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Regular Elements and Von-Neumann Inverses of a class of zero symmetric Local near-rings admitting Frobenius Derivations

Joseph Motanya Abuga [1], Michael Onyango Ojiema [2], Benard Muthiani Kivunge[3]
[1] Department of Mathematics Kisii University
[2] Department of Mathematics Masinde Muliro University of Science and Technology
[3] Department of Mathematics Kenyatta University
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Abstract

Let N be a zero-symmetric local near-ring. An element $x \in N$ is either regular, zero or a zero divisor. In this paper, we construct a class of zero symmetric local near-ring of characteristic p^k; $k \geq 3$ admitting an identity frobenius derivation, characterize the structures and orders of the set $R(N)$, the regular compartment with an aim of advancing the classification problem of algebraic structures. The number theoretic notions relating the number of regular elements to Euler’s phifunction and the arithmetic functions of Galois near-rings are adopted. Using the Fundamental Theorem of finitely generated Abelian groups, the structures of $R(N)$ are proved to be isomorphic to cyclic groups of various orders. The study also extends to the automorphism groups $Aut(R(N))$ of the regular elements.

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1 Introduction

The study of near-rings with identity is very vital in generalizing characterization of commutative rings with identity. The original works on near-rings and their applications are attributed to Pilz[1] who have very good foundations upon which these algebraic structures could be advanced. Much of the recent works on the classification of finite rings with identity have however considered a characterization paradigm using the unit groups, the zero divisor graphs, adjacency and incidence matrices among others. This has left the non-linear aspects fairly untouched. In particular, regular elements and Von-Neumann inverses of near rings admitting derivations hardly exist in the available literature.

Oduor, Ojiema and Mmasi[2] determined construction of idealized local rings of characteristic $p^n : n = 1, 2, 3$ and determined the structures of the unit groups R^*. Osba, et al.

Henriksen and Osama [3] conducted a classification survey on combining local and Von Neumann Regular Rings as a basis upon which the regularity properties of rings and their ideals could be explored. The rings studied in [3] were finite and their Von Neumann inverses gave some asymptotic patterns. Their findings demonstrated how to combine the Von- Neumann inverses of classes of rings such as the power series rings and the ring of integers. They however did not count the number of regular elements in a given finite ring nor did they give the structural formulae for the regular elements and the Von Neumann inverses of the specified classes of rings. In a closely related research, the study on regular elements of Galois rings can be attributed to Osama and Emad [4] where they characterized the regular elements in the ring of integers modulo n, Z_n. Furthermore, they studied the arithmetic functions denoted as $V(n)$ and determined the relationship between $V(n)$ and the Euler's phi function, $\varphi(n)$. This gave an extension of the ring theoretic algebra employed in counting the regular elements of Z_n to the number theoretic methodologies. For instance, the research revealed that if a is a regular element in Z_n, then $a^{(-1)} \equiv a^{\varphi(n)-1} \pmod{n}$. They proposed a criterion for getting the possible Von Neumann inverses in the set of regular elements of Z_n and explored the asymptotic properties of $V(n)$. Their findings did not consider extensions and idealization using maximal submodules of $Z_n \forall n \in Z$.

Closely related works can also be seen in Osba et al [5] and Oduor, Omamo and Musoga[6]. Furthermore, Abujabal et al[7] considered the structure and commutativity
of general near-rings. The ideas postulated in [7] were later improved by Asma and
Inzamam[8] who gave a number of conditions that determine the commutators and anticommutators
of zero symmetric near-rings with Jordan ideals and derivations. Akin[9] studied IFP ideals in near-rings while Ali, Bell and Miyan[10] considered generalized
derivations in rings. In order to advance the problem of classification of algebraic structures,
the paper discovers new classes of near-rings and classifies them via their regular
elements.

2 Zero-Symmetric Local Near-Ring of Characteristic $p^k : k \geq 3$

Let $R_o = GN(p^{kr}, p^k)$. Let $i = 1, ..., h$ and $u_i \in Z_L(N)$ and $M = < u_i >$.
Then,

$$N = R_o \oplus M = R_o \oplus \sum_{i=1}^{h} (R_o/pR_o)_i$$

is a group with respect to addition.
On N , let

$$(r_o, r_1, ..., r_h)(s_o, s_1, ..., s_h) = (r_o s_o, r_o s_1 + r_1 s_o, ..., r_o s_h + r_h s_o)^{\delta} \quad (3)$$

where δ is the identity Frobenius automorphism. The multiplication turns N into a local
zero symmetric near-ring with identity $(1, 0, ..., 0)$.

