Analysis of hybrid stochastic Gompertz model with time delay

Abstract

This study explores a hybrid stochastic delay Gompertz model under regime switching. It is proven that the model has a unique global positive solution. Sufficient conditions for persistence in mean and extinction are obtained. The results demonstrate that the dynamics of the model are intimately associated with the regime switching and random perturbations.

Keywords: delay; persistent in mean; extinction; hybrid stochastic Gompertz model; Markov chains.

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1. Introduction: background and research aims

In mathematical ecology, the Gompertz model is one of the most important models, The Gompertz law of tumour cell growth is given by

$$\mathrm{d}\psi(t) = (a\psi(t) - b\psi(t)\ln\psi(t))\mathrm{d}t, \qquad t > 0 \tag{1.1}$$

*

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here $\psi(t)$ represents the cell number at time t , $\psi(0)$ is the number at the initial time identified as the instant when the cancer is diagnosed. The parameter a is the intrinsic growth rate of the tumour related to the initial mitosis rate and b is the growth deceleration factor. Since time delays are often introduced to the growth models to better reflect reality of considered processes, in Gompertz model delay may represent the time lag in the process of tumor growth/regression. In the paper [1], the authors introduced the time delay to Eq. (1.1) to describe that the growth per capita at present time t depends on the previous time $t-\tau$, that is

$$\mathrm{d}\psi(t) = [a\psi(t) - b\psi(t)\ln\psi(t-\tau)]\mathrm{d}t, \qquad t > 0 \tag{1.2}$$

with the initial date $\psi_0 = \{\varrho(\varsigma), -\tau \leq \varsigma \leq 0\}$ where $\varrho(t)$ is continuous function from $[-\tau, 0]$ to R^+.

There are many random perturbations in the environment [2–4]. Firstly, we consider the small perturbations in the environment. May RM [5] pointed out that the small environmental perturbations mainly affect the growth rate of a species and can be modeled by a white noise. This approach has been widely adopted (see [6–14]). Following this approach, $a \to a + \sigma\dot{B}(t)$, where $\{B(t), t \geq 0\}$ represents standard Brownian motion and real constant σ intensity of the noise, one can get the following stochastic Gompertz model with time delay as

$$\mathrm{d}\psi(t) = [a\psi(t) - b\psi(t)\ln\psi(t-\tau)]\mathrm{d}t + \sigma\psi(t)dB(t). \qquad t > 0 \tag{1.3}$$

Beside small perturbations, there are some environmental perturbations which can cause the growth of a species switches from one state to another. In general, the switching is memoryless. Therefore the regime switching could be modeled by a finite-state Markov chain [15–17]. Let $\xi(t)$ be a right-continuous Markov chain in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$, with the generator $Q = (q_{ij})_{N\times N}$ given by

$$\mathbb{P}\{\xi(t+\Delta t) = j|\xi(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t) & \text{if } i \neq j, \\ 1 + q_{ij}\Delta t + o(\Delta t) & \text{if } i = j, \end{cases} \tag{1.4}$$

where $\Delta t > 0$. Here $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$q_{ii} = -\sum_{j\neq i} q_{ij}. \tag{1.5}$$

In this paper, we mainly consider $a = r, b = r\beta$ of model (1.3), Thus, on the basis of model (1.3), we obtain hybrid stochastic Gompertz model with time delay, i.e.

$$\mathrm{d}\psi(t) = r(\xi(t))[\psi(t) - \beta(\xi(t)))\psi(t)\ln\psi(t-\tau)]\mathrm{d}t + \sigma(\xi(t)))\psi(t)\mathrm{d}B(t), \quad t > 0 \tag{1.6}$$

where $a(i), \beta(i), \sigma(i)$ are all positive for any $i \in \mathbb{S}$, and $\xi(t)$ is $\mathcal{F}_t$ adapted but independent of the Brownian motion $B(t)$.

