

1 Calculation of the radius for any atom based on the Planck 2 constant-free equation, mathematical complexity, and 3 atomism

4 Abstract

5 Concern has been expressed against the new theories, Heisenberg uncertainty principle (HUP)
6 and Schrödinger wave/quantum mechanics (SE) that are purported to have replaced Bohr's
7 theory and equation. The research was undertaken with the following objectives: 1) to review
8 concerns about atomism; 2) appraise the issues of mathematical complexities in HUP and SE; 3)
9 review criticism against SE and HUP; 4) most importantly derive a Planck constant invariant
10 equation for the calculation of any atomic radii; and 5) re-calculate the radii of selected elements
11 chosen for their biological importance. The theoretical research with calculations, showed that
12 the opposing theories are criticised because, they are obsessed with mathematical complexities
13 with ambiguities without common ground that should usher alternative solution to the problem
14 of the size of atom and thus, they cannot be considered as a valid description of reality. Bohr's
15 equation and variants of it and the Planck constant invariant equation (Eq. (15)) derived in this
16 research are regarded as deterministic in nature and were capable of reproducing Bohr's radii for
17 any atom; the radii were also capable of reproducing the average ionisation energies of hydrogen
18 and oxygen atoms, chosen for illustration only, when substituted into derived preliminary
19 equation (Eq. (10)).

20 **Keywords:** *Atomism; Bohr's radii of any element; Criticism of Heisenberg and Schrödinger*
21 *mathematical complexities; Deterministic Planck constant free equation; Ionisation energy.*

22

23

24 1. Introduction

25

26 There is a need to make it abundantly clear unlike almost every other article in the
27 literature, that in this section, every presentation is intended to be comprehensible to all at any
28 level of educational achievement, both formal and informal. Thus, since this research is not
29 intended for those in the field at higher level alone, this section addresses the question of
30 indivisibility of the smallest particle state, even though what may be the smallest cannot for now

31 be strictly defined. A well-known 20th century scientist, Dalton has considered the smallest
32 particle of matter as being the atom including its indivisibility. This is similar to a 21st century
33 observation to the effect that electrons are indivisible at 90 GeV (Laboratoire d'économie de la
34 production et de l'intégration internationale (LEP II)) [1]. Of course this is about electron not the
35 atom, though the issue of indivisibility is pertinent. The atom, hydrogen in particular, and the
36 fundamental particle, the electron, are extremely important biologically and physically. Scholars
37 at the highest level objecting to this position may have to resign; recent challenge to the existent
38 of man was fatally challenged by COVID-19 because the "respiratory chain" where
39 mitochondria are indispensable, was extremely compromised. The hydrogen ion carriers, FADH,
40 NADH, the electron-carriers, the cytochromes and associated enzymes, oxidases and reductases,
41 ensure the generation of molecular energy adenosine triphosphate, ATP, with concomitant
42 release of water vapour. Before this, molecular CO₂ is released. In plants CO₂ is consumed while
43 molecular oxygen is released.

44 Bohr's model and by extension the associated equation for the determination of the
45 orbital radius of revolving electron had always met criticism; for this reason Heisenberg
46 uncertainty principle (HUP) and Schrödinger wave probability equation (SE) evolved as
47 replacement for Bohr's equation, that was seen to be unable to address multi-electron atoms; but
48 with respect to multi-electron atoms, derived alternative equations [2] whose results that are
49 strictly a function of the ionisation energy, upon evaluation has made such concerns of no
50 consequence. In this research the concern is to be made of no consequence again by the
51 calculation of the Bohr's radii of any atoms on the basis of Planck constant-free equation to be
52 derived with issue of atomism and concern for mathematical complexities at the background.
53 The HUP and SE models have also met even the harshest informed criticisms as a result of what

54 they stand for and as a result of their misuse [4]. Thus, reaching out to all academic levels,
55 undergraduates, graduates, etc, as a matter of general interest, the objectives of this research are:
56 1) to carry out an overview of concerns about atomism; 2) to carry out an overview of the issues
57 of mathematical complexities in short theses pertinent to HUP and SE; 3) to carry out an
58 overview of criticism against SE and HUP; 4) most importantly to derive yet again a
59 generalisable Planck constant free alternative equation and 5) re-calculate illustratively the radii
60 of the following atoms: oxygen, hydrogen, carbon, nitrogen, phosphorus, iron, calcium,
61 magnesium, manganese, potassium, sodium, chlorine, fluorine, cobalt, molybdenum, zinc,
62 iodine, and sulphur all being the most abundant elements in the biosphere.

