Original Research Article

Regular Elements and Von-Neumann Inverses of a class of zero symmetric Local near-rings admitting Frobenius Derivations

Abstract

Let $\mathcal{N}$ be a zero-symmetric local near-ring. An element $x \in \mathcal{N}$ is either regular, zero or a zero divisor. In this paper, we construct a class of zero symmetric local near-ring of characteristic p^k; $k \geq 3$ admitting an identity frobenius derivation, characterize the structures and orders of the set $R(\mathcal{N})$, the regular compartment with an aim of advancing the classification problem of algebraic structures. The number theoretic notions relating the number of regular elements to Euler's phi-function and the arithmetic functions of Galois near-rings are adopted. Using the Fundamental Theorem of finitely generated Abelian groups, the structures of $R(\mathcal{N})$ are proved to be isomorphic to cyclic groups of various orders. The study also extends to the automorphism groups $Aut(R(\mathcal{N}))$ of the regular elements.

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1 Introduction

The study of near-rings with identity is very vital in generalizing characterization of commutative rings with identity. Much of the recent works on the classification of finite rings with identity have considered a characterization paradigm using the unit groups, the zero divisor graphs, adjacency and incidence matrices among others. This has left the non-linear aspects fairly untouched. In particular, regular elements and Von-Neumann inverses of near rings admitting derivations hardly exist in the available literature.
Oduor, Ojiema and Mmasi[1] determined construction of idealized local rings of characteristic $p^n : \ n = 1, 2, 3$ and determined the structures of the unit groups R^*. Osba, Henriksen and Osama [2] conducted a classification survey on combining local and Von Neumann Regular Rings as a basis upon which the regularity properties of rings and their ideals could be explored. The rings studied in [2] were finite and their Von Neumann inverses gave some asymptotic patterns. Their findings demonstrated how to combine the

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Von- Neumann inverses of classes of rings such as the power series rings and the ring of integers. They however did not count the number of regular elements in a given finite ring nor did they give the structural formulae for the regular elements and the Von Neumann inverses of the specified classes of rings. In a closely related research, the study on regular elements of Galois rings can be attributed to Osama and Emad [3] where they characterized the regular elements in the ring of integers modulo n, $\mathbb{Z}_n$. Furthermore, they studied the arithmetic functions denoted as $V(n)$ and determined the relationship between $V(n)$ and the Euler's phi function, $\varphi(n)$. This gave an extension of the ring theoretic algebra employed in counting the regular elements of $\mathbb{Z}_n$ to the number theoretic methodologies. For instance, the research revealed that if a is a regular element in $\mathbb{Z}_n$, then $a^{(-1)} \equiv a^{\varphi(n)-1} \ (mod \ \ n)$. They proposed a criterion for getting the possible Von Neumann inverses in the set of regular elements of $\mathbb{Z}_n$ and explored the asymptotic properties of $V(n)$. Their findings did not consider extensions and idealization using maximal submodules of $\mathbb{Z}_n \forall n \in \mathbb{Z}$. Closely related works can also be seen in Osba et al [4] and Oduor, Omamo and Musoga[5] In order to advance the concept of classification of algebraic structures, the paper considers generalized rings, the near rings.

2 Zero-Symmetric Local Near-Ring of Characteristic $p^k: \ k \geq 3$

Let $R_o = G\mathcal{N}(p^{kr}, p^k)$. Let $i = 1, ..., h$ and $u_i \in Z_L(\mathcal{N})$ and $\mathcal{M} = < u_i >$.
Then,

$$\mathcal{N} = R_o \oplus \mathcal{M} = R_o \oplus \sum_{i=1}^{h} (R_o/pR_o)^i$$

is a group with respect to addition.
On $\mathcal{N}$, let

$$(r_o, \overline{r}_1, ..., \overline{r}_h)(s_o, \overline{s}_1, ..., \overline{s}_h) = (r_o s_o, r_o \overline{s}_1 + \overline{r}_1 s_o, ..., r_o \overline{s}_h + \overline{r}_h s_o)^{\delta}$$

where δ is the identity Frobenius automorphism. The multiplication turns $\mathcal{N}$ into a local zero symmetric near-ring with identity $(1, \overline{0}, ..., \overline{0})$.
Indeed $\mathcal{N} = R_o \oplus \mathcal{M}$ is commutative since δ is the identity Frobenius automorphism.

Proposition 2.1. *Consider* $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k)$ *where* $k \geq 3$. *Then,* $char\mathcal{N} = p^k$ *and:*

(i). $Z_L(\mathcal{N}) = pR_o \oplus \sum_{i=1}^{h} (R_o/pR_o)^i$

(ii). $(Z_L(\mathcal{N}))^{k-1} = p^{k-1}R_o \neq (0)$

(iii). $(Z_L(\mathcal{N}))^k = (0)$.