To the best of our knowledge, there are few investigations on model (1.6) in the literatures. In this paper, we will devote our main attention to the study on the dynamics of this model, and establish the threshold between permanence and extinction for the model. The paper is organized as follows: In the next section we show that Eq. (1.6) has a unique positive global solution. In Section 3 we give some long time dynamical properties, such as persistence an extinction. Finally, we conclude the paper by a brief discussion in Section 4.

2. Description of the model and research basis

Through out this paper we always assume that Markov chain $\xi(t)$ is irreducible, that is to say the following linear equation

$$\pi Q = 0, \qquad \sum_{i=1}^{N} = 1, \tag{2.1}$$

has a unique stationary solution $\pi = (\pi_1, \cdots, \pi_N)$ satisfying $\pi_i > 0, i \in \mathbb{S}$.

In this paper, we consider stochastic delay Gompertz population system with regime switching described by (1.6) with the initial value $\psi(0) = \psi_0 > 0, \xi(0) = i \in \mathbb{S}$, where $B(t)$ is 1-dimensional standard Brownian motion and we always suppose that Markov chain $\xi(t)$ is $\mathcal{F}_t$-adapted but independent of the Brownian motion $B(t)$. Eq. (1.6) can be regarded as the results of the following N autonomous equations

$$\mathrm{d}\psi(t) = r(i)[\psi(t) - \beta(i)\psi(t)\ln\psi(t-\tau)]\mathrm{d}t + \sigma(i)\psi(t)\mathrm{d}B(t), \quad i \in \mathbb{S} \tag{2.2}$$

switching from one to the others according to the movement of Markov chain. This switching is without memory and the waiting time for the next switch has an exponential distribution. We will discuss the properties of the solution of Eq. (1.6).

Theorem 2.1. *Let the parameters of Eq. (1.6) satisfy the condition*

$$\max_{i\in\mathbb{S}} |r(i) - \frac{\sigma^2}{2} + r(i)\beta(i)| \le M, \tag{2.3}$$

for some constant $M \ge 0$. Then, there exists a unique positive solution $\psi(t)$ for $t \ge -\tau$ to Eq. (1.6).

Proof. Consider the stochastic delay differential equation

$$\mathrm{d}z(t) = [r(\xi(t)) - \frac{1}{2}\sigma^2(\xi(t)) - r(\xi(t))\beta(\xi(t))z(t-\tau)]\mathrm{d}t + \sigma(i)x(t)\mathrm{d}B(t) \quad (2.4)$$

on $t \geq 0$ with the initial date $z_0 = \ln \psi_0$ Obviously, the coefficients of Eq. (2.4) satisfy the global Lipschitz condition and,under assumption (2.3), the linear growth condition. Thus, for any initial date there exists a unique global solution $z(t)$ on $t \geq -\tau$. Therefore,by using Itô formula, Define $\psi(t) = e^{z(t)}$, we have

$$\begin{aligned} \mathrm{d}\psi(t) &= e^{z(t)}[r(\xi(t)) - \frac{1}{2}\sigma^2(\xi(t)) - r(\xi(t))\beta(\xi(t))z(t-\tau)]\mathrm{d}(t) \\ &+e^{z(t)}\sigma(\xi(t))\mathrm{d}B(t) + \frac{1}{2}e^{z(t)}\sigma^2(\xi(t))\mathrm{d}t \\ &= \psi(t)[r(\xi(t)) - r(\xi(t))\beta(\xi(t))\ln\psi(t-\tau)]\mathrm{d}t + \psi(t)\sigma(\xi(t))\mathrm{d}B(t). \end{aligned} \quad (2.5)$$

The theorem has been proved. □

3. persistent in mean and extinction

Definition 3.1. *Suppose that $\psi(t)$ is a solution of Eq. (1.6), then*
(1) $\psi(t)$ is said to be ***persistent in mean*** *if* $\liminf_{t\to\infty} \frac{1}{t}\int_0^t \psi(s)\mathrm{d}s > 0$ *a.s.*;
(2) $\psi(t)$ is said to be ***extinction*** *if* $\lim_{t\to\infty} \psi(t) = 0$ *a.s.*.

