# On the centralizer of a rhotrix in $R_3(R)$, $R$ a gcd domain

## Abstract

Aims/ objectives: In this paper we determine explicitly the centralizer of a 3-dimensional rhotrix in $R_3(R)$, where $R$ is a gcd domain

*Keywords: Rhotrix; Row-column multiplication on rhotrices; gcd Domain ;Centralizer; Field of fractions.*

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# 1 Introduction

Rhotrix is a new field of research in linear algebra focusing on the rhomboidal representation of arrays of numbers. The idea of a rhotrix, an entity that stands in some respects between 2 by 2 and 3 by 3 matrices, was initially introduced by Ajibade [1] as an extension of the initiative on matrix-tertions and matrix-noitrets proposed in [2]. Ajibade defined the set of rhotrices of size three over reals as follows:

$$R_3(\mathbb{R}) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in \mathbb{R} \right\}$$

The heart of any rhotrix $A_3 \in R_3(\mathbb{R})$ is the entry at the specific intersection of the vertical and horizontal diagonal represented by $h\left(A_3\right) = c$.

Rhotrix theory is currently in the early stages of development. Since its inception in 2003, numerous researchers have expressed interest in developing and expanding this notion, most often by drawing analogies to the concepts of matrices [[10],[18],[13],[14],[3],[15]]. As a result, many applications of rhotrix theory were developed in physics, engineering, cryptography, and coding theory · [[5],[6],[17],[16]].

The binary addition $(+)$, scalar multiplication and multiplication $(\circ)$ operations specified in [1] are listed below, respectively:

$$A=\left\langle\begin{array}{ccc} & a & \\ b & h(A) & d \\ & e & \end{array}\right\rangle \text{ and } B=\left\langle\begin{array}{ccc} & f & \\ g & h(B) & j \\ & k & \end{array}\right\rangle$$

$$A+B=\left\langle\begin{array}{ccc} & a+f & \\ b+g & h(A)+h(B) & d+j \\ & e+k & \end{array}\right\rangle$$

For any $k \in \mathbb{R}$

$$kA=\left\langle\begin{array}{ccc} & ka & \\ kb & kh(A) & kd \\ & ke & \end{array}\right\rangle$$

and

$$A \circ B=\left\langle\begin{array}{ccc} & ah(B)+fh(A) & \\ bh(B)+gh(A) & h(A)h(B) & dh(B)+jh(A) \\ & eh(B)+kh(A) & \end{array}\right\rangle$$

An alternate method of multiplication known as row-column multiplication on rhotrices was proposed by Sani, B [11] . The row-column method of multiplication is described as follows for any two rhotrices with the same size $A$ and $B$: if

$$A=\left\langle\begin{array}{ccc} & a & \\ b & h(A) & d \\ & e & \end{array}\right\rangle \text{ and } B=\left\langle\begin{array}{ccc} & f & \\ g & h(B) & j \\ & k & \end{array}\right\rangle$$

Then

$$A \circ B=\left\langle\begin{array}{ccc} & a & \\ b & h(A) & d \\ & e & \end{array}\right\rangle \circ \left\langle\begin{array}{ccc} & f & \\ g & h(B) & i \\ & j & \end{array}\right\rangle = \left\langle\begin{array}{ccc} & af+dg & \\ bf+eg & h(A)h(B) & ai+dj \\ & bi+ej & \end{array}\right\rangle$$

According to [11], row-column based rhotrix multiplication is non-commutative but associative. The identity rhotrix of size three $I_3$ is given by

$$I_3=\left\langle\begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array}\right\rangle$$

In [12] Sani, B extend the row-column multiplication to higher dimensional rhotrices.
in [[8], p 25] Mohammed, A showed that $(R_n(F), +, \circ)$ is a non-commutative rhotrix ring of size $n$ where $+$ and $\circ$ are the operations of rhotrix addition and row-column based multiplication of rhotrices of the same size respectively and $F$ is an arbitrary ring.

In the follow-up to this paper, $\circ$ refers to the row-column based multiplication of rhotrices of the same size.