63 **1.1 Atomism**

64 This section begins with issues regarding interest in the smallest possible material
65 existence, abiotic or biotic; the question may be why? The answer shall be self-evidence as time
66 goes on. The concern for the smallest possible existence, biotic or abiotic, has been known to
67 man; the notable example is the mustard seed referred to by head of the Christendom, Jesus
68 Christ, even if it is not necessarily the smallest seed on earth, as to imply that science is actually
69 based on four senses, viz.: sight, smell, touch (irritability), and hearing; even a naïve religious
70 leader cannot stand against this view. Still, the atom could not have been the smallest particle
71 despite the view of Dalton. In line with this view is: According to Mills [4] the Copenhagen
72 interpretation of quantum mechanics is: “it asserts that what we observe is all we can know; any
73 speculation about what an electron, photon, atom, or other atomic-sized entity really is or what it
74 is doing when we are not looking is just that—speculation. The postulate of quantum
75 measurement asserts that the process of measuring an observable forces it into a state of reality.
76 In other words, reality is irrelevant until a measurement is made”. “To see is to believe!” Further

77 to this is: “Quantum mechanical theory is not derived from first principles and relies on faith in
78 the infallibility of the Schrödinger equation since it cannot be directly experimentally confirmed”
79 Mills [4]. All we know today about atoms is due to the pioneering work of others who
80 compulsorily are worth mentioning otherwise, the manuscript cannot make progress, in line with
81 ethics.

82 Thus without being unnecessarily historical, with due respect to the pioneers of atomic
83 theory, the following personalities are worth remembering (science.com and en.wikipedia.org):
84 Democritus, the ancient philosopher, who opined that all matter is made of atoms; Abu Bakr
85 Muhammada bin Zakariyya who proposed a theory of atom as space occupying object without
86 dimension, yet having magnitude; Amedeo Avogadro who discovered the mole concept and the
87 basis for the notion of the molecules; Millikan Robert Andrew, who determined the charge of an
88 electron; Dmitri Mendeleev who arranged the elements of the periodic table according to the
89 order of atomic weights and discovered the ionisation energy of the atoms of elements; Goldstein
90 Eugen who observed the positive rays leading to the discovery of the subatomic parts of an atom;
91 Thomson JJ who discovered the electrons and proposed the plum pudding or raisin bun model of
92 the atom; Rutherford Ernest who discovered the dense atomic nucleus; Louis de Broglie who
93 theorised that all matter has wave-property duality; Henry Moseley who arranged the elements in
94 the order of their effective atomic number and has Moseley law to his credit; Niels Bohr who
95 defined the equation for the determination of atomic radius restricted to hydrogenic atom and the
96 planetary model of the atom; Johannes Rydberg who discovered the Rydberg constant, valuable
97 in the study of spectroscopy; Johannes Rydberg and Walter Ritz who formulated the Ritz-
98 Rydberg combination principle; Theodore Lyman who discovered the 1st series called Lyman
99 series, of the hydrogen spectrum (HS); Johann Balmer who discovered the 2nd series called

100 Balmer series, of HS-the visible spectrum; Friedrich Paschen who discovered the 3rd series
101 known as Paschen series; Frederick Summer Brackett who discovered the 4th series known as
102 Brackett series; August Herman Pfund who discovered the 5th series known as Pfund series;
103 Curtis J. Humphreys who discovered the 6th series; Edward Charles Pickering and Alfred Fowler
104 who discovered the 7th series known as the Pickering-Fowler series – the spectral lines arising
105 from **He**⁺ which according to Bohr originally was attributed to unknown form of hydrogen with
106 half-integer transition level; Erwin Schrodinger who introduced the quantum mechanical model
107 of the atom; and Werner Heisenberg who formulated quantum mechanics based on matrices and
108 the controversial “uncertainty relation”. Recent (20th century) concern about Bohr’s theory and
109 equation borders on what makes hydrogen atom stable, or rather what constitutes its ground state
110 [4]. The implication is that the time tested “spectroscopic” data (ionisation energy) values are
111 being put to question but not invalidated.

112 **1.2 Concern about mathematical complexity**

113 Unlike most research papers in physical sciences, Bohr’s equation is regrettably restricted
114 to hydrogen atom and as a consequence other higher elements are left out until recently when
115 modified forms of Bohr’s equation [2, 3] were derived which satisfactorily reproduced Bohr’s
116 radius for hydrogen and there was no reason why those equations cannot be applied to non-
117 hydrogenic atoms. The equations [2, 3] in question, Heisenberg’s equation of uncertainty
118 principle, Millikan’s equation for the calculation of the charge of an electron, equation of mass-
119 energy equivalent by A. Einstein *etc* are all explicit equations showing exact dependent and
120 independent variables. Like in the past, current scientist with interest in most, if not all
121 mathematics based subjects, especially the mathematical fields, *e.g.*, mathematical physics,
122 mathematical chemistry mathematical biology, mathematical economics, *etc* are so imbued with

123 immense knowledge of mathematics beyond mere post graduate level to the point where, a
124 solution to a problem ends up in another problem; this is intended to mean that a lay man may
125 not be able to discern the dependent and independent variable from each other. This is not to say
126 that authors of such papers in those fields containing a lot of mathematical complexities do not
127 know what those variables are, but restriction to them alone is like one “speaking in tongue”
128 which no one but themselves alone understands. Some of those papers can be split into at least
129 three papers, each giving interpretational details of terms, meaning of mathematical terms or
130 forms, with one or two objectives. “Due regard or highest respect must always be given to
131 scientist and social scientist neck-deep in subjects where “highly advanced mathematics” is the
132 case and to the mathematicians in particular”.