Proof. Char $G\mathcal{N}(p^{kr}, p^k) = char\mathcal{N}$ and $id_{\mathcal{N}} = id_{G\mathcal{N}(p^{kr},p^k)}$
Let $a \in R_o$ and a not contained in pR_0 and let $s \in Z_L(\mathcal{N})$.
Then

$$\begin{aligned} (a+s)^{pr} &= a^{pr} + s' : (s' \in Z_L(\mathcal{N})) \\ &= (a+s'')^{p^r-1} : (s'' \in Z_L(\mathcal{N})) \end{aligned}$$

But $(a+s'')^{p^r-1} \equiv 1 + s'''$ with $s''' \in Z_L(\mathcal{N})$ and $(1+s''')^{p^k-1} = 1$. Hence $(a+s)$ is regular and not zero.
Since $\mid Z_L(\mathcal{N}) \mid = p^{(h+k-1)r}$ and
$\mid (R_o/pR_o)^* + Z_L(\mathcal{N}) \mid = (p^r - 1)(p^{(h+k-1)r})$, it follows that
$(R_o/pR_o)^* + Z_L(\mathcal{N}) = \mathcal{N} - Z_L(\mathcal{N})$ and hence all the elements outside $Z_L(\mathcal{N}) \setminus \{0\}$ are regular. □

Remark 2.1. *A regular element $x \in R(\mathcal{N})$ may have more than one Von-Neumann inverse. However, for the classes of near-rings considered in this study, the Von-Neumann inverses are unique.*

Proposition 2.2. *Let $\mathcal{N}$ be a class of near-ring of the construction. For $x \in \mathcal{N}$ and $x_0 \in I(x)$, we have that:*

$$I(x) = \{x_0 + \alpha - x_0 x \alpha x x_0 \mid \alpha \in \mathcal{N}\}$$

Proof. From the construction, if $x \in \mathcal{N}$, then

$$x = (r_0 + (\sum_{i=1}^{h} r_0 + pr^{'})r^{'} \in G\mathcal{N}(p^{kr}, p^k)/pG\mathcal{N}(p^{kr}, p^k)).$$

So the definition of the multiplication in $\mathcal{N}$ gives the desired result. □

Denote by $l(x)$ and $r(x)$ the left and the right annihilator of an element $x \in \mathcal{N}$. So the inner annihilator of $x \in \mathcal{N}$ is: $Iann(x) = \{y \in N : xyx = 0\}$.

Theorem 2.1. *Let $\mathcal{N}$ be the near ring of the construction. If $a \in R(\mathcal{N})$, then for any $b \in \mathcal{N}$, $bI(a)b$ is a singleton set if and only if $b \in \mathcal{N}a \cap a\mathcal{N}$.*

Proof. Suppose there exists $x, y \in \mathcal{N}$ such that $b = xa = ay$ and let $a_o \in I(a)$. We then have that for any $t \in \mathcal{N}$,

$$\begin{aligned} b(a_o + t - a_o a t a a_o)b &= (xaa_o + xat - xataa_o)ay \\ &= xay + xatay - xatay \\ &= xay \end{aligned}$$

Thus the set $bI(a)b = \{xay\}$ is singleton.
Conversely, suppose that $bI(a)b = \{ba_ob\}$.
We then have: $b(a_o + t - a_oataa_o)b = ba_ob$ for any $t \in \mathcal{N}$. This implies that for any $t \in \mathcal{N}$, we have: $b(t - a_oataa_o)b = 0$.............(i). Substituting $(1 - a_oa)t$ for t in this equality yields $b(1 - a_oataa_o)tb = 0$ for any $t \in \mathcal{N}$. But $\mathcal{N}$ constructed is semiprime so that $b(1 - a_oa) = 0 \Rightarrow b = ba_oa \in \mathcal{N}a$ .....(ii)
Similarly, substituting t by $t(1 - aa_o)$ in the equality (i)
gives $b = aa_ob \in a\mathcal{N}$.........(iii)
Comparing (ii) and (iii), we conclude that $b \in \mathcal{N}a \cap a\mathcal{N}$ □

Lemma 2.1. *Let $\mathcal{N}$ be the near ring constructed and let $b, d \in \mathcal{N}$ such that $b + d$ is a Von Neumann regular element. Then the following are equivalent:*

- *(i)* $b\mathcal{N} \oplus d\mathcal{N} = (b + d)\mathcal{N}$
- *(ii)* $\mathcal{N}b \oplus \mathcal{N}d = \mathcal{N}(b + d)$
- *(iii)* $b\mathcal{N}b \cap d\mathcal{N} = \{0\}$ *and* $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.

