

# Generalized Friedrich Numbers

**Abstract.** In this paper, we define and investigate the generalized Friedrich sequences and we deal with, in detail, two special cases, namely, Friedrich and Friedrich-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Friedrich, Friedrich-Lucas and third order Jacobsthal, modified third-order Jacobsthal, third order Jacobsthal-Lucas numbers.

**2020 Mathematics Subject Classification.** 11B37, 11B39, 11B83.

**Keywords.** Friedrich numbers, Friedrich-Lucas numbers, Jacobsthal numbers, third order Jacobsthal numbers, modified third-order Jacobsthal numbers, third order Jacobsthal-Lucas numbers.

## 1. Introduction

Third-order Jacobsthal sequence  $\{J_n\}_{n \geq 0}$  (OEIS: A077947, [32]), modified third-order Jacobsthal sequence  $\{K_n\}_{n \geq 0}$  (OEIS: A186575, [32]) and third-order Jacobsthal-Lucas sequence  $\{j_n\}_{n \geq 0}$  (OEIS: A226308, [32]) are defined, respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2} + J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, J_2 = 1, \quad (1.1)$$

$$K_{n+3} = K_{n+2} + K_{n+1} + 2K_n, \quad K_0 = 3, K_1 = 1, K_2 = 3. \quad (1.2)$$

$$j_{n+3} = j_{n+2} + j_{n+1} + 2j_n, \quad j_0 = 2, j_1 = 1, j_2 = 5, \quad (1.3)$$

The sequences  $\{J_n\}_{n \geq 0}$  and  $\{j_n\}_{n \geq 0}$  are defined in [8] and  $\{K_n\}_{n \geq 0}$  is given in [5]. For more details on the generalized third-order Jacobsthal numbers and its special cases, see [40].

The sequences  $\{J_n\}_{n \geq 0}$ ,  $\{K_n\}_{n \geq 0}$  and  $\{j_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} + \frac{1}{2}J_{-(n-3)}, \\ K_{-n} &= -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)}, \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} + \frac{1}{2}j_{-(n-3)}, \end{aligned}$$

for  $n = 1, 2, 3, \dots$  respectively. Therefore, recurrences (1.1)-(1.3) hold for all integer  $n$ .

Now, we define two sequences related to third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers. Friedrich and Friedrich-Lucas numbers are defined as

$$F_n = F_{n-1} + F_{n-2} + 2F_{n-3} + 1, \quad \text{with } F_0 = 0, F_1 = 1, F_2 = 2, \quad n \geq 3, \quad (1.4)$$

and

$$C_n = C_{n-1} + C_{n-2} + 2C_{n-3} - 3, \quad \text{with } C_0 = 4, C_1 = 2, C_2 = 4, \quad n \geq 3, \quad (1.5)$$

respectively. The first few values of Friedrich and Friedrich-Lucas numbers are

$$0, 1, 2, 4, 9, 18, 36, 73, 146, 292, 585, 1170, 2340, 4681, \dots$$

and

$$4, 2, 4, 11, 16, 32, 67, 128, 256, 515, 1024, 2048, 4099, 8192, \dots$$

respectively. The sequences  $\{F_n\}$  and  $\{C_n\}$  satisfy the following fourth order linear recurrences:

$$F_n = 2F_{n-1} + F_{n-3} - 2F_{n-4}, \quad F_0 = 0, F_1 = 1, F_2 = 2, F_3 = 4, \quad n \geq 4, \quad (1.6)$$

$$C_n = 2C_{n-1} + C_{n-3} - 2C_{n-4}, \quad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 11, \quad n \geq 4. \quad (1.7)$$