133 This section is very important because any mention of Bohr’s theory or rather equation
134 often restricted to hydrogenic atom, must take into cognisance of the theories that seem to go
135 against it; it is also instructive to realise that theories that go against Bohr’s equation and theory
136 are also under severe criticism. Any mention of HUP and SE does not necessarily mean that their
137 formal mathematics has to be adopted; doing so would amount to contradictions because as
138 commented elsewhere [3] SE and HUP mathematics and qualitative arguments and Bohr’s
139 mathematics as well as recent extension of Bohr’s mathematical formalism [5] are strange bed
140 fellows. Most of the criticisms against HUP and SE are based on the difficulties and of course
141 the inherent complexities associated with the mathematics applied in those principles or rather
142 concepts [4]. Yet there may be no end to it any time soon as long as there are publishers with
143 preferences for respected scholars and nationalities of choice where those scholars originate. No
144 one in contemporary “cloud of events” from west to east, north to south and *vice versa* should
145 tragically be in doubt about this comment. Honesty demands that, one open up to say that most

146 papers on HUP and SE are replete with very unfamiliar mathematics such as those where
147 complex calculus and set theories play prominent role. Some of the mathematical exposition
148 such as local fractional continuity of function loaded with set theory, theory of local fractional
149 Fourier analysis, the generalised Fourier transforms in fractal space, HUP in local fractional
150 Fourier analysis with set theory, the mathematical aspect of fractal quantum mechanics (QM)
151 *etc*, can be found in work of Yang *et al.* [6]. A preprint report [7] also showed a lot of
152 mathematics with highly advanced calculus with set theories intended to address what the author
153 felt was vague statements, viz: “It is impossible to prepare states in which position and
154 momentum are simultaneously arbitrarily well localised; it is impossible to measure
155 simultaneously position and momentum, and it is impossible to measure position without
156 disturbing momentum, and vice versa”. In this research those statement are regarded as
157 intelligible.

158 Again the researcher gave as usual, a complex equation each for uncertainty in position
159 and momentum which specifically defined the standard deviations of position and momentum in
160 a state, ψ [7], usually representing wave function. Once again they presented another definition
161 of HUP viz: “According to the uncertainty principle, the qualitative relationship of a strict
162 mutual exclusiveness of sharp preparations or measurements of position and momentum is
163 complemented with a quantitative statement of a trade-off between competing degrees of the
164 concentration of the distributions of these observables in state preparations or between the
165 accuracies in joint measurements” [7]. This, to the layman is vague and clear example of
166 ambiguity often objected to by Mills [4]; what preparation means is unknown, perhaps, it seems
167 to imply the action or procedure taken to measure either position or momentum which could
168 acceptably make it impossible to define initial position of the electron let alone the final position

169 due to almost infinitesimal size of the electron. Action to be carried out remains perhaps, naively,
170 a qualitative statement of intent, until when such is carried out and measurement of effect or
171 outcome such as change of position and momentum with their errors are determined or
172 measured. This position is however, speculative arising from the statement of ambiguity. By the
173 way it would have been better if statistician devoted to the issue of HUP define clearly error and
174 uncertainty in any of those measurements. This follows from what appears to be two aspect of
175 the same issue given by Mill [4]: The Heisenberg Uncertainty Principle is wrongly interpreted
176 as: The uncertainty in the measured momentum times the uncertainty in the measured position
177 must be no less than \hbar , the reduced Planck constant given as $h/2\pi$ [4]. Perhaps the proper
178 presentation should have been $\Delta x \Delta P \leq \hbar$ (or rather “ $\Delta x \Delta P \lesssim \hbar$; $\Delta x \Delta P \leq \hbar$; $\Delta x \Delta P \approx \hbar$ ”, noting is
179 specified). Perhaps this may be in line with the following assertion by Mill [5]: “The HUP gives
180 a lower limit to the product of the *uncertainty* in the momentum and the *uncertainty* in the
181 position—not the product of the momentum and the position”. “The Heisenberg Uncertainty
182 Principle (HUP) is the mathematical expression for the statistical error in the variables of the
183 wave function such as those assigned to the position and momentum of the electron.