This work aims to extend the concept of the centralizer of a matrix which plays an important role in matrix theory, especially in solving some matrix equations, namely the Sylvester matrix equation

$$AX - XB = C$$

to rhotrix.

## 2 Preliminaries

A ring is referred to throughout as a ring with $1 \neq 0$. An integral domain R is a commutative ring with nonzero divisors. The following definition and lemmas are available in [4]

**Definition 2.1.** Let $R$ be a commutative ring and let $a, b \in R$ with $b \neq 0$.

(1) $a$ is said to be a multiple of $b$ if there exists an element $x \in R$ with $a = bx$. In this case $b$ is said to divide $a$ or be a divisor of $a$, written $b \mid a$.

(2) A greatest common divisor of $a$ and $b$ is a nonzero element $d$ such that

  (i) $d \mid a$ and $d \mid b$, and

  (ii) if $d' \mid a$ and $d' \mid b$ then $d' \mid d$.

A greatest common divisor of $a$ and $b$ will be denoted by gcd$(a, b)$

**Definition 2.2.** Let $R$ be an integral domain. $R$ is a $GCD$ domain (or $R$ has the $GCD$ property) if every $a, b \in R$ have a gcd in $R$.

**Lemma 2.1.** *Let $R$ be a $GCD$ domain. For any $a, b, c \in R$ we have: gcd(gcd$(a, b), c)$ = gcd$(a,$ gcd$(b, c))$.*

**Lemma 2.2.** *Let $R$ be a GCD domain, $a, b \in R$. If gcd$(a, b) \neq 0$, then*

$$\left(\frac{a}{\text{gcd}(a,b)}, \frac{b}{\text{gcd}(a,b)}\right) = 1$$

**Lemma 2.3.** *Let $R$ be a GCD domain. For any $a, b, c \in R$, if gcd$(a, b) = 1$ and $a \mid bc$, then $a \mid c$.*

Let $R$ be a ring and $A \in R_3(R)$ be a rhotrix of size 3.
We denote $Cen_{R_3(R)}(A) = \{B \in R_3(R) \mid B \circ A = A \circ B\}$ its centralizer in $R_3(R)$.

## 3 Main results

**Theorem 3.1.** *Let* $A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in R_3(R)$*, where R is a GCD domain. Then*

1. $Cen_{R_3(R)}(A) = R_3(R)$ *if* $a = e$ *and* $b = d = 0$.
2. $Cen_{R_3(R)}(A) = \left\{ \left\langle \begin{array}{ccc} & f & \\ bgq^{-1} & h & dgq^{-1} \\ & f-(a-e)gq^{-1} & \end{array} \right\rangle \mid f, g, h \in R \right\}$ *if*
   - $b = d = 0$ *and* $a \neq e$ *or*
   - $b = 0$ *and* $d \neq 0$ *or*
   - $b \neq 0$

   *where $q^{-1}$ is the inverse of $q = \gcd(a - e, b, d)$ in the quotient field of $R$.*

**Definition 3.1** (See [7])**.** Scalar rhotrices are rhotrices of the form $kI$ , where I is the identity rhotrix and $k$ is a non-zero scalar

**Corollary 3.2.** *Let $A \in R_3(R)$, where R is a GCD domain. If A is a scalar rhotrix then*

$$Cen_{R_3(R)}(A) = R_3(R)$$

To prove the theorem 3.1 we need the following lemmas:

**Lemma 3.3.** *Let* $A = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle \in R_3(F)$*, F a field. Then*

$Cen_{R_3(F)}(A) = R_3(F)$ *if* $a = e$ *and* $b = d = 0$

*Proof.* Assume $a = e$ and $b = d = 0$. Then

$$\left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle \in Cen_{R_3(F)}(A) \Leftrightarrow \left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & e & \\ 0 & c & 0 \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & e & \\ 0 & c & 0 \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle$$

$$\Leftrightarrow \left\langle \begin{matrix} & fe & \\ ge & hc & je \\ & ke & \end{matrix} \right\rangle = \left\langle \begin{matrix} & ef & \\ eg & ch & ej \\ & ek & \end{matrix} \right\rangle$$

Which is always true for all $f, g, h, j, k \in F$. Thus

$$Cen_{R_3(F)}(A) = R_3(F)$$

□

**Lemma 3.4.** *Let* $A = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle \in R_3(F)$*, F a field. Then* $Cen_{R_3(F)}(A) = \left\{ \left\langle \begin{matrix} & f & \\ 0 & h & 0 \\ & k & \end{matrix} \right\rangle \mid f, h, k \in F \right\}$ *if* $a \neq e$*,and* $b = d = 0$.