The next result shows when $I(a) \subseteq I(b)$ necessarily and sufficiently where $a, b \in \mathcal{N}$

Proposition 2.3. *Let $a, b \in R(\mathcal{N})$. Then $I(a) \subseteq I(b)$ if and only if $b\mathcal{N} \cap d\mathcal{N} = \{0\}$ and $\mathcal{N}b \cap \mathcal{N}d = \{0\}$ where $a = d + b$*

Proof. Let $I(a) \subseteq I(b)$. Then by definition, there exists some $x \in I(a)$ such that $bxb = b$.
Now $b \in \mathcal{N}a \cap a\mathcal{N}$.
Write $b = \alpha a = a\beta$ where $\alpha, \beta \in \mathcal{N}$.
Then $bI(a)a = b$.
Next

$$\begin{aligned} bI(a)d &= bI(a)a - bI(a)b \\ &= b - bI(a)b = 0 \end{aligned}$$

Consider now

$$\begin{aligned} dI(a)b &= aI(a)b - bI(a)b \\ &= \alpha\beta - bI(a)b \\ &= b - b = 0 \end{aligned}$$

We thus have $bI(a)d = 0$ and $dI(a)b = 0$...(i)
Then for any $x \in I(a)$ we have;

$$\begin{aligned} b + d = a &= axa \\ &= (b + d)x(b + d) \\ &= bxa + dxb + dxd \\ &= b + 0 + dxd \end{aligned}$$

This yields $dI(a)d = d$..(ii)
To show that $d\mathcal{N} \cap b\mathcal{N} = \{0\}$.
Let $bx = dy \in b\mathcal{N} \cap d\mathcal{N}$.
Multiplying both sides of (ii) by y on the right and using $bx = dy$ yields, $dI(a)bx = dy$
But from above we have that $dI(a)b = 0$ and so $dy = 0$ which clears the proof.
Similarly, we show that $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.
Let $xb = yd \in \mathcal{N}b \cap \mathcal{N}d$. Multiplying both sides of (ii) on the left by y. We get: $ydI(a)d = yd$. This proves that $xbI(a)d = yd$.
Since $bI(a)d = 0$, we obtain $yd = 0$ showing that $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.

□

Theorem 2.2. *Let $a, b \in R(\mathcal{N})$. Then $I(a) = I(b)$ if and only if $a = b$.*

Proof. From the construction, $\mathcal{N} = Z_L(\mathcal{N}) \cup \mathcal{N}^* \cup \{0\}$. Now, assume that $I(a) = I(b)$, we can write $a = b + d$ with $b\mathcal{N} \cap d\mathcal{N} = 0$ and $\mathcal{N}d \cap \mathcal{N}d = 0$. But $(b+d)\mathcal{N} = b\mathcal{N} \oplus d\mathcal{N}$. Since $I(a) = I(b)$, we have that $aI(b)a = \{a\}$ and $bI(a)b = \{b\}$ and therefore it follows that $\mathcal{N}a = \mathcal{N}b$ and $a\mathcal{N} = b\mathcal{N}$ which leads to $a\mathcal{N} = (b+d)\mathcal{N} = b\mathcal{N} \oplus d\mathcal{N}$ giving $d = 0$. Hence $a = b$ as desired.

□

Next, we provide the analogue to the previous theorem by generalizing the case to reflexive inverses:

Theorem 2.3. *Let $a, b \in R(\mathcal{N})$. Then $Ref(a) = Ref(b)$ iff $a = b$*

Proof. Let $a_o \in Ref(a) = Ref(b)$. Since $a = 0$ if and only if $Ref(a) = 0$, assume that $a, b \neq 0$.Since $bRef(a)b = bRe(b)b = b$ and $Ref(a) = I(a)aI(a)$, we have that for any $t \in N$. $b(a_o + t - a_o ataa_o)a(a_o + t - a_o ataa_o)b = b$. Replacing t by $(1 - a_o a)t$ and noting that $a(1 - a_o a) = 0$,we obtain successively
$b(a_o a + (1 - a_o a)ta)(a_o + (1 - a_o a)t)b = b$ and $b(a_o b + (1 - a_o a)ta)(a_o)b = b$ and so $ba_o b + b(1 - a_o a)taa_o)b = b$.
Since $ba_o b = b$ gives $b(1 - a_o a)taa_o b = 0 \ \forall \ t \in \mathcal{N}$, this leads to $aa_o b(1 - a_o a)taa_o b(1 - a_o a) = 0 \ \forall \ t \in \mathcal{N}$.
But we are guaranteed of semi-primeness od $\mathcal{N}$ which then implies that $aa_o b(1 - a_o a) = 0$.
Left multplying by $a_o \in Ref(a)$, we get that
$a_o b(1 - a_o a) = 0$ and hence since $a_o \in I(b)$, we conclude that $b(1 - a_o a) = 0$.
Therefore we obtain that $\mathcal{N}b \subseteq \mathcal{N}a$ and $\mathcal{N}a \subseteq \mathcal{N}b$ which implies that $\mathcal{N}a = \mathcal{N}b$.