184 Since the wave function is interpreted as the probability of the position of the electron
185 which puts it everywhere at once with an infinite number of positions and energies
186 simultaneously including ones with negative kinetic energy, the Heisenberg Uncertainty
187 Principle merely reveals that this model is not a valid physical description of the electron” [4].
188 Yet an opinion in an old literature [8] posits that the wave description is consistent with
189 uncertainty principle according to which the position of an electron in a given energy level
190 cannot be known exactly. One may wish to know if mathematical expression for the statistical
191 error in the variables of the wave function (or rather its square form) assigned to the position and

192 momentum of the electron equivalent to probability density distribution of electron. In all these
193 issues, one should bear in mind what uncertainty stands for viz: In metrology, measurement
194 uncertainty is the expression of the statistical dispersion of the values attributed to a measured
195 quantity; all measurements are subject to associated uncertainty such as standard deviation. On
196 the other hand, observational (or measurement) error is the difference between a measured value
197 of a quantity and its true value (Wikipedia). This is in the light of view that: “Despite its
198 successes, after decades of futility, quantum mechanics and the intrinsic Heisenberg Uncertainty
199 Principle have not yielded a unified theory, are still purely mathematical, and have yet to be
200 shown to be based in reality [5]. Both are based on circular arguments that the electron is a point
201 with no volume with a vague probability wave requiring that the electron have multiple positions
202 and energies including negative and infinite energies simultaneously” [4]. In other words there
203 appears not be a common ground for both theories. So much about HUP but yet to be exhausted

204 Next is the Schrödinger theory and associated equation seen to be suitable to hydrogenic
205 atom. “It fails to predict electron spin and leads to models with nonsensical consequences such as
206 negative energy states of the vacuum, infinities, and negative kinetic energy. In addition to many
207 predictions, which simply do not agree with observations, the Schrödinger equation and
208 succeeding extensions predict noncausality, nonlocality, spooky actions at a distance or quantum
209 telepathy, perpetual motion, and many internal inconsistencies where contradicting statements
210 have to be taken true simultaneously” [1]. QM provides an algorithm for computing probabilities
211 for macroscopic events (‘detector ticks’) that are the consequences of experimental interventions
212 [4].

213 Concluding in part, probabilities have no place in describing atomic energy levels.
214 “Moreover, quantum theory is incompatible with probability theory since it is based on

215 underlying unknown, but determined outcomes” [4]. This paragraph is considered very important
216 because it gives a clear support for the view that HUP is precluded in the procedure used to
217 derive alternative equation that is generalisation to all atoms in the 1990s but was rejected in a
218 manner that was uncomplimentary and reckless; perhaps precluding HUP and not necessarily
219 typographical errors or grammar per se (the grammar could be described as jargon in the first
220 place) may be the reason for the rejection. The paper [2] has been published elsewhere. The
221 wave function must be real and physical in order for it to be considered a valid description of
222 reality. This view by Hill follows from the observation that electrons formed bubbles in
223 superfluid helium apart from being indivisible at 90 (LEP II). Unfortunately these issues
224 described as being inconsistent with physical laws including those emanating from Feynman’s
225 view are largely ignored by the physics community [4]. It may remain so until people like Mills
226 come strongly and until the only motivation to publish for promotion ceases to be.

227 **2.0 Theoretical development**

228 In this section equations of the past are re-appraised and any objections against any of
229 them examined *vis-à-vis* alternatives in the literature. Scientific terms are given general
230 definitions or interpretation in line with broad objective, the aim of this research. Phrases such as
231 after some algebra in statements are strictly forbidden; a step-by-step approach is adopted in the
232 derivation of a new equation that is Planck constant invariant mainly for multi-electron atoms.

233 **2.1 Review of previous mathematical models-equations-and results.**

234 The research in the past has shown that the computed radii of atoms are found to
235 reproduce the expected periodic variation of size in periods and in groups and nicely reproduced
236 the *d*-block and *f*-block contractions in the respective series [9] similar to the claim in paper one
237 that: the calculated set of theoretical atomic radii of 103 elements of the periodic table using

238 Slater orbitals were found to reproduce the periodic law and reproduced the expected vertical and
 239 horizontal trend of variation in atomic size in the periodic table. Though these finding seem very
 240 lofty, it is not subject of verification rather the equations that enabled the calculations may
 241 require further clarification. According to the authors [9] the radial charge density distribution
 242 function (RCDF, or $\rho(r)$ for the symbol), given below is based on what the author called Slater's
 243 analytical form of the radial part of the one-electron function. This is important because it has
 244 been an age-long believe in scientific community that, Bohr's equation and SE was more
 245 successful with hydrogenic atom. Nonetheless recent research has produced variety of equations
 246 that can be used to determine the radii of atoms based on information on the experimentally
 247 determined ionisation energies of elements. Nonetheless the equation of $\rho(r)$ given by Ghosh and
 248 Biswas, [9] is:

$$249 \quad \rho(r) = 4\pi r^{2n} (2\xi)^{2n+1} [(2n)!]^{-1} \exp(-2\xi r) \quad (1)$$

250 In line with broad objective, aim, an explanation to $\rho(r)$ is given herein: As in statistical
 251 mechanics applicable to particles in whatever state, including the leptons, but not limited to
 252 them, shows how the density, ρ varies as a function of distance from another preferred particle
 253 called reference particle (en.wikipedia.org). After differentiating the equation of RCDF, and after
 254 algebraic manipulation or rearrangement, the following result [10] was obtained.

$$255 \quad n r_{max}^{2n-1} - \xi r_{max}^{2n} = 0 \quad (2)$$

256 Where r_{max} , n , and ξ are the atomic or ionic radii, principal quantum number (pqn), and orbital
 257 exponent given as [10]:

$$258 \quad \xi = (Z-S) n^* = Z^* n^* \quad (3)$$

259 Where, Z , Z^* , S , and n^* are the atomic number, effective charge, screening constant and effective
260 pgn. The authors applied Slater rule for the determination of S .

