Analysis of hybrid stochastic Gompertz model with time delay

Abstract

This study explores a hybrid stochastic delay Gompertz model under regime switching. It is proved that the model has a unique global positive solution. Sufficient conditions for persistence in mean and extinction are obtained. The results suggest that the asymptotic nature of the model is closely related to regime switching and random perturbations.

Keywords: Gompertz equation; random perturbations; persistent in mean; extinction; Markov chains.

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1. Introduction: background and research aims

Tumors threaten human health. More than 9.5 million people were counted to have died from tumors in 2018 [1]. Tumor cell growth can be modeled by equations, the most common of which is the Gompertz equation. If

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$\psi(t)$ represents the number of cells, it has the following form

$$\mathrm{d}\psi(\iota) = (a\psi(\iota) - b\psi(\iota)\ln\psi(\iota))\mathrm{d}\iota, \qquad \iota > 0, \tag{1.1}$$

here $a,\ b$ denote the endogenous growth rate and growth deceleration factor of the tumor,respectively.

To better reflect the realism of the considered process, Growth systems usually have a delay term which represents the time lag in the tumor growth/regression process. In [2], the authors studied the previous time $\iota - \tau$ to determine the per capita growth at the current time t, i.e.

$$\mathrm{d}\psi(\iota) = [a\psi(\iota) - b\psi(\iota)\ln\psi(\iota - \tau)]\mathrm{d}\iota, \qquad \iota > 0, \tag{1.2}$$

with the initial date $\psi_0 = \{\varrho(\varsigma), -\tau \leq \varsigma \leq 0\}$ where $\varrho(\iota)$ is continuous function from $[-\tau, 0]$ to R^+.

Stochastic perturbations exist everywhere in everyday life [3–5]. They can usually be divided into large and small perturbations, where May RM [6] stated that disturbances in the environment have an impact on the growth of the species and can be estimated by modeling methods. This Methodology has been Widespread adoption (see [7–13]). According to the white noise modeling approach, $a \to a + \sigma\dot{B}(\iota)$, where $B(\iota)$ denotes the standard Brownian motion and the constant σ represents the strength of the white noise, one can get the following stochastic Gompertz model with time delay as

$$\mathrm{d}\psi(\iota) = [a\psi(\iota) - b\psi(\iota)\ln\psi(\iota - \tau)]\mathrm{d}\iota + \sigma\psi(\iota)dB(\iota), \qquad t > 0. \tag{1.3}$$

In addition to the perturbations described above, there are other perturbations (e.g., drug concentration, oxygen supply) that can cause species to change their state, such as their growth switching from one state to another, however, this variation must not be estimated with white noise [7]. For example, the mortality of hatched hatchlings varies at different temperatures [14]. In general, the next state switch is not the same as the one that occurred before, and the time at which the switch occurs follows an exponential distribution [8–10]. Therefore, Markov chains ϖ can be used to model regime transitions [8–10].

In this thesis, we mainly consider $a = r,\ b = r\beta$ of model (1.3), Thus, with the model (1.3), we can get the hybrid stochastic system, i.e.

$$\mathrm{d}\psi(\iota) = r(\varpi(\iota))[\psi(\iota) - \beta(\varpi(\iota)))\psi(\iota)\ln\psi(\iota-\tau)]\mathrm{d}\iota + \sigma(\varpi(\iota)))\psi(\iota)\mathrm{d}B(\iota), \quad \iota > 0, \tag{1.4}$$

where $a_i > 0,\ \beta_i > 0,\ \sigma_i > 0$ for any $i \in \mathbb{S}$, $\varpi(\iota)$ and $B(\iota)$ do not interfere with each other.

Some integral differential equations are difficult to solve exactly, and in [15, 16] A.Hamoud et al.determine the behavior of the solution by means of an analytic approximation form, [17–20] study the convergence and uniqueness of the solution. However, the solution of stochastic differential equations is also difficult to obtain. We found that there is little investigation of model (1.4) in the literature by looking for reading. In this paper, we can get that the solution of model (1.4) exists and will focus on the asymptotic behavior of this model.

This paper consists of the following structure: Firstly, we show that the solution of Eq. (1.4) exists and is positive. Secondly, in Section 3 we give the asymptotic properties of persistence and extinction. Finally, we conclude the paper with a short example and a brief discussion.