*Proof.* Assume that $b = d = 0$ and $a \neq e$. Then

$$\left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle \in Cen_{R_3(F)}(A) \Leftrightarrow \left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a & \\ 0 & c & 0 \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a & \\ 0 & c & 0 \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle$$

$$\Leftrightarrow \left\langle \begin{matrix} & fa & \\ ga & hc & je \\ & ke & \end{matrix} \right\rangle = \left\langle \begin{matrix} & af & \\ eg & ch & aj \\ & ek & \end{matrix} \right\rangle$$

Hence

$$\begin{cases} ga = eg \\ je = aj \end{cases} \Leftrightarrow \begin{cases} g(a-e) = 0 \\ j(a-e) = 0 \end{cases} \Leftrightarrow \begin{cases} g = 0 \\ j = 0 \end{cases}$$

Thus $Cen_{R_3(F)}(A) \subseteq \left\{ \left\langle \begin{matrix} & f & \\ 0 & h & 0 \\ & k & \end{matrix} \right\rangle : f, h, k \in F \right\}$

Direct calculation shows that for any $f, h, k \in F$,

$$\left\{ \left\langle \begin{matrix} & f & \\ 0 & h & 0 \\ & k & \end{matrix} \right\rangle \mid f, h, k \in F \right\} \subseteq Cen_{R_3(F)}(A)$$

We come to the conclusion that $Cen_{R_3(F)}(A) = \left\{ \left\langle \begin{matrix} & f & \\ 0 & h & 0 \\ & k & \end{matrix} \right\rangle \mid f, h, k \in F \right\}$

□

**Lemma 3.5.** *Let* $A = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle \in R_3(F)$, *F a field. Then*

$$Cen_{R_3(F)}(A) = \left\{ \left\langle \begin{matrix} & f & \\ g & h & 0 \\ & f - b^{-1}(a-e)g & \end{matrix} \right\rangle \mid f, g, h \in F \right\} \text{ if } d = 0 \text{ and } b \neq 0$$

*Proof.* Assume that $d = 0$ and $b \neq 0$ . Then

$$\left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle \in Cen_{R_3(F)}(A) \Leftrightarrow \left\langle \begin{matrix} & a & \\ b & c & 0 \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a & \\ b & c & 0 \\ & e & \end{matrix} \right\rangle$$

$$\Leftrightarrow \left\langle \begin{matrix} & af & \\ bf + eg & hc & aj \\ & bj + ek & \end{matrix} \right\rangle = \left\langle \begin{matrix} & fa + jb & \\ ga + kb & ch & je \\ & ke & \end{matrix} \right\rangle$$

So $\begin{cases} af = fa + jb \\ bf + eg = ga + kb \end{cases} \Leftrightarrow \begin{cases} j = 0 \\ k = f - b^{-1}(a-e)g \end{cases}$

Thus $Cen_{R_3(F)}(A) \subseteq \left\{ \left\langle \begin{matrix} & f & \\ g & h & 0 \\ & f - b^{-1}(a-e)g & \end{matrix} \right\rangle \mid f, g, h \in F \right\}$

Additionally, direct computation demonstrates that for every $f, g, h \in F$,

$$\left\{ \left\langle \begin{matrix} & f & \\ g & h & 0 \\ & f - b^{-1}(a-e)g & \end{matrix} \right\rangle : f, g, h \in F \right\} \subseteq Cen_{R_3(F)}(A)$$

We get to the conclusion that

$Cen_{R_3(F)}(A) = \left\{ \left\langle \begin{matrix} & f & \\ g & h & 0 \\ & f - b^{-1}(a-e)g & \end{matrix} \right\rangle \mid f, g, h \in F \right\}$