□

3 Structures and Orders of Von-Neumann Regular Elements

Definition 3.1. *Let $(\mathcal{N}, +)$ be a group. The exponent of the group is the least common multiple of all the orders of the group elements.*

Remark 3.1. *Let N be a finite near-ring with identity 1 and n be the exponent of $(\mathcal{N}, +)$. Then $ord(1) = n$.*

Let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $\mid \mathbb{Z}_n^* \mid = \varphi(n)$, φ- being the Euler-Phi function. We now give a generalization of this result to an arbitrary case:

Proposition 3.1. *Let $\mathcal{N}$ be the near-ring from classes of near-rings in construction I and II and $\mathcal{N}^*$ be as obtained in the constructions. Let n be the exponent of $(\mathcal{N}, +)$ and φ be the Euler's-Phi function. Then there is a subgroup of order $\varphi(n)$ contained in $\mathcal{N}^*$.*

Proof. We use the fact that the identity $(1, 0, 0, ..., 0) \in \mathcal{N}$ generates a subring of $\mathcal{N}$. Assume the usual $(+)$ and the multiplication $(.)$ defined on $\mathcal{N}$. Consider the cyclic group $< 1, 0, 0, ..., 0 >$, additively generated by 1 where $1 \equiv (1, 0, 0, ..., 0)$. Then $l.1 = \underbrace{1 + 1 + ... + 1}\, l - summands$ and $k.1 = \underbrace{1 + 1 + ... + 1}\, k - summands$ are two elements of $< 1 >$. Since 1 is an identity: $(l.1)(k.1) = (lk.1) \in < 1 >$. Thus $S = (< 1 >, +, .)$ is a sub-near ring containing the identity. Indeed $f : S \longrightarrow \mathbb{Z}_n : f(k.1) = [k]_n$ is a near-ring isomorphism. Thus $\cong \mathbb{Z}_n$. Let S^* be the group of units of S. It follows from the canonical isomorphism above that S^* has $\varphi(n)$ invertible elements. Since S and N have the same identity elements, an element $y \in S : y^{-1} \in S$ implies that $y^{-1} \in N$
$\therefore S^* \subseteq N^*$ and S^* is a subgroup of order $\varphi(n)$. □

We recall some notions in Number Theory:
Let $\mathcal{N} = \mathbb{Z}_{p^k}$. For each natural number n, we have the following functions
$\varphi(n) = \{\sharp x : 1 \leq x \leq n \gcd(x, n) = 1\}$, $\overline{w}(n) =$ number of distinct primes dividing n, $\tau(n) =$ number of the divisors of n and $\sigma(n) =$ sum of the divisors of n.
For example if $p = 2$ and $k = 2 \Rightarrow n = 4$, then: $\varphi(4) = 2$, $\overline{w}(4) = 1$, $\tau(4) = 3$ and $\sigma(4) = 1 + 2 + 4 = 7$

Theorem 3.1. *([3], Theorem 2) Let p be a prime integer and $k \in \mathbb{Z}^+$ then $a \in G\mathcal{N}(p^k, p^k)$ is regular if $a^{p^k - p^{k-1} + 1} \equiv a(mod\, p^k)$*
The element $a^{p^k - p^{k-1} + 1}$ is a Von Neumann inverse of a

Example 3.1. *Let $\mathcal{N} = \mathbb{Z}_4[x]/ < x + 1 >$. Then $\mathcal{N} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$*
From $a \in R(\mathcal{N})$ if and only if $a^{p^k - p^{k-1} + 1} \equiv a(mod\, p^k)$ gives:
If $a = \overline{3}$, then, $\overline{3}^{2^2 - 2^{2-1} + 1} \equiv \overline{3}(mod\, 4)$ which implies that $(\overline{3})^3 \equiv \overline{3}(mod\, 4)$
Thus $\overline{3}$ is a regular element and $(\overline{3})^3$ is a Von-Neumann inverse. Therefore, Von-Neumann inverses of $\overline{1}$, $\overline{3}$ are $\overline{1}$, $\overline{3}$ respectively

Theorem 3.2. *Let $\mathcal{N} = G\mathcal{N}(p^k, p^k)$. Then,*

$$V(p^k) = p^k - p^{k-1} + 1 = \varphi(p^k) + 1.$$

Proof. Since $\mathcal{N} = G\mathcal{N}(p^k, p^k)$ is zero-symmetric local, every element $a \in R(\mathcal{N})$ is either 0 or a unit.

But $\mid \mathcal{N}^* : p^{k-1}+1$ and the zero element is unique, it follows from the arithmetic function formula that:

$$V(p^k) = p^k - p^{k-1} + 1 = \varphi(p^k) + 1.$$

□

Definition 3.2. *Let* $x, y \in \mathbb{Z}^+$. *We say that* x *is a unitary divisor of* y *if* $x \mid y$ *and* $\gcd(x, \frac{y}{x}) = 1$ *and we write* $x \| y$.

The number of regular elements in $\mathcal{N}$ can then be calculated using the unitary divisors of an integer $n = \mid \mathcal{N} \mid$

Proposition 3.2. *Let* $\mathcal{N} = G\mathcal{N}(p^k, p^k)$. *Then* $V(\mathcal{N}) = \Sigma_{x\|p^k}\varphi(x)$ *and* $V(N)/\varphi(p^k) = \Sigma_{x\|p^k}\frac{1}{\varphi(x)}$

Proof. In $\mathcal{N}$ above $x = 1$ and $x = p^k \equiv 0(mod p^k)$.
By definition, $\varphi(1) = 1$. But $\varphi(p^k) = p^k - p^{k-1}$ and

$$\begin{aligned} V(p^k) &= p^k - p^{k-1} + 1 \\ &= \varphi(p^k) + \varphi(1) \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{V(p^k)}{\varphi(p^k)} &= \frac{p^k - p^{k-1} + 1}{p^k - p^{k-1}} \\ &= 1 + \frac{1}{p^k - p^{k-1}} \\ &= \frac{1}{\varphi(1)} + \frac{1}{\varphi(p^k)} \end{aligned}$$