2. The existence of positive solutions

Through out this article we always assume that Markov chain $\varpi(\iota)$ is irreducible, It means the following linear equation (see, [12, 13])

$$\Upsilon Q = 0, \qquad \sum_{i=1}^{N} \Upsilon_i = 1, \tag{2.1}$$

there is a single fixed solution $\Upsilon = (\Upsilon_1, \cdots, \Upsilon_N)$ satisfying $\Upsilon_i > 0,\ i \in \mathbb{S}$.

To proceed with our discussion, we need a few notations:

$$\acute{r} = \max_{1\le i\le N} r_i,\ \grave{r} = \min_{1\le i\le N} r_i,\ \acute{\beta} = \max_{1\le i\le N} \beta_i,$$
$$\grave{\beta} = \min_{1\le i\le N} \beta_i,\ \omega_i = r_i - \frac{1}{2}\sigma_i^2.$$

Theorem 2.1. *Eq. (1.4) satisfies the following conditions*

$$\max_{i\in\mathbb{S}} |\omega_i + r_i\beta_i| \le M, \tag{2.2}$$

and then, the Eq. (1.4) has a unique positive solution $\psi(\iota)$.

Proof. Consider the following differential equation

$$\mathrm{d}z(\iota) = [\omega(\varpi(\iota)) - r(\varpi(\iota))\beta(\varpi(\iota))z(\iota-\tau)]\mathrm{d}\iota + \sigma(\varpi(\iota))x(\iota)\mathrm{d}B(\iota), \tag{2.3}$$

where the initial value $z_0 = \ln\psi_0$. It is easy to see that under certain assumptions, Eq. (2.3) satisfy the global Lipschitz condition and the linear growth condition. Next, define $\psi(\iota) = e^{z(\iota)}$ and using the Itô formula, one obtains

$$\begin{aligned}
\mathrm{d}\psi(\iota) &= e^{z(\iota)}[\omega(\varpi(\iota)) - r(\varpi(\iota))\beta(\varpi(\iota))z(\iota-\tau)]\mathrm{d}(\iota) + e^{z(\iota)}\sigma(\varpi(\iota))\mathrm{d}B(\iota) \\
&\quad + \frac{1}{2}e^{z(\iota)}\sigma^2(\varpi(\iota))\mathrm{d}\iota \\
&= \psi(\iota)[r(\varpi(\iota)) - r(\varpi(\iota))\beta(\varpi(\iota))\ln\psi(\iota-\tau)]\mathrm{d}\iota + \psi(\iota)\sigma(\varpi(\iota))\mathrm{d}B(\iota).
\end{aligned} \tag{2.4}$$

The theorem has been proved. □

3. persistent in mean and extinction

In this section, we are going to study the survival and extinction of species. The definitions of persistence in mean and extinction for stochastic model species were presented in [21, 22].

Definition 3.1. *Suppose that $\psi(\iota)$ is a solution of Eq. (1.4), then*
(i) $\psi(\iota)$ is ***persistent in mean*** *if* $\liminf_{\iota\to\infty} \frac{1}{\iota}\int_0^\iota \psi(s)\mathrm{d}s > 0$ *a.s.*;
(ii) $\psi(\iota)$ is ***extinction*** *if* $\lim_{\iota\to\infty} \psi(\iota) = 0$ *a.s..*

Theorem 3.1. *Suppose Theorem 2.1 holds and*

$$h_* = \sum_{i=1}^{N} \pi_i[\omega_i + r_i\beta_i] > 0, \tag{3.1}$$

then Eq. (1.4) is persistent in mean.