Indeed $N = R_o \oplus M$ is commutative since δ is the identity Frobenius automorphism.

Proposition 2.1. Consider $N = GN(p^{kr}, p^k)$ where $k \geq 3$. Then, $\mathrm{char} N = p^k$ and:

(i). $Z_L(N) = pR_o \oplus \sum_{h}$

$_{i=1}(R_o/pR_o)_i$

(ii). $(Z_L(N))_{k-1} = p_{k-1}R_o \neq (0)$

(iii). $(Z_L(N))_k = (0)$.

Proof. Char $GN(p_{kr}, p_k)$ = charN and $id_N = id_{GN(p_{kr},p_k)}$

Let $a \in R_o$ and a not contained in pR_0 and let $s \in Z_L(N)$.
Then

$(a + s)_{pr} = a_{pr} + s' \ : (s' \in Z_L(N))$

$= (a + s'')_{pr-1} : (s'' \in Z_L(N))$

But $(a + s'')_{pr-1} \equiv 1 + s'''$ with $s''' \in Z_L(N)$ and $(1 + s''')_{pk-1} = 1$. Hence $(a + s)$ is
regular and not zero.

Since $| Z_L(N) | = p_{(h+k-1)r}$ and

$| (R_o/pR_o)_* + Z_L(N) | = (p_r - 1)(p_{(h+k-1)r})$, it follows that

$(R_o/pR_o)_* + Z_L(N) = N - Z_L(N)$ and hence all the elements outside $Z_L(N) \setminus \{0\}$ are
regular.

Remark 2.1. A regular element $x \in R(N)$ may have more than one Von-Neumann inverse. However, for the classes of near-rings considered in this study, the Von-Neumann
inverses are unique.

Proposition 2.2. Let N be a class of near-ring of the construction. For $x \in N$ and
$x_0 \in I(x)$, where $I(x)$ is the inner inverse set, then:

$I(x) = \{x_0 + \alpha - x_0 x \alpha x x_0 \mid \alpha \in N\}$

Proof. From the construction, if $x \in N$, then

$x = (r_0 + ($

$\sum_{i=1}^{h}$

$r_0 + pr$

$'$

$)r$

$'$

$\in GN(p_{kr}, p_k)/pGN(p_{kr}, p_k))$.

So the definition of the multiplication in N gives the desired result.

Denote by $l(x)$ and $r(x)$ the left and the right annihilator of an element $x \in N$. So the inner annihilator of $x \in N$ is: $Iann(x) = \{y \in N : xyx = 0\}$.

Theorem 2.1. Let N be the near ring of the construction. If $a \in R(N)$, then for any $b \in N$, $bI(a)b$ is a singleton set if and only if $b \in Na \cap aN$.

Proof. Suppose there exists $x, y \in N$ such that $b = xa = ay$ and let $a_o \in I(a)$. We then have that for any $t \in N$,

$b(a_o + t - a_o ataa_o)b = (xaa_o + xat - xataa_o)ay$

$= xay + xatay - xatay$

$= xay$

Thus the set $bI(a)b = \{xay\}$ is singleton.

Conversely, suppose that $bI(a)b = \{ba_o b\}$.

We then have: $b(a_o + t - a_o ataa_o)b = ba_o b$ for any $t \in N$. This implies that for any $t \in N$, we have: $b(t - a_o ataa_o)b = 0$ ....(i). Substituting $(1 - a_o a)t$ for t in this equality yields $b(1 - a_o ataa_o)tb = 0$ for any $t \in N$. But N constructed is semiprime so that $b(1 - a_o a) = 0 \Rightarrow b = ba_o a \in Na$ .....(ii)

Similarly, substituting t by $t(1 - aa_o)$ in the equality (i) gives $b = aa_o b \in aN$ ....(iii)

Comparing (ii) and (iii), we conclude that $b \in Na \cap aN$

Lemma 2.1. Let N be the near ring constructed and let $b, d \in N$ such that $b + d$ is a Von Neumann regular element. Then the following are equivalent:

(i) $bN \oplus dN = (b + d)N$

(ii) $Nb \oplus Nd = N(b + d)$

(iii) $bNb \cap dN = \{0\}$ and $Nb \cap Nd = \{0\}$.