The summatory function:

$$\begin{aligned} K(p^k) &= \sum_{x\|(p^k)} V(x) \\ &= \sum_{i=0}^{k} V(p^i) \\ &= V(1) + \sum_{i=1}^{k} V(p^i) \\ &= V(1) + \sum_{i=1}^{k} [(p^i - p^{i-1}) + 1] \\ &= 1 + (p + p^2 + ... + p^k) - (1 + p + p^2 + ... + p^{k-1}) + k \end{aligned}$$

$K(p^k) = p^k + k$ □

Example 3.2. *Consider* $\mathcal{N} = GR(2^2, 2^2)$ *, then*

$$\begin{aligned} V(2^2) &= \sum_{t||} \varphi(t) \\ &= \varphi(1) + \varphi(4) \\ &= 1 + 2 = 3. \end{aligned}$$

Thus the number of regular elements are 3.

Theorem 3.3. *Let* $\mathcal{N} = GR(p^k, p^k)$ *and* $\sigma(p^k)$ *be the sums of the divisors of* p^k *. Then*

$$\begin{aligned} \sigma(p^k) &= \sum_{i=0}^{k} p^i and \\ V(p^k)\sigma(p^k) &= [p^k - p^{k-1}][\sum_{i=0}^{k} p^i] \end{aligned}$$

Proof. Clearly,

$$\begin{aligned} V(p^k)\sigma(p^k) &= [p^k - p^{k-1}][\sum_{i=0}^{k} p^i] \\ &= p^k(1 - \frac{1}{p} + \frac{1}{p^k})(\sum_{i=1}^{k} p^i) \\ &= p^k(1 - \frac{1}{p} + \frac{1}{p^k})(1 + p + p^2 + ... + p^k) \\ &= p^k[1 + p + p^2 + ... + p^k - \frac{1}{p} - 1 - p - ...p^{k-1} + \frac{1}{p^k} + \frac{1}{p^{k-1}} + \frac{1}{p^2} + \frac{1}{p} + 1] \\ &= p^k[1 + p^k + p^{-2} + p^{-3} + ... + p^{2-k} + p^{1-k} + p^k] \\ &= p^k[1 + p^k + \sum_{i=2}^{k} p^{-i}] \\ &= p^{2k}[1 + p^{-k} + \sum_{i=2}^{k} p^{-(k+i)}] \end{aligned}$$

which implies that

$$\frac{V(p^k)\sigma(p^k)}{p^{2k}} = 1 + p^{-k} + \sum_{i=2}^{k} p^{-(k+i)}$$

as required □

Theorem 3.4. *Let* $\mathcal{N} = GR(p^k, p^k)$ *. Then* $\sigma(p^k) + \varphi(p^k) \leq p^k \tau(p^k)$

Proof. Let $k=1$. Then $\sigma(p^k)=p+1$ and $\varphi(p)=p-1$ so that $\sigma(p)+\varphi(p)=2p$. Since p has only two divisors 1 and p, this implies that $2p=p(p\tau)$. Thus $\sigma(p)+\varphi(p)=2p$. Now suppose that $k>1$, then,

$$\sigma(p^k) \quad = \quad \sum_{i=1}^{k} p^i$$

and $\varphi(p^k)=p^k-p^{k-1}$ so that

$$\begin{aligned}\sigma(p^k)+\varphi(p^k) &= 1+p+...+p^k+p^k+p^{k-1}\\ &= 2p^k+p^{k-2}+...+p+1<(k+1)p^k\end{aligned}$$

But p^k has $(k+1)$ divisors so that $(k+1)p^k=p^k\tau(p^k)$
thus $\sigma(p^k)+\varphi(p^k)<p^k\tau(p^k)$ ☐

Example 3.3. *Let* $\mathcal{N}=\mathbb{Z}_4[x]/<x+1>=GR(2^2,2^2)$

$$\begin{aligned}\sigma(2^2)+\varphi(2^2) &\leq 2^2\tau(2^2)\\ \Rightarrow \sigma(4)+\varphi(4) &\leq 4\tau 4\\ \Rightarrow 7+2 &\leq 4\times 3.\end{aligned}$$

Thus the result of $\sigma(p^k)+\varphi(p^k)<p^k\tau(p^k)$ holds.