There are close relations between Friedrich, Friedrich-Lucas and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} 3F_n &= J_{n+2} + 2J_n - 1, \\ 2C_n &= -J_{n+2} + 7J_{n+1} - 3J_n + 2, \\ 147F_n &= 17K_{n+2} + 10K_{n+1} - 4K_n - 49, \\ C_n &= K_n + 1, \\ 18F_n &= j_{n+2} + 3j_{n+1} - j_n - 6, \\ 24C_n &= 11j_{n+2} - 21j_{n+1} + 19j_n + 24, \end{aligned}$$

and

$$\begin{aligned}
J_{n+1} &= F_{n+1} - F_n, \\
147J_n &= 19C_{n+2} - 9C_{n+1} - 16C_n + 6, \\
4K_n &= F_{n+2} + 13F_{n+1} - 23F_n - 3, \\
3K_n &= C_{n+3} - C_{n+2} - C_{n+1} + C_n, \\
j_n &= -F_{n+3} + 3F_{n+2} - 2F_n, \\
49j_n &= -5C_{n+2} + 23C_{n+1} + 30C_n - 48.
\end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Friedrich, Friedrich-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized  $(r, s, t, u)$  sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence)  $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n \geq 0}$  (or shortly  $\{W_n\}_{n \geq 0}$ ) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \quad (1.8)$$

where  $W_0, W_1, W_2, W_3$  are arbitrary complex (or real) numbers and  $r, s, t, u$  are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [16,22,24,30,35,37,38,62,63]. The sequence  $\{W_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for  $n = 1, 2, 3, \dots$  when  $u \neq 0$ . Therefore, recurrence (1.8) holds for all integers  $n$ .

As  $\{W_n\}$  is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \quad (1.9)$$

whose roots are  $\alpha, \beta, \gamma, \delta$ . Note that we have the following identities

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= r, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\
\alpha\beta\gamma\delta &= -u.
\end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows (which can be found in the literature, for completeness, we include the proof):

THEOREM 1. (Four Distinct Roots Case:  $\alpha \neq \beta \neq \gamma \neq \delta$ ) For all integers  $n$ , Binet's formula of generalized Tetranacci numbers is

$$W_n = \frac{p_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{p_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{p_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{p_4 \delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)} \quad (1.10)$$

where

$$\begin{aligned} p_1 &= W_3 - (\beta + \gamma + \delta)W_2 + (\beta\gamma + \beta\delta + \gamma\delta)W_1 - \beta\gamma\delta W_0, \\ p_2 &= W_3 - (\alpha + \gamma + \delta)W_2 + (\alpha\gamma + \alpha\delta + \gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha + \beta + \delta)W_2 + (\alpha\beta + \alpha\delta + \beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - (\alpha + \beta + \gamma)W_2 + (\alpha\beta + \alpha\gamma + \beta\gamma)W_1 - \alpha\beta\gamma W_0. \end{aligned}$$

Proof. If the roots  $\alpha, \beta, \gamma, \delta$  of (1.10) are distinct, then (the sequences  $(\alpha^n)_{n \geq 0}$ ,  $(\beta^n)_{n \geq 0}$ ,  $(\gamma^n)_{n \geq 0}$  and  $(\delta^n)_{n \geq 0}$  are solutions of (1.8) and) the general formula of  $W_n$  is in the following form:

$$W_n = A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n + A_4 \delta^n$$

where the coefficients  $A_1, A_2, A_3$  and  $A_4$  are determined by the system of linear equations

$$\begin{aligned} W_0 &= A_1 + A_2 + A_3 + A_4 \\ W_1 &= A_1 \alpha + A_2 \beta + A_3 \gamma + A_4 \delta \\ W_2 &= A_1 \alpha^2 + A_2 \beta^2 + A_3 \gamma^2 + A_4 \delta^2 \\ W_3 &= A_1 \alpha^3 + A_2 \beta^3 + A_3 \gamma^3 + A_4 \delta^3 \end{aligned}$$

Solving these four simultaneous equations for  $W_0, W_1, W_2$  and  $W_3$ , we obtain the required result.  $\square$

Usually, it is customary to choose  $\alpha, \beta, \gamma, \delta$  so that the Equ. (1.9) has at least one real (say  $\alpha$ ) solutions. Note that the Binet form of a sequence satisfying (1.9) for non-negative integers is valid for all integers  $n$  (see [17]).