The next result shows when $I(a) \subseteq I(b)$ necessarily and sufficiently where $a, b \in N$

Proposition 2.3. Let $a, b \in R(N)$. Then $I(a) \subseteq I(b)$ if and only if $bN \cap dN = \{0\}$ and $Nb \cap Nd = \{0\}$ where $a = d + b$

Proof. Let $I(a) \subseteq I(b)$. Then by definition, there exists some $x \in I(a)$ such that $bxb = b$.

Now $b \in Na \cap aN$.

Write $b = \alpha a = a\beta$ where $\alpha, \beta \in N$.

Then $bI(a)a = b$.

Next

$bI(a)d = bI(a)a - bI(a)b$

$= b - bI(a)b = 0$

Consider now

$dI(a)b = aI(a)b - bI(a)b$

$= \alpha\beta - bI(a)b$

$= b - b = 0$

We thus have $bI(a)d = 0$ and $dI(a)b = 0$ ....(i)

Then for any $x \in I(a)$ we have;

$b + d = a = axa$

$= (b + d)x(b + d)$

$= bxa + dxb + dxd$

$= b + 0 + dxd$

This yields $dI(a)d = d$...(ii)

To show that $dN \cap bN = \{0\}$.

Let $bx = dy \in bN \cap dN$.

Multiplying both sides of (ii) by y on the right and using $bx = dy$ yields,

$dI(a)bx = dy$

But from above we have that $dI(a)b = 0$ and so $dy = 0$ which clears the proof.

Similarly, we show that $Nb \cap Nd = \{0\}$.

Let $xb = yd \in Nb \cap Nd$. Multiplying both sides of (ii) on the left by y. We get:

$ydI(a)d = yd$. This proves that $xbI(a)d = yd$.

Since $bI(a)d = 0$, we obtain $yd = 0$ showing that $Nb \cap Nd = \{0\}$.

Theorem 2.2. Let $a, b \in R(N)$. Then $I(a) = I(b)$ if and only if $a = b$.

Proof. From the construction, $N = Z_L(N) \cup N_* \cup \{0\}$. Now, assume that $I(a) = I(b)$,

we can write $a = b+d$ with $bN \cap dN = 0$ and $Nd \cap Nd = 0$. But $(b+d)N = bN \oplus dN$.

Since $I(a) = I(b)$, we have that $aI(b)a = \{a\}$ and $bI(a)b = \{b\}$ and therefore it follows

that $Na = Nb$ and $aN = bN$ which leads to $aN = (b+d)N = bN \oplus dN$ giving $d = 0$.

Hence $a = b$ as desired.

Next, we provide the analogue to the previous theorem by generalizing the case to reflexive inverses:

Theorem 2.3. Let $a, b \in R(N)$. Then $Ref(a) = Ref(b)$ iff $a = b$

Proof. Let $a_o \in Ref(a) = Ref(b)$. Since $a = 0$ if and only if $Ref(a) = 0$, assume that

$a, b \neq 0$. Since $bRef(a)b = bRe(b)b = b$ and $Ref(a) = I(a)aI(a)$, we have that for any

$t \in N$. $b(a_o+t-a_oataa_o)a(a_o+t-a_oataa_o)b = b$. Replacing t by $(1-a_oa)t$ and noting

that $a(1 - a_oa) = 0$, we obtain successively

$b(a_oa + (1 - a_oa)ta)(a_o + (1 - a_oa)t)b = b$ and $b(a_ob + (1 - a_oa)ta)(a_o)b = b$ and so

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$ba_ob + b(1 - a_oa)taa_o)b = b$.

Since $ba_ob = b$ gives $b(1 - a_oa)taa_ob = 0 \ \forall \ t \in N$, this leads to $aa_ob(1 - a_oa)taa_ob(1 - a_oa) = 0 \ \forall \ t \in N$.

But we are guaranteed of semi-primeness od N which then implies that $aa_ob(1- a_oa) = 0$.

Left multplying by $a_o \in Ref(a)$, we get that

$a_ob(1 - a_oa) = 0$ and hence since $a_o \in I(b)$, we conclude that $b(1 - a_oa) = 0$.

Therefore we obtain that $Nb \subseteq Na$ and $Na \subseteq Nb$ which implies that $Na = Nb$.

3 Structures and Orders of Von-Neumann Regular Elements

Definition 3.1. Let $(N, +)$ be a group. The exponent of the group is the least common multiple of all the orders of the group elements.

Remark 3.1. Let N be a finite near-ring with identity 1 and n be the exponent of $(N, +)$. Then $ord(1) = n$.