To proceed with our discussion, we need a few notations:

$$\begin{aligned} &\check{r} = \max_{1\leq i\leq N} r(i),\ \hat{r} = \min_{1\leq i\leq N} r(i),\ \check{\beta} = \max_{1\leq i\leq N} \beta(i), \\ &\hat{\beta} = \min_{1\leq i\leq N} \beta(i),\ \omega(i) = r(i) - \frac{1}{2}\sigma^2(i). \end{aligned}$$

Theorem 3.1. *Suppose Theorem 2.1 holds and*

$$h_* = \sum_{i=1}^{N} \pi_i[\omega(i) + r(i)\beta(i)] > 0, \quad (3.1)$$

then Eq. (1.6) is persistent in mean.

Proof. By using generalised Itô formula to Eq. (1.6), we can get that

$$\ln\psi(t) = \ln\psi(0) + \int_0^t [\omega(\xi(s)) - r(\xi(s))\beta(\xi(s))\ln\psi(s-\tau)]\mathrm{d}s + \int_0^t \sigma(\xi(s))\mathrm{d}B(s). \tag{3.2}$$

Elementary inequality $\ln\psi \leq \psi - 1$ for $\psi > 0$, implies

$$\begin{aligned} \ln\psi(t) \quad &+ \int_0^t r(\xi(s))\beta(\xi(s))\ln\psi(s-\tau)\mathrm{d}s \\ &\leq \psi(t) + \int_0^t r(\xi(s))\beta(\xi(s))\psi(s-\tau)\mathrm{d}s - \int_0^t r(\xi(s))\beta(\xi(s))\mathrm{d}s \\ &\leq \psi(t) + \check{r\beta}\int_0^t \psi(s)\mathrm{d}s - \int_0^t r(\xi(s))\beta(\xi(s))\mathrm{d}s + \check{r\beta}\int_{-\tau}^0 \varrho(\xi(s))\mathrm{d}s \\ &= e^{-\check{r\beta}t}\frac{d}{dt}\Big(e^{\check{r\beta}t}\int_0^t \psi(s)\mathrm{d}s\Big) - \int_0^t r(\xi(s))\beta(\xi(s))\mathrm{d}s + \check{r\beta}\int_{-\tau}^0 \varrho(\xi(s))\mathrm{d}s. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\begin{aligned} \ln\psi(0) \quad &+ \int_0^t \omega(\xi(s))\mathrm{d}s + \int_0^t \sigma(\xi(s))\mathrm{d}B(s) \\ &\leq \psi(t) + \check{r\beta}\int_0^t \psi(s)\mathrm{d}s - \int_0^t r(\xi(s))\beta(\xi(s))\mathrm{d}s + \check{r\beta}\int_{-\tau}^0 \varrho(\xi(s))\mathrm{d}s. \end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned} e^{-\check{r\beta}t}\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{\check{r\beta}t}\int_0^t \psi(s)\mathrm{d}s\Big) \quad &\geq \ln\psi(0) + \int_0^t [\omega(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s \\ &\quad + \int_0^t \sigma(\xi(s))\mathrm{d}B(s) - \check{r\beta}\int_{-\tau}^0 \varrho(\xi(s))\mathrm{d}s. \end{aligned} \tag{3.5}$$

Now, integrating both sides of (3.5), it yields

$$\begin{aligned} \int_0^t \psi(s)\mathrm{d}s \quad &\geq \frac{C}{\check{r\beta}}(1 - e^{-\check{r\beta}t}) + \int_0^t \Big(e^{\check{r\beta}(s-t)}\int_0^s [\omega(\xi(u)) + r(\xi(u))\beta(\xi(u))]\mathrm{d}u\Big)\mathrm{d}s \\ &\quad + \int_0^t \Big(e^{\check{r\beta}(s-t)}\int_0^s \sigma(\xi(u))\mathrm{d}B(u)\Big)ds \\ &= \frac{C}{\check{r\beta}}\big(1 - e^{-\check{r\beta}t}\big) + \frac{1}{\check{r\beta}}\int_0^t [\omega(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&-\frac{1}{\check{r\beta}}\int_0^t e^{\check{r\beta}(s-t)}[\omega(\xi(s))+r(\xi(s))\beta(\xi(s))]\mathrm{d}s \\
&+\frac{1}{\check{r\beta}}\int_0^t \sigma(\xi(s))\mathrm{d}B(s)-\frac{1}{\check{r\beta}}\int_0^t e^{\check{r\beta}(s-t)}\sigma(\xi(s))\mathrm{d}B(s),
\end{aligned}
$$