Proposition 3.3. *Consider* $\mathcal{N}=GR(p^{kr},p^k)$ *where* $kr=n>1$. *Then* $\sigma(p^n)+V(p^n)<p^n\tau(p^n)$

Proof. $1+\frac{1}{p}+\frac{1}{p^2}+...+p^n<n=(n+1)-1=\tau(p^n)-1$ Now

$$\begin{aligned}\frac{\sigma(p^n)}{p^n} &= \frac{1+p+p^2+...+p^n}{p^n}<\tau(p^n)-1\\ \Rightarrow \sigma(p^n) &< \sigma p^n[\tau(p^n)-1]\\ &= p^n\tau(p^n)-p^n\end{aligned}$$

Since $V(p^n)<p^n$, we clear that $\sigma(p^n)+V(p^n)<p^n\tau(p^n)$. However, if $n=1$, then $\sigma(p)+V(p)>p\tau(p)$. Let

$$\begin{aligned}\mathcal{N} &= \mathbb{Z}_2[x]/<x^2+x+1>: p=2, r=2, k=1, n=kr>1\\ &= \{\overline{0},\overline{1},\overline{x},\overline{x+1}\}\end{aligned}$$

We notice that,

$$\begin{aligned}\sigma(p) &= \sigma(2)=1+2=3\\ V(p) &= V(2)=2\\ \tau(p) &= \tau(2)=2\end{aligned}$$

$$\Rightarrow \sigma(p)+V(p)>p\tau(p) i.e. 5>4.$$

But, if $\mathcal{N} = \mathbb{Z}_2[x]/ < x^2 + x + 1 > \cong GR(p^{kr}, p^k), k = 2, r = 2, p = 2$,
$\sigma(p^k) = \sigma(4) = 2$, $V(4) = 4$, $p^k\tau(p^k) = 4\tau(4) = 4 \times 3 = 12$
Therefore $\sigma(p^k) + V(p^k) < p^k\tau(p^k)(6 < 12)$ which justifies the previous result. □

Lemma 3.1. *Let* $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k) \oplus \mathcal{M}$ *where* p *is prime* k *and* r *are positive integers and* $\mathcal{M}$ *is a* h*-dimensional module over* $\mathcal{N}$*. Then if* $h = 0$,

(i) $R(\mathcal{N}) \cong (1 + Z(\mathcal{N})) \cup \{0\}$ *and*

(ii) $\mid R(\mathcal{N}) \mid = (p^{(k-1)r})(p^r - 1) + 1$

Proof. Let $a \in R(\mathcal{N}) \cong (1 + Z(\mathcal{N}))$. Then a is invertible or 0. But $\mathcal{N}$ is local means that a is regular i.e. $a \in R(\mathcal{N})$.
Thus $R(\mathcal{N}) \subseteq [< a > \times 1 + Z(\mathcal{N}))] \cup \{0\}$..(i)
Conversely, let $a \in R(\mathcal{N})$. Then by definition $\exists$ an element $b \in R(\mathcal{N})$ such that $a = a^2b \Rightarrow a(1 - ab) = 0$.
If $a \in (\mathcal{N}^*)$ then $1 - ab = 0 \Rightarrow ab = 1$.
Hence b is a Von Neumann inverse of a. If is not a member of $\mathcal{N}^*$ then ab is not a member of $\mathcal{N}^*$ but $ab = aabb = a^2b^2 = abab = (ab)^2$.
Since $\mathcal{N}$ commutes $\Rightarrow ab = (ab)^2 \Rightarrow ab(1 - ab) = 0$.
Now $\Rightarrow 1 - ab$ is a unit and $ab = 0$ so that $a = 0$ because b is its Von Neumann inverse.
$[\{< a > \times 1 + Z(\mathcal{N})\} \cup \{0\}] \subseteq R(\mathcal{N})$..(ii)
Combining (i) and (ii) gives

$$\begin{aligned} R(\mathcal{N}) &\cong [1 + Z(\mathcal{N})] \cup \{0\} \\ &= < a > \times [1 + Z(\mathcal{N})] \cup \{0\} \end{aligned}$$

Next,

$$\begin{aligned} \mathcal{N}^* &= (\mathcal{N}^*/1 + Z(\mathcal{N})) \times 1 + Z(\mathcal{N}) \\ &\cong < a > \times [1 + Z(\mathcal{N})] \\ &= \mathbb{Z}_{p^r - 1} \times [1 + Z(\mathcal{N})] \end{aligned}$$

But

$$\begin{aligned} \mid [1 + Z(\mathcal{N})] \mid &= \mid Z(\mathcal{N}) \mid \\ &= p^{(k-1)r} \end{aligned}$$

Therefore $\mid \mathcal{N}^* \mid = (p^r - 1)(p^{(k-1)r})$
But $R(\mathcal{N}) = \mathcal{N}^* \cup \{0\}$ $\mid R(\mathcal{N}) \mid = (p^r - 1)(p^{(k-1)r}) + 1$ as required. □

Theorem 3.5. *Let* $\mathcal{N}$ *be the near-ring constructed and* $R(\mathcal{N})$ *be the set of all the regular elements. Then*

$$R(\mathcal{N}) = \begin{cases} \mathbb{Z}_{2^r - 1} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-2}} \times \mathbb{Z}^{r-1}_{2^{k-1}} \times (\mathbb{Z}_2)^h \cup \{0\} & p = 2; \\ \mathbb{Z}_{p^r - 1} \times \mathbb{Z}^r_{p^k - 1} \times (\mathbb{Z}^r_p)^h \cup \{0\} & p \neq 2 : Char\mathcal{N} = p^k : k \geq 3. \end{cases}$$