Let Z_n be the ring of integers modulo n. Then $|Z_n^*| = \varphi(n)$, φ- being the Euler-Phi function. We now give a generalization of this result to an arbitrary case:

Proposition 3.1. Let N be the near-ring from classes of near-rings in construction I and II and N^* be as obtained in the constructions. Let n be the exponent of $(N, +)$ and φ be the Euler's-Phi function. Then there is a subgroup of order $\varphi(n)$ contained in N^* .

Proof. We use the fact that the identity $(1, 0, 0, \ldots, 0) \in N$ generates a subring of N .

Assume the usual (+) and the multiplication (.) defined on N . Consider the cyclic group $< 1, 0, 0, \ldots, 0 >$, additively generated by 1 where $1 \equiv (1, 0, 0, \ldots, 0)$.

Then $l.1 = \underbrace{1 + 1 + \ldots + 1}_{l}$ and $k.1 = \underbrace{1 + 1 + \ldots + 1}_{k}$ are two elements of $< 1 >$. Since 1 is an identity:

$(l.1)(k.1) = (lk.1) \in < 1 >$. Thus $S = (< 1 >, +, .)$ is a sub-near ring containing the identity. Indeed $f : S \longrightarrow Z_n : f(k.1) = [k]_n$ is a near-ring isomorphism. Thus $\cong Z_n$.

Let S_* be the group of units of S. It follows from the canonical isomorphism above that

S_* has $\varphi(n)$ invertible elements. Since S and N have the same identity elements, an

element $y \in S : y^{-1} \in S$ implies that $y^{-1} \in N$

$\therefore S_* \subseteq N_*$ and S_* is a subgroup of order $\varphi(n)$.

In the sequel, we recall some notions in Number Theory: Let $N = Z_{p^k}$. For each natural number n, we have the following functions:

$\varphi(n) = \{\sharp x : 1 \leq x \leq n \gcd(x, n) = 1\}$, $w(n)$ = number of distinct primes dividing

n, $\tau(n)$ = number of the divisors of n and $\sigma(n)$ = sum of the divisors of n.

For example if $p = 2$ and $k = 2 \Rightarrow n = 4$, then: $\varphi(4) = 2$, $w(4) = 1$, $\tau(4) = 3$ and

$\sigma(4) = 1 + 2 + 4 = 7$

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Theorem 3.1. ([4], Theorem 2) Let p be a prime integer and $k \in Z_+$ then $a \in GN(p^k, p^k)$ is regular if $a^{p^k - p^{k-1} + 1} \equiv a (\mathrm{mod}\ p^k)$

The element $a^{p^k - p^{k-1} + 1}$ is a Von Neumann inverse of a

Example 3.1. Let $N = Z_4[x] \big/ < x + 1 >$. Then $N = \{0, 1, 2, 3\}$. By definition, an element a is a memeber of $R(N)$ if and only if $a^{p^k - p^{k-1} + 1} \equiv a (\mathrm{mod}\ p^k)$. Thus, if $a = \overline{3}$,

then, $\overline{3}^{2^2 - 2^{2-1} + 1} \equiv \overline{3} (\mathrm{mod}\ 4)$ which implies that $(\overline{3})^3 \equiv \overline{3} (\mathrm{mod}\ 4)$

Therefore, $\overline{3}$ is a regular element and $(\overline{3})^3$ is a Von-Neumann inverse. So, the Von-Neumann inverses of $\overline{1}, \overline{3}$ are $\overline{1}, \overline{3}$ respectively

Theorem 3.2. Let $N = GN(p^k, p^k)$. Then,

$V(p^k) = p^k - p^{k-1} + 1 = \varphi(p^k) + 1.$

Proof. Since $N = GN(p^k, p^k)$ is zero-symmetric local, every element $a \in R(N)$ is either

0 or a unit.

But $| N_* : p^{k-1} + 1$ and the zero element is unique, it follows from the arithmetic function formula that:

$V(p^k) = p^k - p^{k-1} + 1 = \varphi(p^k) + 1.$

Definition 3.2. Let $x, y \in Z_+$. We say that x is a unitary divisor of y if $x \mid y$ and

$\gcd(x, \frac{y}{x}) = 1$ and we write $x /\!/ y$.

The number of regular elements in N can then be calculated using the unitary divisors of an integer $n = |N|$

Proposition 3.2. Let $N = GN(p_k, p_k)$. Then $V(N) = \sum_{x /\!/ p_k} \varphi(x)$ and $V(N)/\varphi(p_k) = \sum_{x /\!/ p_k} \frac{1}{\varphi(x)}$

Proof. In N above $x = 1$ and $x = p_k \equiv 0 (\mathrm{mod}\, p_k)$.