where $C=\ln x(0)-\check{r\beta}\int_{-\tau}^0 \varrho(\xi(s))\mathrm{d}s$. Since

$$
\limsup_{t\to\infty}\frac{|C\big(1-e^{-\check{r\beta}t}\big)|}{t}=0, \tag{3.7}
$$

$$
\limsup_{t\to\infty}\frac{\Big|\int_0^t e^{\check{r\beta}(s-t)}[\omega(\xi(s))+r(\xi(s))\beta(\xi(s))]\mathrm{d}s\Big|}{t}\le\limsup_{t\to\infty}\frac{M(1-e^{-\check{r\beta}t})}{\check{r\beta}t}=0, \tag{3.8}
$$

$$
\limsup_{t\to\infty}\frac{\Big|\int_0^t \sigma(\xi(s))\mathrm{d}B(s)\Big|}{t}=0 \quad \text{and} \quad \limsup_{t\to\infty}\frac{\Big|\int_0^t e^{\check{r\beta}(s-t)}\sigma(\xi(s))\mathrm{d}B(s)\Big|}{t}=0 \tag{3.9}
$$

holds a.s., then from (3.6)

$$
\liminf_{t\to\infty}\frac{1}{t}\int_0^t\psi(s)\mathrm{d}s\ge\liminf_{t\to\infty}\frac{1}{t}\int_0^t[\omega(\xi(s))+r(\xi(s))\beta(\xi(s))]\mathrm{d}s=\sum_{i=1}^N\pi_i[\omega(i)+r(i)\beta(i)]=h_*. \tag{3.10}
$$

Now, if $h_*>0$, we have

$$
\liminf_{t\to\infty}\frac{1}{t}\int_0^t\psi(s)\mathrm{d}s>0 \quad a.s.. \tag{3.11}
$$

This completes the proof.

□

Theorem 3.2. *Suppose*

$$
\max_{i\in\mathbb{S}}|r(i)-\frac{\sigma^2}{2}|\le A, \tag{3.12}
$$

and

$$
\rho=\lim_{t\to\infty}\int_0^t r(\xi(u))\beta(\xi(u))\mathrm{d}s<1, \tag{3.13}
$$

hold, and for all $t \geq 0$

$$r(i) - \frac{\sigma^2}{2} \leq -\theta < 0, \tag{3.14}$$

where θ *is a constant such that* $\theta > \frac{\rho A}{1-\rho}$*. Then Eq. (1.6) is extinction with probability* 1*.*

Proof. By using Itô formula to Eq. (1.6), we show that

$$\ln\psi(t) = \ln\psi(0) + \int_0^t [r(\xi(s)) - \frac{1}{2}\sigma^2(\xi(s)) - \beta(\xi(s))r(\xi(s))\ln\psi(s-\tau)]\mathrm{d}s + \int_0^t \sigma(\xi(s))\mathrm{d}B(s). \tag{3.15}$$

Consequently,

$$\begin{aligned}
|\ln\psi(t)| &\leq |\ln\psi(0)| + \int_0^t |r(\xi(s)) - \frac{1}{2}\sigma^2(\xi(s))|\mathrm{d}s \\
&+ \int_0^t \beta(\xi(s))r(\xi(s))|\ln\psi(s-\tau)|\mathrm{d}s + |\int_0^t \sigma(\xi(s))\mathrm{d}B(s)| \\
&\leq |\ln\psi(0)| + At + \sup_{u\in[-\tau,t]}\{|\ln\psi(u)|\}\int_0^t \beta(\xi(s))r(\xi(s))\mathrm{d}s \\
&+ |\int_0^t \sigma(\xi(s))\mathrm{d}B(s)| \\
&\leq |\ln x(0)| + At + \rho\sup_{u\in[-\tau,t]}\{|\ln\psi(u)|\} + |\int_0^t \sigma(\xi(s))\mathrm{d}B(s)|.
\end{aligned} \tag{3.16}$$