261 From Eq. (1) the authors [10] obtained:

$$262 \quad r_{\max} = n / \xi \quad (4)$$

263 The concern in this research is that, it is not certain how dimensionless variables such as “ ξ ”, Z ,
264 Z^* , S , and n^* can be used to calculate values such as radii whose SI unit is metres. Noting in
265 papers one and two points to the equation **into** which the variable can be substituted.

266
267 Owolabi *et al* [10] explored computer-based programme, otherwise called artificial intelligence
268 using support vector regression that predicts atomic radii of elements above 99 % of accuracy
269 and with associated set theory and matrix formalism were able generate results for many
270 elements. However, the report for hydrogen is far from what Mills [4] says about constant
271 parameters of the hydrogen atom which are known to 10 figure accuracy. The computer
272 “language” and associated mathematics, matrix, set theory *etc* may be of restricted interest,
273 reserved for high-level specialist.

274 A hybrid-density functional calculation, perhaps, on atomic scale and Dirac-Breit
275 methods were also explored for the calculation of radii of atoms. According to the authors [11]
276 the results from the two methods were in satisfactory agreement. The Dirac-Breit equation is a
277 relativistic wave equation derived by Breit based on Dirac equation which describes two or more
278 massive spin-1/2 particles (*e.g.* electrons) interacting electromagnetically to the 1st order in
279 perturbation (or rather, perhaps, during perturbation); this accounts for magnetic interactions.
280 Dirac equation describes all spin-1/2 massive particles such as electrons and quarks (elementary
281 particles that makeup each nucleon) for which parity is a symmetry; it is consistent with both

282 principle of quantum mechanics and theory of special relativity (en.wikipedia.org). Spin being
283 seen as a conserved quantity carried by elementary particles (en.wikipedia.org), is also one of the
284 2 kinds of angular momentum (the 2nd being orbital angular momentum); it is the rotational
285 motion of electron about its axis passing through its centre of mass, producing in the process,
286 tiny magnetic fields as previously observed by Otto Stern and Walther Gerlach of the University
287 of Hamburg, Germany (Scientific America).

288
289 Next in line is the original Bohr's equation which attracted criticism for some reason seen to be
290 unnecessary. The Bohr atomic model or theory considers electrons to have both a known
291 distance from the nucleus – the radius of a circle (or orbit) – and orbit *i.e.* known position and
292 momentum at the same time, which is impossible according to HUP [12]. This shows that one
293 cannot extricate Bohr's original equation from his planetary model: The most important issue is
294 the electrostatic influence of the nucleus on the electron (s), be it oscillatory, vibrational or
295 rotating in motion. This is where Bohr's equation remains very relevant but regrettably he had no
296 knowledge of how to determine the effective nuclear charge, even if the data for ionisation
297 energies for all elements were available in his days which postdate the days of Dmitri Mendeleev
298 who discovered ionisation energy. "The effective nuclear charge, Z_{eff} , is a measure of the average
299 nuclear charge felt by the outermost electron in the various orbitals, considering the inter-
300 electronic repulsions and its penetration capability" [13]. The only reasonable challenge in old
301 Bohr's equation is the lack of know-how for the determination of effective nuclear charge which
302 however, is not the interest of this research, otherwise with the information about the Z_{eff} of any
303 element the radius of any element can be calculated. Nonetheless there are methods in the
304 literature for calculating Z_{eff} [2]; ionisation energy dependent approach and another approach that

305 relates with the fine structure constant had been derived [14]. This makes Bohr's equation that
306 has the character of specificity, for the calculation of the radius of any atom to assume a
307 universal appeal. Recent researches [2, 3] have shown that the radii of all atoms off all elements
308 can be determined.

309 **2.2 Derivation of yet another alternative equation to the original Bohr's equation.**

310

311 In this section, another equation which can be used to calculate the radii of some selected
312 elements motivated by their biological importance is to be derived. This is against the backdrop
313 of the description of electron as a wave function, a mathematical description of a quantum state
314 (the mathematical entity that provides a probability distribution for the outcomes of each
315 possible measurement on a system), of an isolated system, instead of a point charge as claimed
316 by Fattah [15] with reference to the literature [16]. The consequence of describing electrons as
317 waveforms is that, it is mathematically impossible to simultaneously derive the position and
318 momentum of an electron [16], a view that has also met opposition [4, 17, 18]. It seems the
319 "accusers" of Bohr's theory and equation are meeting exceedingly harsher criticism and flawed
320 theories. The first of this Bohr's equation written in different forms is given as:

$$321 \quad a_0 = \frac{n^2 h^2 \epsilon_0}{\pi m_e e^2 Z_{\text{eff}}} \quad (5)$$

