigg) \\
&= \frac{1}{n_1+n_1n_2}\Bigg(\frac{2n_1m_2}{(d_G+n_2)(d_G+n_2-1)} + \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)} \\
&\quad \cdot \sum_{u\in V(G)} Cc_u(G) + \frac{2n_1n_2}{d_H+1} + \frac{n_1(d_H-1)}{d_H+1}\sum_{v\in V(H)} Cc_v(H)\Bigg) \\
&= \frac{1}{n_1(1+n_2)}\Bigg(\frac{2n_1m_2}{(d_G+n_2)(d_G+n_2-1)} + \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)} \\
&\quad \cdot n_1Cc(G) + \frac{2n_1n_2}{d_H+1} + \frac{n_1(d_H-1)}{d_H+1}\cdot n_2Cc(H)\Bigg) \\
&= \frac{1}{1+n_2}\Bigg(\frac{2m_2}{(d_G+n_2)(d_G+n_2-1)} + \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)}\cdot Cc(G) \\
&\quad + \frac{2n_2}{d_H+1} + \frac{d_H-1}{d_H+1}\cdot n_2Cc(H)\Bigg) \\
&= \frac{1}{1+n_2}\Bigg(\frac{2m_2}{(d_G+n_2)(d_G+n_2-1)} + \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)}\cdot Cc(G) \\
&\quad + n_2\left(\frac{2}{d_H+1} + \frac{d_H-1}{d_H+1}\cdot Cc(H)\right)\Bigg) \\
&= \frac{1}{1+n_2}\Bigg(\frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)}\cdot Cc(G) + \frac{2m_2}{(d_G+n_2)(d_G+n_2-1)} \\
&\quad + n_2\left(\frac{d_H-1}{d_H+1}\cdot Cc(H) + \frac{2}{d_H+1}\right)\Bigg).
\end{aligned}
$$

Clearly,

$$
Cc(G\circ H) = \frac{1}{n_2+1}\left(\mathring{f}\cdot Cc(G) + \mathring{g} + n_2\left(Cc(H)\cdot\frac{d_H-1}{d_H+1} + \frac{2}{d_H+1}\right)\right),
$$

where $\mathring{f} = \frac{d_G(d_G-1)}{(d_G+n_2)(d_G+n_2-1)}$ and $\mathring{g} = \frac{2m_2}{(d_G+n_2)(d_G+n_2-1)}$. □

The figure below shows the graphs $P_5\circ P_7$ and $C_5\circ C_7$. Observe that for the graph $P_5\circ P_7$, $Cc_{u_i}(P_5\circ P_7) = 1/6$ for every $i\in\{2,3,4\}$, while $Cc_{u_1}(P_5\circ P_7) = Cc_{u_5}(P_5\circ P_7) = 3/14$ and $Cc_{v_j}(P_5\circ P_7) = 2/3$ for every $j\in\{2,3,4,5,6\}$, while $Cc_{v_1}(P_5\circ P_7) = Cc_{v_7}(P_5\circ P_7) = 1$. As for the graph $C_5\circ C_7$, $Cc_{u_i}(C_5\circ C_7) = 7/36$ for every $u_i\in V(G)$, while $Cc_{v_i}(C_5\circ C_7) = 2/3$ for every $v_i\in V(H)$.

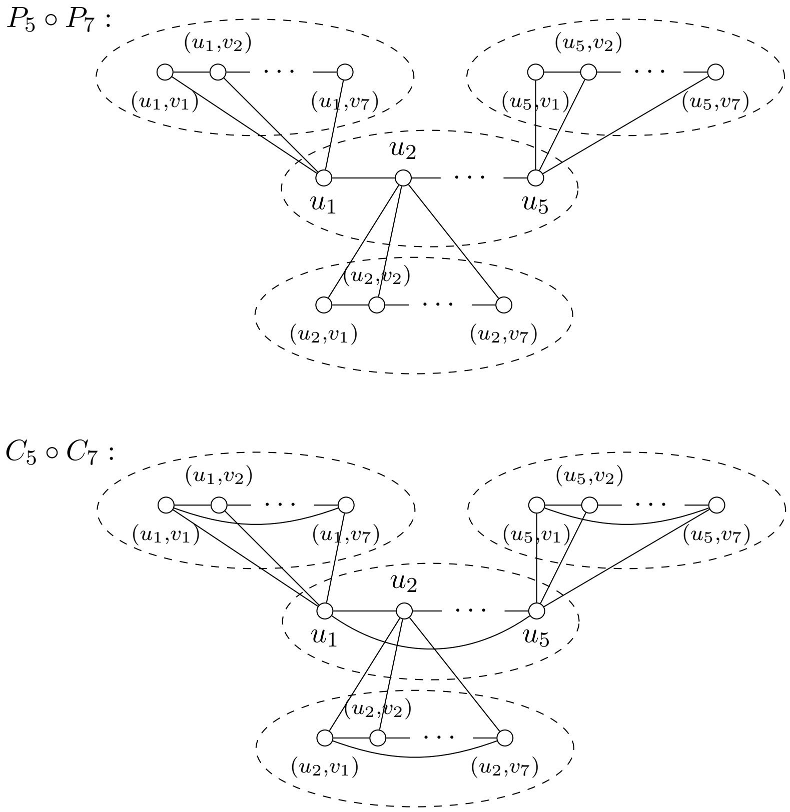

Figure 7: The graphs $P_5 \circ P_7$ and $C_5 \circ C_7$

Corollary 3.5. *For the corona $P_m \circ P_m$ and $C_m \circ C_m$, $Cc(P_m \circ P_m) \to 2/3$ and $Cc(C_m \circ C_m) \to 2/3$ as m increases without bound.*

Proof: Using Corollary 3.3 and Corollary 3.4, we have

$$Cc(P_m \circ P_m) = \frac{2m^3 - 2m^2 + 8m - 8}{m^5 + 4m^4 + 5m^3 + 2m^2} + \frac{2m-2}{3m+3}$$

and

$$Cc(C_m \circ C_m) = \frac{2m}{m^3 + 4m^2 + 5m + 2} + \frac{2m}{3m+3},$$

respectively. Observe that as m increases without bound, both $Cc(P_m \circ P_m)$ and $Cc(C_m \circ C_m)$ approach to zero. □

4 Conclusion

Our inner goal in this work was to determine whether the parameters $Cc(G \vee H)$ and $Cc(G \circ H)$ could be expressed meaningfuly in terms of $Cc(G)$ and $Cc(H)$, similar to our motives in [5, 6, 7, 8, 11].

We succeeded in generating many positive results in this direction. In addition, we were also able to show that as m increases without bound, $Cc(P_m \vee P_m)$ and $Cc(C_m \vee C_m)$ approach to 0, while $Cc(F_m)$, $Cc(W_m)$, $Cc(P_m \circ P_m)$, and $Cc(C_m \circ C_m)$ approach to 2/3. There are still some concerns to be addressed in the clustering coefficient of graphs; we hope that this paper could further stimulate research efforts in this area.

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