Proof. By using generalised Itô formula to Eq. (1.4), we can get that

$$\ln\psi(\iota) = \ln\psi(0)+\int_0^\iota [\omega(\varpi(\hbar))-r(\varpi(\hbar))\beta(\varpi(\hbar))\ln\psi(\hbar-\tau)]\mathrm{d}\hbar+\int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar). \tag{3.2}$$

Elementary inequality $\ln\psi \le \psi - 1$ for $\psi > 0$, implies

$$\begin{aligned} \ln\psi(\iota) \quad &+ \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\ln\psi(\hbar-\tau)\mathrm{d}\hbar \\ &\le \psi(\iota) + \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\psi(\hbar-\tau)\mathrm{d}\hbar - \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\mathrm{d}\hbar \\ &\le \psi(\iota) + \acute{r}\acute{\beta}\int_0^\iota \psi(\hbar)\mathrm{d}\hbar - \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\mathrm{d}\hbar + \acute{r}\acute{\beta}\int_{-\tau}^0 \varrho(\varpi(\hbar))\mathrm{d}\hbar \\ &= e^{-\acute{r}\acute{\beta}\iota}\frac{d}{d\iota}\Big(e^{\acute{r}\acute{\beta}\iota}\int_0^\iota \psi(\hbar)\mathrm{d}\hbar\Big) - \int_0^\iota r(\varpi(\hbar))\beta(\varpi(\hbar))\mathrm{d}\hbar + \acute{r}\acute{\beta}\int_{-\tau}^0 \varrho(\varpi(\hbar))\mathrm{d}\hbar. \end{aligned} \tag{3.3}$$

Joining (3.2) and (3.3), we obtain

$$\begin{aligned}\ln\psi(0) \quad &+ \int_0^{\iota}\omega(\varpi(\hbar))\mathrm{d}\hbar + \int_0^{\iota}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar)\\ &\leq \psi(\iota) + \acute{r}\beta\int_0^{\iota}\psi(\hbar)\mathrm{d}\hbar - \int_0^{\iota} r(\varpi(\hbar))\beta(\varpi(\hbar))\mathrm{d}\hbar + \acute{r}\beta\int_{-\tau}^{0}\varrho(\varpi(\hbar))\mathrm{d}\hbar.\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}e^{-\acute{r}\beta\iota}\frac{\mathrm{d}}{\mathrm{d}\iota}\Big(e^{\acute{r}\beta\iota}\int_0^{\iota}\psi(\hbar)\mathrm{d}\hbar\Big) \quad &\geq \ln\psi(0) + \int_0^{\iota}[\omega(\varpi(\hbar)) + r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar\\ &+ \int_0^{\iota}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar) - \acute{r}\beta\int_{-\tau}^{0}\varrho(\varpi(\hbar))\mathrm{d}\hbar.\end{aligned} \tag{3.5}$$

Now, integrating both sides of (3.5), it yields

$$\begin{aligned}\int_0^{\iota}\psi(\hbar)\mathrm{d}\hbar \quad &\geq \frac{C}{\acute{r}\beta}(1-e^{-\acute{r}\beta\iota}) + \int_0^{\iota}\Big(e^{\acute{r}\beta(\hbar-\iota)}\int_0^{\hbar}[\omega(\varpi(u)) + r(\varpi(u))\beta(\varpi(u))]\mathrm{d}u\Big)\mathrm{d}\hbar\\ &+ \int_0^{\iota}\Big(e^{\acute{r}\beta(\hbar-\iota)}\int_0^{\hbar}\sigma(\varpi(u))\mathrm{d}B(u)\Big)d\hbar\\ &= \frac{C}{\acute{r}\beta}\big(1-e^{-\acute{r}\beta\iota}\big) + \frac{1}{\acute{r}\beta}\int_0^{\iota}[\omega(\varpi(\hbar)) + r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar\\ &- \frac{1}{\acute{r}\beta}\int_0^{\iota}e^{\acute{r}\beta(\hbar-\iota)}[\omega(\varpi(\hbar)) + r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar\\ &+ \frac{1}{\acute{r}\beta}\int_0^{\iota}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar) - \frac{1}{\acute{r}\beta}\int_0^{\iota}e^{\acute{r}\beta(\hbar-\iota)}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar),\end{aligned} \tag{3.6}$$

where $C = \ln\psi(0) - \acute{r}\beta\int_{-\tau}^{0}\varrho(\varpi(\hbar))\mathrm{d}\hbar$. On the other hand, let

$$M_1(\iota) = \int_0^{\iota}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar), M_2(\iota) = \int_0^{\iota}e^{\acute{r}\beta(\hbar-\iota)}\sigma(\varpi(\hbar))\mathrm{d}B(\hbar). \tag{3.7}$$