322 where, n , h , m_e , e , ϵ_0 and Z_{eff} are the principal quantum number (otherwise called energy level,
323 pqn), Planck constant, rest mass of an electron, charge of an electron, and permittivity in free
324 space (electric constant) and effective nuclear charge respectively. Most high quality text books
325 [8, 19] present Bohr's equation for hydrogen as:

$$326 \quad a_0 = \frac{h^2 \epsilon_0}{\pi m_e e^2} \quad (6)$$

327 Guess, simply because the Z_{eff} is = 1 and $n = 1$. Another way is to write the so-called “primitive”
328 equation, the Coulomb equation such as:

$$329 \quad a_0 = \frac{Z_{\text{eff}} e^2}{8 \pi \epsilon_0 \xi_{\text{H}}} \quad (7)$$

330 where ξ_i is average ionisation energy of any atom other than hydrogen.

331 Again most text books present, Eq. (7) as:

$$332 \quad a_0 = \frac{e^2}{8 \pi \epsilon_0 \xi_{\text{H}}} \quad (8)$$

333 where ξ_{H} is the average ionisation energy of hydrogen. The reason is as written earlier. It is very
334 clear that the only challenge is the lack of information about Z_{eff} in Eqs (5) and (7) for elements
335 whose Z_{eff} is > 1 and n is ≥ 1 .

336 Meanwhile, an equation for the determination of the radius of any atom (with due respect
337 for N. Bohr, all radii regardless of element are referred to as Bohr’s radii, a_i and a_0 for any other
338 element other than hydrogenic atom or ion and for hydrogen respectively.) had been derived as
339 shown in the literature [2]. The equation is given as:

$$340 \quad a_i = \frac{n h}{\pi^2 \sqrt{8 m_e \xi_i}} \quad (9)$$

341 Although Eq. (9) seems to be tied down to heavier atoms, be it hydrogenic and non-hydrogenic,
342 it is nevertheless a general one; substitution of accurate value of the average ionisation energy of
343 say hydrogen gives exactly the same value known in the literature, CODATA [20] for instance.

344 Taking the square of Eq. (9) and rearranging gives:

$$345 \quad \xi_i = \frac{n^2 h^2}{8 \pi^2 m_e a_i^2} \quad (10)$$

346

347 Meanwhile, it has been shown that [15]:

348
$$\xi_H = \frac{n^2 \xi_i}{Z_{\text{eff}}^2} \quad (11)$$

349 The equation for Z_{eff} is given as [15]:

350
$$Z_{\text{eff}} = \sqrt{\left(\frac{8 \xi_i}{m_e}\right)} \cdot \frac{nh\varepsilon_0}{e^2} \quad (12)$$

351 Then, take the reciprocal of Eq. (11) to give:

352
$$\frac{1}{\xi_H} = \frac{Z_{\text{eff}}^2}{n^2 \xi_i} \quad (13)$$

353 After substituting Eq. (10) and the square of Eq. (12) into Eq. (13) one gets:

354

355
$$\frac{1}{\xi_H} = \frac{8 \pi^2 a_i^2 m_e}{n^4 h^2} \cdot \frac{8 \xi_i n^2 h^2 \varepsilon_0^2}{m_e e^4} \quad (14a)$$

356 Simplification gives:

357
$$\frac{1}{\xi_H} = \frac{64 \pi^2 \varepsilon_0^2 a_i^2 \xi_i}{n^2 e^4} \quad (14b)$$

358

359 Making a_i subject of the formula in Eq. (14b) gives:

360

361
$$a_i = \frac{e^2 n}{8 \sqrt{\xi_H \xi_i} \cdot \pi \varepsilon_0} \quad (15)$$

362 Equation (15) represents an equation independent (or rather free) of Planck constant for the

363 determination of the radius of any atom: Thus if ξ_i and ξ_H are equal, then n should be

364 =1, such that: $a_0 = \frac{e^2}{8 \xi_H \pi \varepsilon_0}$. Nonetheless n may be equal to 1 even if ξ_H is not equal to ξ_i as

365 applicable to He and monovalent ion (Li^+). Note however, that n may be = 1 for multi-electron

366 atoms and ions if, $\frac{A}{Z}X$ has only 2 electrons as an atom or as an ion if it has already lost 1 electron

367 leaving behind $(Z - 1)$ where $Z = 3$ or 2 electrons leaving behind 2 electrons where $Z = 4$. Thus,
368 with carefulness and deliberate interest one can obtain a well-known Bohr's radius for hydrogen,
369 and consequently there is no justification to stop midway thinking that Eq. (15) should
370 exceptionally yield incorrect value for any multi-electron atom as shown and confirmed in Table
371 1 for the selected elements. Note however, that the unnumbered equation constitute a
372 reproduction of Eq. (8) because it is unavoidable, in that, Z_{eff} for hydrogen and any other larger
373 hydrogenic ion, ${}^A_ZX^{(Z-1)+}$ is 1. The only innovation lies in Eq. (15) which is generally applicable
374 to higher atoms and non-hydrogenic ions whose ionisation energy and the quantum number of
375 energy level are known for the calculation of the radius without the need for h . If an atom of
376 higher elements has 2 or more electrons, information about effective nuclear charge is needed for
377 the calculation of the radius; but such information is not applicable to Eq. (15) and, unlike Eq.
378 (9), h is not needed as written earlier.