By definition, $\varphi(1) = 1$. But $\varphi(p_k) = p_k - p_{k-1}$ and

$$V(p_k) = p_k - p_{k-1} + 1 = \varphi(p_k) + \varphi(1)$$

Moreover,

$$\frac{V(p_k)}{\varphi(p_k)} = \frac{p_k - p_{k-1} + 1}{p_k - p_{k-1}} = 1 + \frac{1}{p_k - p_{k-1}} = \frac{1}{\varphi(1)} + \frac{1}{\varphi(p_k)}$$

The summatory function:

$$K(p_k) = \sum_{x /\!/ (p_k)} V(x)$$

$$= \sum_{i=0}^{k} V(p^i) = V(1) + \sum_{i=1}^{k} V(p^i) = V(1) + \sum_{i=1}^{k} [(p^i - p^{i-1}) + 1]$$

$$= 1 + (p + p^2 + \ldots + p^k) - (1 + p + p^2 + \ldots + p^{k-1}) + k$$

$$K(p^k) = p^k + k$$

Example 3.2. Consider $N = GR(2^2, 2^2)$, then

$$V(2^2) = \sum_{t||} \varphi(t) = \varphi(1) + \varphi(4) = 1 + 2 = 3.$$

Thus the number of regular elements are 3.

Theorem 3.3. Let $N = GR(p^k, p^k)$ and $\sigma(p^k)$ be the sums of the divisors of p^k. Then

$$\sigma(p^k) = \sum_{i=0}^{k} p^i$$ and

$$V(p^k)\sigma(p^k) = [p^k - p^{k-1}][\sum_{i=0}^{k} p^i]$$

Proof. Clearly,

$$V(p^k)\sigma(p^k) = [p^k - p^{k-1}][\sum_{i=0}^{k}$$

$p_i]$

$= p_k(1 -$

1

p

$+$

1

$p_k)($

$\sum_{i=1}^{k}$

$p_i)$

$= p_k(1 -$

1

p

$+$

1

$p_k)(1 + p + p_2 + ... + p_k)$

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$= p_k[1 + p + p_2 + ... + p_k -$

1

p

$- 1 - p - ...p_{k-1} +$

1

$p_k +$

1

$p_{k-1} +$

1

$p_2 +$

1

p

$+ 1]$

$= p_k[1 + p_k + p_{-2} + p_{-3} + ... + p_{2-k} + p_{1-k} + p_k]$

$= p_k[1 + p_k +$

$\sum_{i=2}^{k}$

$p_{-i}]$

$= p^{2k}[1 + p^{-k} + \sum_{i=2}^{k} p^{-(k+i)}]$

which implies that

$$\frac{V(p^k)\sigma(p^k)}{p^{2k}} = 1 + p^{-k} + \sum_{i=2}^{k} p^{-(k+i)}$$

as required

Theorem 3.4. Let N = $GR(p^k, p^k)$. Then $\sigma(p^k) + \varphi(p^k) \leq p^k\tau(p^k)$

Proof. Let $k = 1$. Then $\sigma(p^k) = p + 1$ and $\varphi(p) = p - 1$ so that

$\sigma(p) + \varphi(p) = 2p$. Since p has only two divisors 1 and p , this implies that

$2p = p(p\tau)$. Thus $\sigma(p) + \varphi(p) = 2p$. Now suppose that $k > 1$, then,

$$\sigma(p^k) = \sum_{i=1}^{k} p^i$$

and $\varphi(p^k) = p^k - p^{k-1}$ so that

$\sigma(p^k) + \varphi(p^k) = 1 + p + ... + p^k + p^k + p^{k-1}$

$= 2p^k + p^{k-2} + ... + p + 1 < (k + 1)p^k$

But p^k has $(k + 1)$ divisors so that $(k + 1)p^k = p^k\tau(p^k)$

thus $\sigma(p^k) + \varphi(p^k) < p^k\tau(p^k)$

Example 3.3. Let N = $Z_4[x]/ < x + 1 >= GR(2^2, 2^2)$

$\sigma(2^2) + \varphi(2^2) \leq 2^2\tau(2^2)$

$\Rightarrow \sigma(4) + \varphi(4) \leq 4\tau 4$

$\Rightarrow 7 + 2 \leq 4 \times 3.$

Thus the result of $\sigma(p^k) + \varphi(p^k) < p^k\tau(p^k)$ holds.