Next, we consider two special cases of the generalized  $(r, s, t, u)$  sequence  $\{W_n\}$  which we call them  $(r, s, t, u)$ -Fibonacci and  $(r, s, t, u)$ -Lucas sequences.  $(r, s, t, u)$ -Fibonacci sequence  $\{G_n\}_{n \geq 0}$  and  $(r, s, t, u)$ -Lucas sequence  $\{H_n\}_{n \geq 0}$  are defined, respectively, by the fourth-order recurrence relations

$$G_{n+4} = rG_{n+3} + sG_{n+2} + tG_{n+1} + uG_n, \quad (1.11)$$

$$G_0 = 0, G_1 = 1, G_2 = r, G_3 = r^2 + s,$$

$$H_{n+4} = rH_{n+3} + sH_{n+2} + tH_{n+1} + uH_n, \quad (1.12)$$

$$H_0 = 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t.$$

The sequences  $\{G_n\}_{n \geq 0}$  and  $\{H_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)} - \frac{s}{u}G_{-(n-2)} - \frac{r}{u}G_{-(n-3)} + \frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)} - \frac{s}{u}H_{-(n-2)} - \frac{r}{u}H_{-(n-3)} + \frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for  $n = 1, 2, 3, \dots$  respectively. Therefore, recurrences (1.11) and (1.12) hold for all integers  $n$ .

For all integers  $n$ ,  $(r, s, t, u)$ -Fibonacci and  $(r, s, t, u)$ -Lucas numbers (using initial conditions in (1.11) or (1.12)) can be expressed using Binet's formulas as in the following corollary (by setting  $W_n = G_n$  and  $W_n = H_n$  in Theorem 1, respectively).

**COROLLARY 2.** *(Four Distinct Roots Case:  $\alpha \neq \beta \neq \gamma \neq \delta$ ) Binet's formula of  $(r, s, t, u)$ -Fibonacci and  $(r, s, t, u)$ -Lucas numbers are*

$$G_n = \frac{\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^{n+2}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^{n+2}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^{n+2}}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

respectively.

Next, we give the ordinary generating function  $\sum_{n=0}^{\infty} W_n z^n$  of the sequence  $W_n$ .

**LEMMA 3.** [35, Lemma 1] *Suppose that  $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$  is the ordinary generating function of the generalized  $(r, s, t, u)$  sequence  $\{W_n\}_{n \geq 0}$ . Then,  $\sum_{n=0}^{\infty} W_n z^n$  is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{1 - rz - sz^2 - tz^3 - uz^4}. \quad (1.13)$$

The following theorem presents Simson's formula of generalized  $(r, s, t, u)$  sequence (generalized Tetranacci sequence)  $\{W_n\}$ .

**THEOREM 4** (Simson's Formula of Generalized  $(r, s, t, u)$  Numbers). [34] *For all integers  $n$ , we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \quad (1.14)$$

The following theorem shows that the generalized Tetranacci sequence  $W_n$  at negative indices can be expressed by the sequence itself at positive indices.

THEOREM 5. [36, Theorem 1.] For  $n \in \mathbb{Z}$ , for the generalized Tetranacci sequence (or generalized  $(r, s, t, u)$ -sequence or 4-step Fibonacci sequence) we have the following:

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_nW_{2n} - 3H_n^2W_n + 3H_{2n}W_n + W_0H_n^3 + 2W_0H_{3n} - 3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n} - H_nW_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Using Theorem 5, we have the following corollary.

COROLLARY 6. [36, Corollary 4] For  $n \in \mathbb{Z}$ , we have

$$\begin{aligned} \text{(a): } 2(-u)^{n+4}G_{-n} &= -(3ru^2 + t^3 - 3stu)^2G_n^3 - (2su - t^2)^2G_{n+3}^2G_n - (-rt^2 - tu + 2rsu)^2G_{n+2}^2G_n \\ &\quad - (-st^2 + 2s^2u + 4u^2 + rtu)^2G_{n+1}^2G_n + 2(3ru^2 + t^3 - 3stu)((-2su + t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + \\ &\quad (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1})G_n^2 + 2(2su - t^2)(-rt^2 - tu + 2rsu)G_{n+3}G_{n+2}G_n + 2(2su - t^2)(-st^2 + \\ &\quad 2s^2u + 4u^2 + rtu)G_{n+3}G_{n+1}G_n - 2(-st^2 + 2s^2u + 4u^2 + rtu)(-rt^2 - tu + 2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + \\ &\quad u^2(-2su + t^2)G_{2n+3}G_n + u^2(-rt^2 - tu + 2rsu)G_{2n+2}G_n + u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n+1}G_n - \\ &\quad 2u^2(2su - t^2)G_{2n}G_{n+3} + 2u^2(-rt^2 - tu + 2rsu)G_{2n}G_{n+2} + 2u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n}G_{n+1} - \\ &\quad 3u^2(3ru^2 + t^3 - 3stu)G_{2n}G_n. \end{aligned}$$