Note that $M(\iota)$ is a martingale with quadratic variation

$$\langle M_1(\iota), M_1(\iota)\rangle = \int_0^{\iota}\sigma^2(\varpi(\hbar))\mathrm{d}\hbar \leq \acute{\sigma}\iota, \tag{3.8}$$

$$\langle M_2(\iota), M_2(\iota)\rangle = \int_0^{\iota}e^{2\acute{r}\beta(\hbar-\iota)}\sigma^2(\varpi(\hbar))\mathrm{d}\hbar \leq \acute{\sigma}\iota. \tag{3.9}$$

Using a strong law of large numbers for local martingales (see, e.g., [23]), we have

$$\lim_{\iota\to\infty}\frac{M_i(\iota)}{\iota}=0 \qquad a.s.,\ i=1,2. \tag{3.10}$$

From this we see that

$$\begin{aligned}\liminf_{\iota\to\infty}\frac{1}{\iota}\int_0^\iota\psi(\hbar)\mathrm{d}\hbar \ &\geq \liminf_{\iota\to\infty}\frac{1}{\iota\acute{r\beta}}\int_0^\iota[\omega(\varpi(\hbar))+r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar\\ &-\liminf_{\iota\to\infty}\frac{1}{\iota\acute{r\beta}}\int_0^\iota e^{\acute{r\beta}(\hbar-\iota)}[\omega(\varpi(\hbar))+r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar.\end{aligned} \tag{3.11}$$

Since

$$\lim_{\iota\to\infty}\frac{\left|\int_0^\iota e^{\acute{r\beta}(\hbar-\iota)}[\omega(\varpi(\hbar))+r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar\right|}{\iota}\leq\lim_{\iota\to\infty}\frac{M(1-e^{-\acute{r\beta}\iota})}{\acute{r\beta}\iota}=0, \tag{3.12}$$

holds a.s., then from (3.11)

$$\liminf_{\iota\to\infty}\frac{1}{\iota}\int_0^\iota\psi(\hbar)\mathrm{d}\hbar\geq\liminf_{\iota\to\infty}\frac{1}{\iota}\int_0^\iota[\omega(\varpi(\hbar))+r(\varpi(\hbar))\beta(\varpi(\hbar))]\mathrm{d}\hbar=\sum_{i=1}^N\pi_i[\omega_i+r_i\beta_i]=h_*. \tag{3.13}$$

Now, if $h_*>0$, we have

$$\liminf_{\iota\to\infty}\frac{1}{\iota}\int_0^t\psi(\hbar)\mathrm{d}\hbar>0 \ \ a.s.. \tag{3.14}$$

This completes the proof.

□

Theorem 3.2. *Suppose*

$$\max_{i\in\mathbb{S}}|r_i-\frac{\sigma_i^2}{2}|\leq A, \tag{3.15}$$

and

$$\eta=\lim_{\iota\to\infty}\int_0^\iota r(\varpi(u))\beta(\varpi(u))\mathrm{d}u<1, \tag{3.16}$$

hold, and for all $\iota \geq 0$

$$r - \frac{\sigma^2}{2} \leq -\theta < 0, \tag{3.17}$$

where θ *is a constant such that* $\theta > \frac{\eta A}{1-\eta}$. *Then Eq. (1.4) is extinction with probability* 1.

Proof. By using Itô formula to Eq. (1.4), we show that

$$\ln\psi(\iota) = \ln\psi(0) + \int_0^\iota [r(\varpi(\hbar)) - \frac{1}{2}\sigma^2(\varpi(\hbar)) - \beta(\varpi(\hbar))r(\varpi(\hbar))\ln\psi(\hbar-\tau)]\mathrm{d}\hbar + \int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar). \tag{3.18}$$

Consequently,

$$\begin{aligned} |\ln\psi(\iota)| &\leq |\ln\psi(0)| + \int_0^\iota |r(\varpi(\hbar)) - \frac{1}{2}\sigma^2(\varpi(\hbar))|\mathrm{d}\hbar \\ &+ \int_0^\iota \beta(\varpi(\hbar))r(\varpi(\hbar))|\ln\psi(\hbar-\tau)|\mathrm{d}\hbar + |\int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar)| \\ &\leq |\ln\psi(0)| + A\iota + \eta \sup_{u\in[-\tau,\iota]}\{|\ln\psi(u)|\} + |\int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar)|. \end{aligned} \tag{3.19}$$