□

**Lemma 3.6.** *Let* $A = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle \in R_3(F)$, *F a field. Then*

$$Cen_{R_3(F)}(A) = \left\{ \left\langle \begin{matrix} & f & \\ d^{-1}bj & h & j \\ & f - d^{-1}(a-e)j & \end{matrix} \right\rangle \mid j, f, h \in F \right\} \text{ if } d \neq 0$$

*Proof.* Assume that $d \neq 0$. Then

$$\left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle \in Cen_{R_3(F)}(A) \Leftrightarrow \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & f & \\ g & h & j \\ & k & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle$$

$$\Leftrightarrow \left\langle \begin{matrix} & af + dg & \\ bf + eg & hc & aj + dk \\ & bj + ek & \end{matrix} \right\rangle = \left\langle \begin{matrix} & fa + jb & \\ ga + kb & ch & fd + je \\ & gd + ke & \end{matrix} \right\rangle$$

So $\begin{cases} af + dg = fa + jb \\ aj + dk = fd + je \end{cases} \Leftrightarrow \begin{cases} g = d^{-1}jb \\ k = f - d^{-1}(a-e)j \end{cases}$

Thus $Cen_{R_3(F)}(A) \subseteq \left\{ \left\langle \begin{array}{ccc} & f & \\ d^{-1}jb & h & j \\ & f - d^{-1}(a-e)j & \end{array} \right\rangle \mid j, f, h \in F \right\}$

Direct verification also demonstrates that for every arbitrary $j, f, h \in F$,

$$\left\{ \left\langle \begin{array}{ccc} & f & \\ d^{-1}jb & h & j \\ & f - d^{-1}(a-e)j & \end{array} \right\rangle \mid j, f, h \in F \right\} \subseteq Cen_{R_3(F)}(A)$$

We conclude that $Cen_{R_3(F)}(A) = \left\{ \left\langle \begin{array}{ccc} & f & \\ d^{-1}jb & h & j \\ & f - d^{-1}(a-e)j & \end{array} \right\rangle \mid j, f, h \in F \right\}$

□

**Lemma 3.7.** *Let $A \in R_3(R)$ , $R \subset F$, where F is a field and R is a subring of F. Then*

$$Cen_{R_3(R)}(A) = Cen_{R_3(F)}(A) \cap R$$

*Proof.*

$$B \in \mathrm{Cen}_{R_3(R)}(A) \Leftrightarrow B \in R \subseteq F \text{ and } B \circ A = A \circ B$$
$$\Leftrightarrow B \in R \cap Cen_F(A)$$

□

Proof of Theorem 3.1:

Let $R$ be a GCD domain and $F$ its field of fractions(quotient field) .

1. If $a = e$ and $b = d = 0$. Then by Lemma 3.3 and Lemma 3.7

$$Cen_{R_3(R)}(A) = Cen_{R_3(F)}(A) \cap R = R_3(F) \cap R = R_3(R)$$

2. (i) If $b = d = 0$ and $a \neq e$, in this case $q := \gcd(a-e, 0, 0) = a - e$. Consequently, it follows from Lemma 3.4 and Lemma 3.7 that

$$\begin{aligned} Cen_{R_3(R)}(A) &= \left\{ \left\langle \begin{array}{ccc} & f & \\ 0 & h & 0 \\ & k & \end{array} \right\rangle \mid f, h; k \in F \right\} \cap R_3(R) \\ &= \left\{ \left\langle \begin{array}{ccc} & f & \\ 0 & h & 0 \\ & f-g & \end{array} \right\rangle \mid f, g, h \in F \right\} \cap R_3(R) \\ &= \left\{ \left\langle \begin{array}{ccc} & f & \\ 0 & h & 0 \\ & f-g & \end{array} \right\rangle \mid f, g, h \in R \right\} \\ &= \left\{ \left\langle \begin{array}{ccc} & f & \\ 0g(a-e)^{-1} & h & 0g(a-e)^{-1} \\ & f-(a-e)(a-e)^{-1}g & \end{array} \right\rangle \mid f, g, h \in R \right\} \\ &= \left\{ \left\langle \begin{array}{ccc} & f & \\ bgq^{-1} & h & dgq^{-1} \\ & f-(a-e)(a-e)^{-1}g & \end{array} \right\rangle \mid f, g, h \in R \right\}. \end{aligned}$$