Proposition 3.3. Consider N = $GR(p^{kr}, p^k)$ where $kr = n > 1$. Then $\sigma(p^n) + V(p^n) < p^n\tau(p^n)$

Proof. $1 + \frac{1}{p} + \frac{1}{...}$

$p_2 + \ldots + p_n < n = (n+1) - 1 = \tau(p_n) - 1$ Now

$\sigma(p_n)$

$p_n =$

$1 + p + p_2 + \ldots + p_n$

$p_n < \tau(p_n) - 1$

$\Rightarrow \sigma(p_n) < \sigma p_n[\tau(p_n) - 1]$

$= p_n\tau(p_n) - p_n$

Since $V(p_n) < p_n$, we clear that $\sigma(p_n) + V(p_n) < p_n\tau(p_n)$. However, if $n = 1$, then

$\sigma(p) + V(p) > p\tau(p)$. Let

$N = Z_2[x]/ < x_2 + x + 1 >$: $p = 2, r = 2, k = 1, n = kr > 1$

$= \{0, 1, x, x + 1\}$

We notice that,

$\sigma(p) = \sigma(2) = 1 + 2 = 3$

$V(p) = V(2) = 2$

$\tau(p) = \tau(2) = 2$

$\Rightarrow \sigma(p) + V(p) > p\tau(p)$ i.e. $5 > 4$.

But, if $N = Z_2[x]/ < x_2 + x + 1 > \cong$

$GR(p_{kr}, p_k), k = 2, r = 2, p = 2$,

$\sigma(p_k) = \sigma(4) = 2$, $V(4) = 4$, $p_k\tau(p_k) = 4\tau(4) = 4 \times 3 = 12$

Therefore $\sigma(p_k) + V(p_k) < p_k\tau(p_k) (6 < 12)$ which justifies the previous result.

Lemma 3.1. Let $N = GN(p_{kr}, p_k) \oplus M$ where p is prime k and r are positive integers and M is a h-dimensional module over N. Then if $h = 0$,

(i) $R(N) \cong$

$(1 + Z(N)) \cup \{0\}$ and

(ii) $| R(N) | = (p_{(k-1)r})(p_r - 1) + 1$

Proof. Let $a \in R(N) \cong (1 + Z(N))$. Then a is invertible or 0. But N is local means

that a is regular i.e. $a \in R(N)$.

Thus $R(N) \subseteq [< a > \times 1 + Z(N))] \cup$

$\{0\}$...(i)

Conversely, let $a \in R(N)$. Then by definition $\exists$ an element $b \in R(N)$ such that

$a = a^2b \Rightarrow a(1 - ab) = 0$.

If $a \in (N^*)$ then $1 - ab = 0 \Rightarrow ab = 1$.

Hence b is a Von Neumann inverse of a. If is not a member of N^* then ab is not a member of N^* but $ab = aabb = a^2b^2 = abab = (ab)^2$.

Since N commutes $\Rightarrow ab = (ab)^2 \Rightarrow ab(1 - ab) = 0$.

Now $\Rightarrow 1-ab$ is a unit and $ab = 0$ so that $a = 0$ because b is its Von Neumann inverse.

$[\{< a > \times 1 + Z(N)\} \cup \{0\}] \subseteq R(N)$ ....(ii)

Combining (i) and (ii) gives

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$R(N) \cong [1 + Z(N)] \cup \{0\}$

$= < a > \times [1 + Z(N)] \cup \{0\}$

Next,

$N^* = (N^*/1 + Z(N)) \times 1 + Z(N)$

$\cong < a > \times [1 + Z(N)]$

$= Z_{p^r-1} \times [1 + Z(N)]$

But

$| [1 + Z(N)] | = | Z(N) |$

$= p^{(k-1)r}$

Therefore $| N^* | = (p^r - 1)(p^{(k-1)r})$

But $R(N) = N^* \cup \{0\}$ $| R(N) | = (p^r - 1)(p^{(k-1)r}) + 1$ as required.

Theorem 3.5. Let N be the near-ring constructed and $R(N)$ be the set of all the regular elements. Then

$R(N) =$

$\overline{Z_{2^r-1} \times Z_2 \times Z_{2^{k-2}} \times Z_{r-1}}$

$_{2^{k-1}} \times (Z_2)^h \cup \{0\}$ $p = 2$;

$Z_{p^r-1} \times Z^r_{p^{k-1}} \times (Z^r_p)^h \cup \{0\}$ $p \neq 2$: $CharN = p^k : k \geq 3$.