It follows from (3.19) that

$$\begin{aligned} \sup_{u\in[-\tau,\iota]}\{|\ln\psi(u)|\} &\leq \sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} + \sup_{u\in[0,\iota]}\{|\ln\psi(u)|\} \\ &\leq 2\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} + A\iota + \eta\sup_{u\in[-\tau,\iota]}\{|\ln\psi(u)|\} \\ &+ \sup_{u\in[0,\iota]}|\int_0^u \sigma(\varpi(\hbar))\mathrm{d}B(\hbar)|, \end{aligned} \tag{3.20}$$

and then

$$\begin{aligned} \sup_{u\in[-\tau,\iota]}\{|\ln\psi(u)|\} &\leq \frac{2}{1-\eta}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} \\ &+ \frac{A}{1-\eta}\iota + \frac{1}{1-\eta}\sup_{u\in[0,\iota]}|\int_0^u \sigma(\varpi(\hbar))\mathrm{d}B(\hbar)|. \end{aligned} \tag{3.21}$$

This, together with (3.18), gives that

$$\begin{aligned}\ln\psi(\iota) &\leq \ln\psi(0) + (-\theta\iota) + \int_0^\iota \sigma(\varpi(\hbar))\mathrm{d}B(\hbar) + \frac{2\eta}{1-\eta}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} + \frac{A\eta}{1-\eta}\iota \\ &+\frac{\eta}{1-\eta}\sup_{u\in[0,\iota]}|\int_0^u \sigma(\xi(\hbar))\mathrm{d}B(\hbar)| \\ &\leq \frac{1+\eta}{1-\eta}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} + (-\theta + \frac{A\eta}{1-\eta}) + \frac{1}{1-\eta}\sup_{u\in[0,\iota]}|\int_0^u \sigma(\varpi(\hbar))\mathrm{d}B(\hbar)|.\end{aligned} \tag{3.22}$$

Then by use of the strong law of large numbers for martingales, from (3.22) we obtain that

$$\limsup_{\iota\to\infty}\frac{\ln\psi(\iota)}{\iota} \leq -\theta + \frac{\eta A}{1-\eta} < 0. \tag{3.23}$$

That is

$$\lim_{\iota\to\infty}\psi(\iota) = 0 \quad a.s.. \tag{3.24}$$

The proof is complete. □

4. Conclusions

Example: Suppose that the irreducible Markov chain $\varpi(\iota)$ takes values in $\mathbb{S} = 1, 2$ and the transition probability matrix is $q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$. Thus, its stationary distribution is $\Upsilon = (\Upsilon_1, \Upsilon_2) = (\frac{2}{3}, \frac{1}{3})$. To verify the result of Theorem 3.1, let

$$\begin{aligned}&r_1 = 4,\ r_2 = 1,\ \sigma_1 = \sqrt{5},\ \sigma_2 = \sqrt{3}, \\ &\beta_1 = 1/2,\ \beta_2 = 1,\ \omega_1 = 1.5,\ \omega_2 = -0.5,\ M = 3.5892.\end{aligned}$$

By calculation, we can get

$$h_* = \pi_1[\omega_1 + r_1\beta_1] + \pi_2[\omega_2 + r_2\beta_2] > 0,\ \omega_2 + r_2\beta_2 \leq M.$$

Therefore, by Theorem 3.1, Eq. (1.4) is persistence in mean.

An important topic in ecology is the effect of various perturbations on the persistence in mean and extinction of stochastic Gompertz models [6]. In this paper we present and explore stochastic Gompertz model with two perturbations and with time lags. Theorem 3.1 and Theorem 3.2 establish sufficient conditions for persistence in mean and extinction of the species.

The topics that trigger our further research through this article are as follows: fist what happens if Eq. (1.4) is perturbed by the intrinsic growth rate r while β is also perturbed by white noise. Secondly what happens to the asymptotic properties of Eq. (1.4) if both distribution delay and Lvy jumps are introduced (for more details of Lvy jumps, see [11, 24, 25]).

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