379 **3. Materials and Methods**

380 The research is purely theoretical and calculational without any measurement.

381 **4. Results and Discussion**

382 A very simple equation such as Eq. (4) does not offer any clue as to how the important
383 periodic property such Bohr's radius for any atom can be calculated; the denominator and
384 nominator are dimensionless. Equations (5) to (16) do not present any dimensional
385 inconsistencies. Of particular interest is Eq. (15) which is the ultimate result of this research.
386 Equation (16) as a corollary validates the procedures of arriving at Eq. (15). The radii of 103
387 elements as a function of their 1st average ionisation energies had been calculated based on Eq.
388 (9) as in the literature [2] and another equation in the literature [3]. However, in other to evaluate

389 Eq. (15), the radii at the highest energy level of elements commonly encountered in the
 390 biosphere were calculated and the values are shown in Table 1.

391 A close examination of Table 1 reveals that the old result (values of ionisation energies)
 392 are very similar to values calculated in this research; differences may be as a result of
 393 approximations and the use of fundamental constants whose values were not exactly CODATA
 394 [20] values. Typical examples of this issue are the adoption of the following values of
 395 fundamental constants, viz: e ($1.6021 \text{ exp. } (-19) \text{ C}$); ξ_{H} ($(1312000 / N_{\text{A}}) \text{ J}$ where $N_{\text{A}} = 6.02252$
 396 $\text{exp. } (23)/\text{mol.}$); ϵ_0 ($8.854 \text{ exp. } (-12) \text{ C}^2/\text{N.m}^2$); the corresponding 2016 CODATA [20] values
 397 are: e ($1.6021766208 \text{ exp. } (-19) \text{ C}$); ξ_{H} ($(1312035.26 / N_{\text{A}}) \text{ J}$ where $N_{\text{A}} = 6.022140857 \text{ exp.}$
 398 $(23)/\text{mol.}$); ϵ_0 ($8.854187817 \text{ exp. } (-12) \text{ C}^2/\text{N.m}^2$). It is pointless recalculating the radii of all the
 399 atoms in this research using Eq. (9) except an isolated case in which hydrogen is considered for
 400 the purpose of showing that Eq. (9) and Eq. (15) would always give the same value of Bohr's
 401 radii. If so, there is no reason whatsoever, why it should not be so for other multi-electron atoms.

402 **Table 1. First Bohr's radii of selected bioactive elements**

S/N	Elements	Electronic configuration	Bohr's radius* exp. (-11) m	Bohr's radius# exp. (-11) m
1	H	$1s^1$	5.292523983	5.29
2	C	$[\text{He}] 2s^2 2p^2$	11.6319013	11.6
3	N	$[\text{He}] 2s^2 p^3$	10.2390291	10.2
4	O	$[\text{He}] 2s^2 2p^4$	10.57786666	10.6
5	F	$[\text{He}] 2s^2 2p^5$	9.351509449	9.4
6	Na	$[\text{Ne}] 3s^1$	25.82957393	25.8

7	Mg	[Ne] 3s ²	21.17468415	21.2
8	P	[Ne] 3s ² 3p ³	18.08046082	18.1
9	S	[Ne] 3s ² 3p ⁴	18.19046115	18.2
10	Cl	[Ne] 3s ² 3p ⁵	16.25898646	16.3
11	K	[Ar] 4s ¹	37.4707368	37.5
12	Ca	[Ar] 4s ²	31.57498735	31.6
13	Mn	[Ar] 3d ⁵ 4s ²	28.6315694	28.6
14	Fe	[Ar] 3d ⁶ 4s ²	27.76998501	27.8
15	Co	[Ar] 3d ⁷ 4s ²	27.80830482	27.8
16	Zn	[Ar] 3d ¹⁰ 4s ²	25.47039153	25.5
17	Se	[Ar] 3d ¹⁰ 4s ² 4p ⁴	24.99774063	25.0
18	Mo	[Kr] 4d ⁵ 5s ¹	36.6422651	36.7
19	I	[Kr] 4d ¹⁰ 5s ² 5p ⁵	30.18485986	30.2

403 The average 1st ionisation energies used for calculations were obtained from the literature
404 (en.wikipedia.org/wiki/ionisation energies of the elements); the ionisation energy of hydrogen was however,
405 calculated using 2016, CODATA [20] values of fundamental physical constants. The value calculated for hydrogen
406 using the equation, $\xi_H = Z_{\text{eff}}^2 e^4 m_e / 8 \epsilon_0^2 n^2 h^2$ is: 2.179872321 exp. (–18) J (using rest mass) and 2.178685772 exp.
407 (–18) J (using reduced mass = 9.104425137 exp. (–31) kg.). The asterisks (*) and ash (#) denote values calculated
408 using reduced mass in this research and values obtained in earlier research [2, 3] respectively.