Proof. Let char $N = p^k : k \geq 3$. We provide the general case using $p =$ odd. Notice that every $l = 1, ..., r;$ $(1 + p\tau_1)^{p^{k-1}} = 1$

$(1 + \tau_l u_1)^{p^k} = 1, ..., (1 + p\tau_L u_1 + \tau_l u_2 + ... + \tau_l u_n)^{p^k} = 1$.

Let $a_l, b_{1l}, ..., b_{hl} \in Z_+$ with $a_l \leq p^{k-1}$, $b_{il} \leq p^k : 1 \leq i \leq h$. We notice that

$$\prod_{l=1}^{r} \{(1 + p\tau_L)^{a_L}\} \cdot \prod_{l=1}^{r} \{(1 + \tau_l u_1)^{b_{1l}}\} \cdot \prod_{l=1}^{r} \{(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)\} = 1$$

which implies that $a_l = p^{k-1}$, $b_{1l} = p^k = \cdots = b_{hl} = p^k$. Set

$T_l = < \{(1 + p\tau_l)^a \mid a = 1, ..., p^{k-1}\} >$

$S_{1l} = < \{(1 + \tau_l u_1)^{b_1} \mid b_1 = 1, \cdots , p^k\} >$

...

$S_{hl} = < \{(1 + \tau_l u_1 + \cdots + \tau_l u_n)^{b_h} \mid b_h = 1, \cdots , p^k\} >$

The sets defined are all cyclic subgroups of the group 1+Z(N) and they are of the indicated orders. Furthermore, the intersection of any pair of the cyclic subgroups indicated gives an identity group and the product of the $(h + 1)r$ subgroups gives:

$| T_l \times S_{1L} \times S_{hl} |= p^{k((h+1)r)-1}$ exhausting $1 + Z(N)$.

Thus $1 + Z(N) \cong Z^r_{p^{k-1}} \times (Z^r_p)^h$.

Therefore

$R(N) =< \alpha > \ltimes (1 + (Z(N))) \cup \{0\}$

$= Z_{p^r-1} \times Z^r$

$p^{k-1} \times (Z_{p^r})^h \cup \{0\}$.

Theorem 3.6. Let $N = R_o \oplus M$ where $r = 1$ and p -prime, $k \in Z_+$. If $M = R_0/pR_0 \oplus \dots \oplus R_0/pR_0$. Let $r_0 \in R(R_0)$ then, its Von-Neumann inverse is

$$r_0^{-1} = r_0^{p^k-p^{k-1}-1}$$

and $(r_0, \dots, r_h)^{-1} = (r_0^{p^k-p^{k-1}-1}, -r_1 t_0 r_0^{-1}, \dots, -r_h t_0 r_0^{-1})$

Proof. We know that if $a \in R_0 = GN(p^{kr}, p^k)$ and $a \in R_0$ then, the Von-Neumann inverse of a is given by: $a^{-1} \equiv a^{p^{(k-1)r}(p^r-1}(\mod p^k)$ therefore

$$r_0^{-1} \equiv r_0^{p^k-p^{k-1}-1}$$

as required in step 1

Now let $(t_0, \dots, t_h) = (r_0, \dots, r_h)^{-1}$, then

$(r_0, r_1, \dots, r_h) = (r_0, \dots, r_h)^2(t_0, \dots, t_h)$

$= (r_0^2, r_0r_1 + r_1r_0, \dots, r_0r_h + r_hr_0)(t_0, \dots, t_h)$

$= (r_0^2t_0, r_0^2t_1 + (r_0r_1 + r_1r_0)t_0, \dots, r_0^2t_h + (r_0r_h + r_hr_0)t_0)$

therefore $r_0 = r_0^2t_0 \Rightarrow r_0t_0 = 1 \Rightarrow t_0 = r_0^{-1} = r_0^{p^k-p^{k-1}-1}$