Proof. Let char $\mathcal{N} = p^k : k \geq 3$. We provide the general case using p = odd. Notice that every $l = 1, ..., r; (1 + p\tau_1)^{p^{k-1}} = 1$
$(1 + \tau_l u_1)^{p^k} = 1, ..., (1 + p\tau_L u_1 + \tau_l u_2 + ... + \tau_l u_n)^{p^k} = 1$.
Let $a_l, b_{1l}, ..., b_{hl} \in \mathbb{Z}^+$ with $a_l \leq p^{k-1}$, $b_{il} \leq p^k : 1 \leq i \leq h$. We notice that

$$\prod_{l=1}^{r}\{(1 + p\tau_L)^{a_L}\} \cdot \prod_{l=1}^{r}\{(1 + \tau_l u_1)^{b_{1l}}\} \cdot \prod_{l=1}^{r}\{(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)\} = 1$$

which implies that $a_l = p^{k-1}$, $b_{1l} = p^k = \cdots = b_{hl} = p^k$. Set

$$\begin{aligned} T_l &= < \{(1 + p\tau_l)^a \mid a = 1, ..., p^{k-1}\} > \\ S_{1l} &= < \{(1 + \tau_l u_1)^{b_1} \mid b_1 = 1, \cdots, p^k\} > \\ &\vdots \\ S_{hl} &= < \{(1 + \tau_l u_1 + \cdots + \tau_l u_n)^{b_h} \mid b_h = 1, \cdots, p^k\} > \end{aligned}$$

The sets defined are all cyclic subgroups of the group $1 + Z(\mathcal{N})$ and they are of the indicated orders. Furthermore, the intersection of any pair of the cyclic subgroups indicated gives an identity group and the product of the $(h + 1)r$ subgroups gives:
$\mid T_l \times S_{1L} \times S_{hl} \mid = p^{k((h+1)r)-1}$ exhausting $1 + Z(\mathcal{N})$.
Thus $1 + Z(\mathcal{N}) \cong \mathbb{Z}^r_{p^{k-1}} \times (\mathbb{Z}^r_p)^h$.
Therefore

$$R(\mathcal{N}) = < \alpha > \ltimes (1 + (Z(\mathcal{N}))) \cup \{0\}$$

$$= \mathbb{Z}_{p^r-1} \times \mathbb{Z}^r_{p^k-1} \times (\mathbb{Z}^r_p)^h \cup \{0\}.$$

□

Theorem 3.6. *Let $\mathcal{N} = R_o \oplus \mathcal{M}$ where $r = 1$ and p-prime, $k \in \mathbb{Z}^+$. If $\mathcal{M} = R_0/pR_0 \oplus ... \oplus R_0/pR_0$. Let $r_0 \in R(R_0)$ then, its Von-Neumann inverse is $r_0^{-1} = r_0^{p^k-p^{k-1}-1}$ and $(r_0, ..., r_h)^{-1} = (r^{p^k-p^{k-1}-1}, -r_1 t_0 r_0^{-1}, ..., -r_h t_0 r_0^{-1})$*

Proof. We know that if $a \in R_0 = G\mathcal{N}(p^{kr}, p^k)$ and $a \in R_0$ then, the Von-Neumann inverse of a is given by: $a^{-1} \equiv a^{p^{(k-1)r}(p^r-1}(mod p^k)$ therefore

$$r_0^{-1} \equiv r_0^{p^k-p^{k-1}-1}$$

as required in step 1
Now let $(t_0, ..., t_h) = (r_0, ..., r_h)^{-1}$, then

$$\begin{aligned} (r_0, r_1, ..., r_h) &= (r_0, ..., r_h)^2(t_0, ..., t_h) \\ &= (r_0^2, r_0r_1 + r_1r_0, ..., r_0r_h + r_hr_0)(t_0, ..., t_h) \\ &= (r_0^2t_0, r_0^2t_1 + (r_0r_1 + r_1r_0)t_0, ..., r_0^2t_h + (r_0r_h + r_hr_0)t_0) \end{aligned}$$

therefore $r_0 = r_0^2 t_0 \Rightarrow r_0 t_0 = 1 \Rightarrow t_0 = r_0^{-1} = r_0^{p^k - p^{k-1} - 1}$
For $i = 1, ..., h, r_i = r_0^2 t_i + (r_0 r_i + r_i r_0) t_0$

$$\begin{aligned} \Rightarrow r_0^2 t_i &= r_i - (r_0 r_i + r_i r_0) t_0 \\ \Rightarrow t_i = & \frac{r_i - 2 r_0 r_i t_0}{r_0^2} (\therefore \mathcal{N} commutative) \\ \Rightarrow t_i &= \frac{r_i}{r_0^2} - \frac{2 r_i t_0}{r_0} \end{aligned}$$

But $t_0 = r_0^{-1}$

$$\begin{aligned} \Rightarrow t_i &= \frac{r_i}{r_0^2} - \frac{2 r_i}{r_0^2} \\ &= -\frac{r_i}{r_0^2} = -r_i r_0^{-2} \end{aligned}$$