409
410 The Coulomb equation ($\xi_i = Z_{\text{eff}} e^2 / 8 \pi \epsilon_0 a_i$) is one in which *ab initio* the Planck constant does not
411 appear. Where Z_{eff} is equal to one, then hydrogen atom is the case. Since ξ_H is, experimentally
412 determinable, a_0 for hydrogen can be calculated without the Planck constant; but this is not

413 possible if multi-electron atoms are the case because, Z_{eff} should be greater than one and needs to
414 be separately determined. The original Bohr's equation, Eq. (5) and Eq. (6) contain Planck
415 constant. This is applicable to other variants of the equation given as
416 (en.wikipedia.org/wiki/Bohr's radius) $a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2$; $a_0 = \hbar/m_e c\alpha$ and the equation (Eq. (9))
417 derived elsewhere [2]. Like the original Bohr's equation, the presence of reduced Planck
418 constant, $\hbar (= h/2\pi)$, in the latter equations means that the Bohr's radius of hydrogen and any
419 hydrogenic ion with $Z>1$ cannot be calculated without Planck constant. This is however, unlike
420 Eq. (15).

421 Worthy of mention is Eq. (10) which shows that if the atomic radius of any atom is
422 determined correctly by experiment or theory, the average ionisation energy of an element can be
423 determined. Adopting hydrogen and oxygen as reference elements, this assumption is tested as
424 follows: By substituting known values of a_i for multi-electron atom like oxygen and a_0 for
425 hydrogen into Eq. (10) the values of calculated average ionisation energies are respectively,
426 1313817.284 J/mol. and 1312035.26 J/mol. En.wikipedia.org value is 1313900 and 1312000
427 J/mol. respectively. Ghosh and Biswas [9] reported radii values equal to 5.292 exp. (-11) m and
428 4.652 exp. (-11) m for hydrogen and oxygen respectively. Substituting these values into Eq. (15)
429 gives after calculation the following values respectively for hydrogen and oxygen: 1312377.653
430 J/mol. and 3396636.57 J/mol. The value for hydrogen calculated using radius reported in the
431 literature [9] is very similar to values calculated in this research and recorded elsewhere
432 (en.Wikipedia.org/wiki/ionisation energy). However, the calculated value for oxygen in this
433 research is much more similar to the value elsewhere [3] than the value calculated using the
434 radius determined for oxygen in the literature [9]. The adoption of Slater rule for the
435 determination of effective nuclear charge [9] may account for larger values of the latter leading

436 to shorter atomic radius. It is interesting to note that the derivation of equations in this research
437 takes its root in the original Bohr's mathematical formalism and there are definite Bohr's
438 equations for the radius of hydrogen and any other hydrogenic multi-electron ion and also for
439 ionisation energy.

440 **5. Conclusion**

441 The fact that a material or matter in existence has an infinitesimal magnitude has been
442 reemphasised by evidence from both nonscientific and scientific perspectives. Modern atomic
443 theories, the HUP and SE, that seem to repudiate Bohr's deterministic theory and cognate
444 equation, have been seen to be excessively loaded with mathematical complexities with
445 associated ambiguities without common ground that should usher in alternative solutions to the
446 problem of the size of the atom. From the literature, it is obvious that quantum mechanics (QM)
447 only provides an algorithm for computing probabilities for macroscopic events, though
448 probabilities have no place in describing atomic energy levels. The wave function must be real
449 and physical rather than being mathematical in order for it to be considered a valid description of
450 reality. The derived equation in this research reproduced the values of the radii of atoms,
451 including hydrogen, that were very similar to values reported in the older and much newer
452 literature. The new equation in question is not restricted to multi-electron atoms and ions; it is
453 applicable to hydrogen and other hydrogen-like ions, that is, those ions that possess a single
454 electron. The only experimental variable needed is the average ionisation energy. Substitution of
455 calculated radii for hydrogen ($5.292523983 \times 10^{-11}$ m) and oxygen ($10.57786666 \times 10^{-11}$
456 m) into the derived preliminary equation reproduced after calculation the average ionisation
457 energies for hydrogen (1312035.26 J/mol.) and oxygen (1313817.284 J/mol.). The Bohr's radii
458 of all atoms can be calculated independent of the Planck constant. An alternative method for the

459 determination of the definite three-dimensional space of an atom, excluding interparticle space,
460 should be reserved for future research.

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466 There is no competing interest of any kind.

467 **Authors' contributions**

468 The sole author designed, analysed and interpreted and prepared the manuscript.

469 **COMPETING INTERESTS DISCLAIMER:**

470 Authors have declared that they have no known competing financial interests OR non-financial interests
471 OR personal relationships that could have appeared to influence the work reported in this paper.

472

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