(ii) If $d = 0$ and $b \neq 0$, then it follows from Lemma 3.5 and Lemma 3.7 that

$$\begin{aligned} Cen_{R_3(R)}(A) &= \left\{ \left\langle \begin{array}{ccc} & f & \\ g & h & 0 \\ & f-b^{-1}(a-e)g & \end{array} \right\rangle \mid f, g, h \in F \right\} \cap R_3(R) \\ &= \left\{ \left\langle \begin{array}{ccc} & f & \\ g & h & 0 \\ & f-b^{-1}(a-e)g & \end{array} \right\rangle \mid f, g, h \in R \right\}. \end{aligned}$$

Let $B$ be any element of $Cen_{R_3(R)}(A)$. It follows from above that

$$B = \left\langle \begin{array}{ccc} & f & \\ g & h & 0 \\ & f - b^{-1}(a-e)g & \end{array} \right\rangle \in R_3(R)$$

for some $f, g, h \in R$. We will now demonstrate that

$$B = \left\langle \begin{array}{ccc} & f & \\ buq^{-1} & h & 0 \\ & f - (a-e)uq^{-1} & \end{array} \right\rangle$$

for some $u \in R$. Since

$$q := \gcd(a-e, d, b) = \gcd(a-e, 0, b) = \gcd(a-e, b)$$

it follows from Lemma 2.2 that

$$\begin{cases} b = qb' \\ a - e = ql \end{cases}$$

for some $b', l \in R$ such that $\gcd(b', l) = 1$. Because $g(a-e)b^{-1} \in R$, it follows that

$$g(a-e)b^{-1} = gql\left(qb'\right)^{-1} = \mathrm{gl}\left(b'\right)^{-1} \in R.$$

since $\gcd(b', l) = 1$ it follows from Lemma 2.3 that $b' \mid g$, which implies that

$$g = ub'$$

for some $u \in R$. Hence, it follows that

$$\begin{aligned} B &= \left\langle \begin{array}{ccc} & f & \\ ub' & h & 0 \\ & f - b^{-1}(a-e)g & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & f & \\ ub' & h & 0 \\ & f - u(a-e)q^{-1} & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & f & \\ ubq^{-1} & h & 0 \\ & f - u(a-e)q^{-1} & \end{array} \right\rangle \end{aligned}$$

Since

$$\begin{aligned} g(a-e)b^{-1} &= gql\left(qb'\right)^{-1} \\ &= gl\left(b'\right)^{-1} \\ &= ub'l\left(b'\right)^{-1} \\ &= ul \\ &= u(a-e)q^{-1} \end{aligned}$$

Thus,

$$Cen_{R_3(R)}(A) \subseteq \left\{ \left\langle \begin{array}{ccc} & f & \\ bgq^{-1} & h & 0 \\ & f - g(a-e)q^{-1} & \end{array} \right\rangle \mid f, g, h \in R \right\}$$

and by letting $c = bgq^{-1}$, then $f - g(a-e)q^{-1} = f - (a-e)b^{-1}c$, hence

$$\left\{ \left\langle \begin{matrix} & f & \\ bgd^{-1} & \alpha & 0 \\ & f-g(a-e)q^{-1} & \end{matrix} \right\rangle \mid f,g,h \in R \right\} \subseteq \left\{ \left\langle \begin{matrix} & f & \\ c & h & 0 \\ & f-(a-e)b^{-1}c & \end{matrix} \right\rangle \mid f,c,h \in R \right\}$$
$$= Cen_{R_3(R)}(A)$$

Therefore, we conclude that

$$\begin{aligned} Cen_{R_3(R)}(A) &= \left\{ \left\langle \begin{matrix} & f & \\ gbq^{-1} & h & 0 \\ & f-(a-e)gq^{-1} & \end{matrix} \right\rangle \mid f,g,h \in R \right\} \\ &= \left\{ \left\langle \begin{matrix} & a & \\ bgq^{-1} & h & dgq^{-1} \\ & f-(a-e)gq^{-1} & \end{matrix} \right\rangle \mid f,g,h \in R \right\} \end{aligned}$$