$\therefore t_1 = -r_1 r_0^{-2} ... t_h = -r_h r_0^{-2}$
$\Rightarrow (r_0, ..., r_h)^{-1} = (r_0^{p^k - p^{k-1} - 1}, ..., -r_h r_0^{-2})$ as required ☐

Example 3.4. $\mathcal{N} = \mathbb{Z}_9 \oplus \mathbb{Z}_9 / 3\mathbb{Z}_9 \oplus ... \oplus \mathbb{Z}_9 / 3\mathbb{Z}_9$
Then

$$\begin{aligned} (2, \overline{2}, ..., \overline{2})^{-1} &= (2^{9-3-1}, (-2)(5)^2, ..., (-2)(5)^2) \\ &= (5, \overline{1}, \overline{1}, ..., \overline{1}) \end{aligned}$$

$(5, \overline{1}, \overline{1}, ..., \overline{1})(2, \overline{2}, ..., \overline{2}) = (1, \overline{0}, ..., \overline{0})$

Example 3.5. *Consider* $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k) \cong \mathbb{Z}_2[x] / < x^2 + x + 1 >$ *where* $p = 2, k = 1, r = 2$.
Now $G\mathcal{N} = \{0, 1, x, x+1\}$ *and* $R(\mathcal{N}) = \{0, 1, x, x+1\}$.

Let $\mathcal{N} = G\mathcal{N}(4, 2) \oplus G\mathcal{N}(4, 2)$ *with* $G\mathcal{N}(4, 2)$ *as defined above, then:*

$$\mathcal{N} = \{0, 1, x, x+1\} \oplus \{0, 1, x, x+1\}$$

$= \{(0,0), (0,1), (0,x), (0,x+1), (1,0), (1,1), (1,x), (1,x+1), (x,0), (x,1), (x,x), (x,x+1), (x+1,0), (x+1,1), (x+1,x), (x+1,x+1)\}$
So $\mid \mathcal{N} \mid = 16$, $Z_L(\mathcal{N}) = \{(0,0), (0,1), (0,x), (0,x+1)\}$. *Since* $\mathcal{N}$ *is an extension of* $G\mathcal{N}(4, 2)$,

$$\mid R(N) \mid = 13 = (p^r - 1)(p^{kr}) + 1$$

Applying $(r_0, r_1)^{-1} = (r_0^{p^k - p^{k-1} - 1}, -r_1 r_0^{-2})$, *we can find the Von Neumann inverses of all the members of* $R(\mathcal{N})$.
For instance,

$$R(\mathcal{N}) = \{(1,0), (1,1), (1,x), (1,x+1), (x,0), (x,1), (x,x), (x,x+1),$$

$(x+1,0),(x+1,1),(x+1,x),(x+1,x+1)\}$.
So $(1,0)^{-1} = (1^{2^1-2^0-1}, -01^{-1}) = (1^2,0) = (1,0)$, $(x,x)^{-1} = (x^{-2}, x^{-1})$
This can be done in the same manner for the other members of $R(\mathcal{N})$. *The next result gives the structures and orders of the automorphism groups of the regular elements,* $R(\mathcal{N})$.

Theorem 3.7. *Let* $\mathcal{N}$ *be a near-ring of construction* $R(\mathcal{N})$ *be the set of all the regular elements including 0. Then if*
$Aut : R(\mathcal{N}) \rightarrow R(\mathcal{N})$ *we have that*

$$Aut(R(\mathcal{N})) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{(k-1)r}(GN(p^{kr},p^k))] \times GL_{hr}(GN(p^{kr},p^k))] \cup \triangle$$

Theorem 3.8. *Let* $\mathcal{N}$ *be a zero symmetric local near-rings from the class of near-rings of the construction. Then:*

$$\mid Aut(R(\mathcal{N})) \mid \;=\; [\varphi(p^r-1) \cdot \prod_{k=1}^{(k-1)r} (p^k - p^{k-1}) \cdot \prod_{k=1}^{hr} (p^k - p^{k-1})] + 1$$

when $char\mathcal{N} = p^k : k \geq 3$

4 Conclusion

This study was set up with an aim of determining and classifying the regular elements and Von-Neumann inverses of the zero symmetric local near-rings with n-nilpotent radical of Jordan ideals admitting Frobenius derivations. The study gave a general construction representing the classes of the near-rings under investigations whose algebraic structures assumed commutation checks attributed the Theorem of Asma and Inzamam . The structures and orders of $R(\mathcal{N})$ were then characterized in a case by case basis using the FundamentalTtheorem of Finitely Generated Abelian Groups and the properties of the general linear groups in the endomorphism of $R(\mathcal{N})$ respectively. The structures of $V(\mid R(\mathcal{N}) \mid)$ followed asymptotic patterns proposed by Osama and Emad [3] using the properties of $V(n)$, $\tau(n)$, $\overline{\omega}(n)$, $\sigma(n)$ and $K(n)$. The results reveal unique algebraic structures.

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