$$\text{(b): } H_{-n} = \frac{1}{6}(-u)^{-n}(H_n^3 + 2H_{3n} - 3H_{2n}H_n).$$

Note that  $G_{-n}$  and  $H_{-n}$  can be given as follows by using  $G_0 = 0$  and  $H_0 = 4$  in Theorem 5,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n} + 6H_nG_{2n} - 3H_n^2G_n + 3H_{2n}G_n), \quad (1.15)$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}(H_n^3 + 2H_{3n} - 3H_{2n}H_n), \quad (1.16)$$

respectively.

If we define the square matrix  $A$  of order 4 as

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (1.17)$$

and also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix} \quad (1.18)$$

and

$$U_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix} \quad (1.19)$$

then we get the following Theorem.

**THEOREM 7.** [35, Theorem 19] For all integers  $m, n$ , we have

(a):  $B_n = A^n$ , i.e.,

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}. \quad (1.20)$$

(b):  $U_1 A^n = A^n U_1$ .

(c):  $U_{n+m} = U_n B_m = B_m U_n$ .

**THEOREM 8.** [35, Theorem 20] For all integers  $m, n$ , we have

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \quad (1.21)$$

In the next sections, we present new results.

## 2. Generalized Friedrich Sequence

In this paper, we consider the case  $r = 2, s = 0, t = 1, u = -2$ . A generalized Friedrich sequence  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n \geq 0}$  is defined by the fourth-order recurrence relation

$$W_n = 2W_{n-1} + W_{n-3} - 2W_{n-4} \quad (2.1)$$

with the initial values  $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$  not all being zero. The sequence  $\{W_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$W_{-n} = \frac{1}{2}W_{-(n-1)} + W_{-(n-3)} - \frac{1}{2}W_{-(n-4)}$$

for  $n = 1, 2, 3, \dots$ . Therefore, recurrence (2.1) holds for all integers  $n$ .

Characteristic equation of  $\{W_n\}$  is

$$z^4 - 2z^3 - z + 2 = (z^3 - z^2 - z - 2)(z - 1) = (z^2 + z + 1)(z - 2)(z - 1) = 0$$

whose roots are

$$\begin{aligned}\alpha &= 2, \\ \beta &= \frac{-1 + i\sqrt{3}}{2}, \\ \gamma &= \frac{-1 - i\sqrt{3}}{2}, \\ \delta &= 1.\end{aligned}$$

Note that

$$\begin{aligned}\alpha + \beta + \gamma + \delta &= 2, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 0, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\ \alpha\beta\gamma\delta &= 2.\end{aligned}$$

Note also that

$$\begin{aligned}\alpha + \beta + \gamma &= 1, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 2.\end{aligned}$$

The first few generalized Friedrich numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Friedrich numbers