(iii) If $d \neq 0$ then, it follows from Lemma 3.6 and Lemma 3.7 that

$$Cen_{R_3(R)}(A) = \left\{ \left\langle \begin{matrix} & a & \\ d^{-1}bj & h & j \\ & f-d^{-1}(a-e)j & \end{matrix} \right\rangle \mid f,h,j \in F \right\} \cap R_3(R)$$

$$= \left\{ \left\langle \begin{matrix} & f & \\ d^{-1}bj & h & j \\ & f-d^{-1}(a-e)j & \end{matrix} \right\rangle \mid f,h,j \in R \right\}.$$

Let $B$ be an arbitrary element of $Cen_{R_3(R)}(A)$. Then it follows that

$$B = \left\langle \begin{matrix} & f & \\ d^{-1}bj & h & j \\ & f-(a-e)jd^{-1} & \end{matrix} \right\rangle$$

for some $f,h,j \in R$. We now show that

$$B = \left\langle \begin{matrix} & f & \\ bgq^{-1} & h & dgq^{-1} \\ & f-(a-e)gq^{-1} & \end{matrix} \right\rangle$$

for some $f,g,h \in R$. Now, let

$$d_1 := \gcd(b,d).$$

Then by Lemma 2.2

$$\begin{cases} b = d_1 b' \\ d = d_1 d' \end{cases}$$

for some $b', d' \in R$ such that $\gcd(b', d') = 1$. Since,

$$bjd^{-1} = d_1 b' j \left(d_1 d'\right)^{-1} = b' j \left(d'\right)^{-1} \in R$$

hence, $b'j = \alpha d', \alpha \in R$ and since $\gcd(b', d') = 1$, it follows by Lemma 2.3 that $d' \mid j$. Thus

$$j = d'j'$$

for some $j' \in R$. Hence, it follows that

$$bjd^{-1} = d_1 b' d' j' \left(d_1 d'\right)^{-1} = b'j'.$$

Furthermore, it follows that

$$(a-e)jd^{-1} = (a-e)j'd'\left(d_1 d'\right)^{-1} = (a-e)j'd_1^{-1}$$

and so, that

$$\begin{aligned} B &= \left\langle \begin{matrix} & f & \\ d^{-1}bj & h & j \\ & f-(a-e)jd^{-1} & \end{matrix} \right\rangle \\ &= \left\langle \begin{matrix} & f & \\ b'j' & h & d'j' \\ & f-(a-e)j'd_1^{-1} & \end{matrix} \right\rangle \end{aligned}$$

Since $q := \gcd(b,d,a-e)$ and by Lemma 2.1 $q = \gcd(\gcd(b,d), a-e) = \gcd\left(d_1, a-e\right)$, it follows by Lemma 2.2 that

$$\begin{cases} d_1 = d_1'q \\ (a-e) = lq \end{cases}$$

for some $d_1', l \in R$ such that $\gcd\left(d_1', l\right) = 1$. Since,

$$(a-e)j'd_1^{-1} = (a-e)j'\left(d_1'q\right)^{-1} = lqj'q^{-1}(d_1')^{-1} = lj'\left(d_1'\right)^{-1} \in R$$

hence,

$$lj' = d_1'\alpha$$

for some $\alpha \in R$ thus $d_1' \mid lj'$ and since $\gcd\left(d_1', l\right) = 1$, it follows by Lemma 2.3 that $d_1' \mid j'$. Therefore

$$j' = cd_1'$$

for some $c \in R$. Thus,

$$(a-e)j'd_1^{-1} = lqj'\left(d_1'\right)^{-1} = lqcd_1'\left(d_1'q\right)^{-1} = lc.$$

since $\begin{cases} d_1 = d_1'q \\ a-e = lq \end{cases}$ implies $\begin{cases} d_1' = d_1q^{-1} \\ l = (a-e)q^{-1} \end{cases}$ it follows that