It follows from (3.16) that

$$\begin{aligned}
\sup_{u\in[-\tau,t]}\{|\ln\psi(u)|\} &\leq \sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} + \sup_{u\in[0,t]}\{|\ln\psi(u)|\} \\
&\leq \sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} + |\ln\psi(0)| + At + \rho\sup_{u\in[-\tau,t]}\{|\ln\psi(u)|\} \\
&+ \sup_{u\in[0,t]}|\int_0^u \sigma(\xi(s))\mathrm{d}B(s)| \\
&\leq 2\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\} + At + \rho\sup_{u\in[-\tau,t]}\{|\ln\psi(u)|\} \\
&+ \sup_{u\in[0,t]}|\int_0^u \sigma(\xi(s))\mathrm{d}B(s)|,
\end{aligned} \tag{3.17}$$

and then

$$
\begin{aligned}
\sup_{u\in[-\tau,t]}\{|\ln\psi(u)|\} \quad &\le \frac{2}{1-\rho}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\}\\
&+\frac{A}{1-\rho}t+\frac{1}{1-\rho}\sup_{u\in[0,t]}|\int_0^u\sigma(\xi(s))\mathrm{d}B(s)|.
\end{aligned}
\tag{3.18}
$$

This, together with (3.15), gives that

$$
\begin{aligned}
\ln\psi(t) \quad &\le \ln\psi(0)+\int_0^t[r(\xi(s))-\frac{1}{2}\sigma^2(\xi(s))]\mathrm{d}s+\rho\sup_{u\in[-\tau,t]}\{|\ln\psi(u)|\}+\int_0^t\sigma(\xi(s))\mathrm{d}B(s)\\
&\le \ln\psi(0)+(-\theta t)+\int_0^t\sigma(\xi(s))\mathrm{d}B(s)+\frac{2\rho}{1-\rho}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\}+\frac{A\rho}{1-\rho}t\\
&+\frac{\rho}{1-\rho}\sup_{u\in[0,t]}|\int_0^u\sigma(\xi(s))\mathrm{d}B(s)|\\
&\le \frac{1+\rho}{1-\rho}\sup_{u\in[-\tau,0]}\{|\ln\psi(u)|\}+(-\theta+\frac{A\rho}{1-\rho})+\frac{1}{1-\rho}\sup_{u\in[0,t]}|\int_0^u\sigma(\xi(s))\mathrm{d}B(s)|.
\end{aligned}
\tag{3.19}
$$

Then by use of the strong law of large numbers for martingales, from (3.19) we obtain that

$$
\limsup_{t\to\infty}\frac{\ln\psi(t)}{t}\le-\theta+\frac{\rho A}{1-\rho}<0. \tag{3.20}
$$

That is

$$
\lim_{t\to\infty}\psi(t)=0 \quad a.s.. \tag{3.21}
$$

The proof is complete. □

4. Conclusions

The effect of stochasticity on the persistence in mean and extinction of stochastic Gompertz models is a key topic in ecology [5]. This paper has been formulated and explored a stochastic Gompertz model with Markovian switching. Theorem 3.1 and Theorem 3.2 have established sufficient conditions for persistence in mean and extinction of the species. Several topics deserve further research. Eq. (1.6) assumes that only the intrinsic growth rate r is perturbed by the white noise. An interesting topic is to explore what happens if β are also perturbed by the white noise. Another topic is to explore Eq. (1.6) is also perturbed by the Lvy jumps (for more details of Lvy jumps, see [12, 19, 20]).

References

[1] Piotrowska MJ, Forys U. *The nature of Hopf bifurcation for the Gompertz model with delays*. Math Comput Model. 2011;54(9-10):2183-2198. doi:10.1016/j.mcm.2011.05.027.