For $i = 1, \dots, h$, $r_i = r_0^2t_i + (r_0r_i + r_ir_0)t_0$

$\Rightarrow r_0^2t_i = r_i - (r_0r_i + r_ir_0)t_0$

$\Rightarrow t_i =$

$r_i -\ 2r_0r_it_0$

r_2

0

$(\therefore N commutative)$

$\Rightarrow\ t_i =$

r_i

r_2

0

$-$

$2r_it_0$

r_0

But $t_0 = r_{-1}$

0

$\Rightarrow\ t_i =$

r_i

r_2

0

$-$

$2r_i$

r_2

0

$= -$

r_i

r_2

0

$= -r_ir_{-2}$

0

$\therefore t_1 = -r_1r_{-2}$

0 $\ldots t_h = -r_hr_{-2}$

0

$\Rightarrow\ (r_0, \ldots, r_h)_{-1} = (r_{pk-pk-1-1}$

0 $, \ldots, -r_hr_{-2}$

0) as required

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Example 3.4. $N = Z_9 \oplus Z_9/3Z_9 \oplus \ldots \oplus Z_9/3Z_9$

Then

$(2, 2, ..., 2)^{-1} = (2^{9-3-1}, (-2)(5)^2, ..., (-2)(5)^2)$

$= (5, 1, 1, ..., 1)$

$(5, 1, 1, ..., 1)(2, 2, ..., 2) = (1, 0, ..., 0)$

Example 3.5. Consider $N = GN(p^{kr}, p^k) \cong Z_2[x] / < x^2 + x + 1 >$ where $p = 2, k = 1, r = 2$.

Now $GN = \{0, 1, x, x + 1\}$ and $R(N) = \{0, 1, x, x + 1\}$.

Let $N = GN(4, 2) \oplus GN(4, 2)$ with $GN(4, 2)$ as defined above, then:

$N = \{0, 1, x, x + 1\} \oplus \{0, 1, x, x + 1\}$

$= \{(0, 0), (0, 1), (0, x), (0, x + 1), (1, 0), (1, 1), (1, x), (1, x + 1), (x, 0), (x, 1), (x, x),$

$(x, x + 1), (x + 1, 0), (x + 1, 1), (x + 1, x), (x + 1, x + 1)\}$

So $| N |= 16$, $Z_L(N) = \{(0, 0), (0, 1), (0, x), (0, x + 1)\}$. Since N is an extension of $GN(4, 2)$,

$| R(N) |= 13 = (p^r - 1)(p^{kr}) + 1$

Applying $(r_0, r_1)^{-1} = (r_0^{p^k - p^{k-1} - 1}, -r_1 r_0^{-2})$, we can find the Von Neumann inverses of all the members of $R(N)$.

For instance,

$R(N) = \{(1, 0), (1, 1), (1, x), (1, x + 1), (x, 0), (x, 1), (x, x), (x, x + 1),$

$(x + 1, 0), (x + 1, 1), (x + 1, x), (x + 1, x + 1)\}$.

So $(1, 0)^{-1} = (1^{2^1 - 2^0 - 1}, -0 1^{-1}) = (1^2, 0) = (1, 0)$, $(x, x)^{-1} = (x^{-2}, x^{-1})$

This can be done in the same manner for the other members of $R(N)$. The next result gives the structures and orders of the automorphism groups of the regular elements, $R(N)$.

Theorem 3.7. Let N be a near-ring of construction $R(N)$ be the set of all the regular elements including 0. Then if

$Aut : R(N) \to R(N)$ we have that

$Aut(R(N)) \cong$

$[(Z_{p^r-1})^* \times GL_{(k-1)r}(GN(p^{kr}, p^k))] \times GL_{hr}(GN(p^{kr}, p^k))] \cup \triangle$

Theorem 3.8. Let N be a zero symmetric local near-rings from the class of near-rings
of the construction. Then:

$$| Aut(R(N)) | = [\varphi(p^r - 1) \prod_{k=1}^{(k-1)r} (p^k - p^{k-1}) \prod_{k=1}^{hr} (p^k - p^{k-1})] + 1$$

when $charN = p^k : k \geq 3$

4 Conclusion

This study was set up with an aim of determining and classifying the regular elements and Von-Neumann inverses of the zero symmetric local near-rings with n-nilpotent radical of Jordan ideals admitting Frobenius derivations. The study gave a general construction representing the classes of the near-rings under investigations whose algebraic structures assumed commutation checks attributed the Theorems of Asma and Inzamam in [8] .
The structures and orders of $R(N)$ were then characterized in a case by case basis using the Fundamental Theorem of Finitely Generated Abelian Groups and the properties of the general linear groups in the endomorphism of $R(N)$ respectively. The structures of $V(| R(N) |)$ followed asymptotic patterns proposed by Osama and Emad [4] using the properties of $V(n)$, $\tau(n)$, $\omega(n)$, $\sigma(n)$ and $K(n)$. The results reveal unique algebraic structures.

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