$$\begin{aligned} B &= \left\langle \begin{matrix} & f & \\ b'cd_1' & h & d'cd_1' \\ & f-lc & \end{matrix} \right\rangle \\ &= \left\langle \begin{matrix} & f & \\ b'cq^{-1}d_1 & h & d'cq^{-1}d_1 \\ & f-(a-e)cq^{-1} & \end{matrix} \right\rangle \\ &= \left\langle \begin{matrix} & f & \\ (b'd_1)cq^{-1} & h & (d'd_1)cq^{-1} \\ & f-(a-e)cq^{-1} & \end{matrix} \right\rangle \\ &= \left\langle \begin{matrix} & f & \\ bcq^{-1} & h & dcq^{-1} \\ & f-(a-e)cq^{-1} & \end{matrix} \right\rangle \end{aligned}$$

Thus,

$$Cen_{R_3(R)}(A) \subseteq \left\{ \left\langle \begin{matrix} & f & \\ bgq^{-1} & h & dgq^{-1} \\ & f-(a-e)gq^{-1} & \end{matrix} \right\rangle \mid f,g,h \in R \right\}$$

and

$$\left\{ \left\langle \begin{array}{ccc} & f & \\ bgq^{-1} & h & dgq^{-1} \\ & f-(a-e)gq^{-1} & \end{array} \right\rangle \mid f,g,h \in R \right\} =$$

$$\left\{ \left\langle \begin{array}{ccc} & f & \\ bd^{-1}(dgq^{-1}) & h & dgq^{-1} \\ & f-(a-e)gq^{-1} & \end{array} \right\rangle \mid f,g,h \in R \right\} \subseteq$$

$$\left\{ \left\langle \begin{array}{ccc} & f & \\ bd^{-1}j & h & j \\ & f-(a-e)d^{-1}j & \end{array} \right\rangle \mid f,h,j \in R \right\} =$$

$$Cen_{R_3(R)}(A)$$

Hence we conclude that

$$Cen_{R_3(R)}(A) = \left\{ \left\langle \begin{array}{ccc} & f & \\ gbq^{-1} & h & dgq^{-1} \\ & f-(a-e)gq^{-1} & \end{array} \right\rangle \mid f,g,h \in R \right\}$$

# 4 Examples

**Example 4.1.** *Let $R$ be the gcd domain $\mathbb{Z}$ and let* $A = \left\langle \begin{array}{ccc} & 2 & \\ 6 & 5 & 3 \\ & 8 & \end{array} \right\rangle$. *It follows from Theorem 3.1 that*

$$\mathrm{Cen}_{R_3(\mathbb{Z})}(A) = \left\{ \left\langle \begin{array}{ccc} & a & \\ \frac{6}{3}b & c & \frac{3}{3}b \\ & a+\frac{6}{3}b & \end{array} \right\rangle \mid a,b,c \in \mathbb{Z} \right\} = \left\{ \left\langle \begin{array}{ccc} & a & \\ 2b & c & b \\ & a+2b & \end{array} \right\rangle \mid a,b,c \in \mathbb{Z} \right\}.$$

**Example 4.2.** *Let $R$ be the gcd domain $\mathbb{Z}$ and let* $A = \left\langle \begin{array}{ccc} & 6 & \\ 0 & 1 & 0 \\ & 8 & \end{array} \right\rangle$. *It follows from Theorem 3.1 that*

$$\mathrm{Cen}_{R_3(\mathbb{Z})}(A) = \left\{ \left\langle \begin{array}{ccc} & a & \\ 0 & c & 0 \\ & a-(6-8)\frac{1}{2}b & \end{array} \right\rangle \mid a,b,c \in \mathbb{Z} \right\} = \left\{ \left\langle \begin{array}{ccc} & a & \\ 0 & c & 0 \\ & a+b & \end{array} \right\rangle \mid a,b,c \in \mathbb{Z} \right\}.$$

# 5 Conclusion

In this paper, the concept of the centralizer of a matrix has been expanded to rhotrix. We also describe with more details the centralizer of a 3-dimensional rhotrix with entries from a field and a gcd domain with some examples.

# 6 Future Works

- Generalizing the above work to higher dimensional rhotrices ($n > 3$)
- Solving the rhotrix equation

$$A \circ X - X \circ B = C$$

where $A, B, C \in R_n(K)$

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