[2] Krstic M, Jovanovic M. *On stochastic population model with the Allee effect*. Math Comput Modelling. 2010;52(1-2):370-379. doi:10.1016/j.mcm.2010.02.051.

[3] Carlos C, Braumann CA. *General population growth models with Allee effects in a random environment*. Ecol Complex. 2017;30:26-33. doi:10.1016/j.ecocom.2016.09.003.

[4] Zhang B, Wang H, Lv G. *Exponential extinction of a stochastic predatorCprey model with Allee effect*. Physica A. 2018;507:192-204. doi:10.1016/j.physa.2018.05.073.

[5] May RM. *Stability and Complexity in Model Ecosystems*. Princeton Univ Press. 1973.

[6] Wang K. *Stochastic Mathematical Biology Models*. Science Press Beijing. 2010.

[7] Li XY, Jiang DQ, Mao XR. *Population dynamical behavior of Lotka-Volterra system under regime switching*. J Comput Appl Math. 2009;232(2):427-448. doi:10.1016/j.cam.2009.06.021.

[8] Li XY, Gray A, Jiang DQ, Mao XR. *Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching*. J Math Anal Appl. 2011;376(1):11-28. doi:10.1016/j.jmaa.2010.10.053.

[9] Wu R, Zou X, Wang K. *Asymptotic properties of stochastic hybrid Gilpin-Ayala system with jumps*. Appl Math Comput. 2014;249:53-66. doi:10.1016/j.amc.2014.10.043.

[10] Liu M, Du C, Deng M. *Persistence and extinction of a modified Leslie-Gower Holling-type II stochastic predator-prey model with impulsive toxicant input in polluted environments*. Nonlinear Anal Hybrid Syst. 2018;27:177-190. doi:10.1016/j.nahs.2017.08.001.

[11] Liu M, Yu Y, Mandal P. *Dynamics of a stochastic delay competitive model with harvesting and Markovian switching*. Appl Math Comput. 2018;337:335-349. doi:10.1016/j.amc.2018.03.044.

[12] Liu M, Zhu Y. *Stationary distribution and ergodicity of a stochastic hybrid competition model with Lvy jumps*. Nonlinear Anal Hybrid Syst. 2018;30:225-239. doi:10.1016/j.nahs.2018.05.002.

[13] Liu M, Deng M. *Permanence and extinction of a stochastic hybrid model for tumor growth*. Appl Math Lett. 2019;94:66-72. doi:10.1016/j.aml.2019.02.016.

[14] Liu M. *Dynamics of a stochastic regime–switching predator-prey model with modified Leslie-Gower Holling-type II schemes and prey harvesting.* Nonlinear Dynam. 2019;96:417-442. doi:10.1007/s11071-019-04797-x.

[15] Li D, Liu M. *Invariant measure of a stochastic food-limited population model with regime switching*. Math Comput Simul. 2020;178:16-26. doi:10.1016/j.matcom.2020.06.003.

[16] Liu M, Bai C. *Optimal harvesting of a stochastic mutualism model with regime-switching*. Appl Math Comput. 2020;375:125040. doi:10.1016/j.amc.2020.125040.

[17] Wu R, Zou X, Wang K. *Asymptotic properties of stochastic hybrid Gilpin-Ayala system with jumps*. Appl Math Comput. 2014;249:53-66. doi:10.1016/j.amc.2014.10.043.

[18] Smith HL, Thieme HR. *Dynamical Systems and Population Persistence.* American Mathematical Society, NY: 2011.

[19] Bao J, Mao X, Yin G, Yuan C. *Competitive Lotka-Volterra population dynamics with jumps*. Nonlinear Anal. 2011;74(17):6601-6616. doi:10.1016/j.na.2011.06.043.

[20] Zhang X, Chen F, Wang K, Du H. *Stochastic SIRS model driven by Lvy noise*. Acta Math Sci. 2016;36(3):740-752. doi:10.1016/s0252-9602(16)30036-4.