$n$	$W_n$	$W_{-n}$
0	$W_0$	$W_0$
1	$W_1$	$\frac{1}{2}(W_0 + 2W_2 - W_3)$
2	$W_2$	$\frac{1}{4}(W_0 + 4W_1 - W_3)$
3	$W_3$	$\frac{1}{8}(9W_0 - W_3)$
4	$W_1 - 2W_0 + 2W_3$	$\frac{1}{16}(9W_0 + 16W_2 - 9W_3)$
5	$W_2 - 4W_0 + 4W_3$	$\frac{1}{32}(9W_0 + 32W_1 - 9W_3)$
6	$9W_3 - 8W_0$	$\frac{1}{64}(73W_0 - 9W_3)$
7	$W_1 - 18W_0 + 18W_3$	$\frac{1}{128}(73W_0 + 128W_2 - 73W_3)$
8	$W_2 - 36W_0 + 36W_3$	$\frac{1}{256}(73W_0 + 256W_1 - 73W_3)$
9	$73W_3 - 72W_0$	$\frac{1}{512}(585W_0 - 73W_3)$
10	$W_1 - 146W_0 + 146W_3$	$\frac{1}{1024}(585W_0 + 1024W_2 - 585W_3)$
11	$W_2 - 292W_0 + 292W_3$	$\frac{1}{2048}(585W_0 + 2048W_1 - 585W_3)$
12	$585W_3 - 584W_0$	$\frac{1}{4096}(4681W_0 - 585W_3)$
13	$W_1 - 1170W_0 + 1170W_3$	$\frac{1}{8192}(4681W_0 + 8192W_2 - 4681W_3)$

Note that the sequences  $\{F_n\}$  and  $\{C_n\}$  which are defined in the section Introduction, are the special cases of the generalized Friedrich sequence  $\{W_n\}$ . For convenience, we can give the definition of these two special cases of the sequence  $\{W_n\}$  in this section as well. Friedrich sequence  $\{F_n\}_{n \geq 0}$  and Friedrich-Lucas sequence  $\{C_n\}_{n \geq 0}$  are defined, respectively, by the fourth-order recurrence relations

$$F_n = 2F_{n-1} + F_{n-3} - 2F_{n-4}, \quad F_0 = 0, F_1 = 1, F_2 = 2, F_3 = 4, \quad n \geq 4, \tag{2.2}$$

$$C_n = 2C_{n-1} + C_{n-3} - 2C_{n-4}, \quad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 11, \quad n \geq 4. \tag{2.3}$$

The sequences  $\{F_n\}_{n \geq 0}$  and  $\{C_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$F_{-n} = \frac{1}{2}F_{-(n-1)} + F_{-(n-3)} - \frac{1}{2}F_{-(n-4)},$$

$$C_{-n} = \frac{1}{2}C_{-(n-1)} + C_{-(n-3)} - \frac{1}{2}C_{-(n-4)},$$

for  $n = 1, 2, 3, \dots$  respectively.

Next, we present the first few values of the Friedrich and Friedrich-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$F_n$	0	1	2	4	9	18	36	73	146	292	585	1170	2340	4681
$F_{-n}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{9}{16}$	$-\frac{9}{32}$	$-\frac{9}{64}$	$-\frac{73}{128}$	$-\frac{73}{256}$	$-\frac{73}{512}$	$-\frac{585}{1024}$	$-\frac{585}{2048}$
$C_n$	4	2	4	11	16	32	67	128	256	515	1024	2048	4099	8192
$C_{-n}$	4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{25}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{193}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1537}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{12289}{4096}$	$\frac{1}{8192}$

Theorem 1 can be used to obtain the Binet formula of generalized Friedrich numbers. Using these (the above) roots and the recurrence relation, Binet's formula of generalized Friedrich numbers can be given as follows:

**THEOREM 9.** *(Four Distinct Roots Case:  $\alpha \neq \beta \neq \gamma \neq \delta = 1$ ) For all integers  $n$ , Binet's formula of generalized Friedrich numbers is*

$$\begin{aligned} W_n = & \frac{(\alpha W_3 - \alpha(2 - \alpha)W_2 + (-\alpha^2 + \alpha + 2)W_1 - 2W_0)\alpha^n}{2\alpha^2 + 5\alpha - 4} \\ & + \frac{(\beta W_3 - \beta(2 - \beta)W_2 + (-\beta^2 + \beta + 2)W_1 - 2W_0)\beta^n}{2\beta^2 + 5\beta - 4} \\ & + \frac{(\gamma W_3 - \gamma(2 - \gamma)W_2 + (-\gamma^2 + \gamma + 2)W_1 - 2W_0)\gamma^n}{2\gamma^2 + 5\gamma - 4} \\ & + \frac{W_3 - W_2 - W_1 - 2W_0}{-3}. \end{aligned}$$

Friedrich and Friedrich-Lucas numbers can be expressed using Binet's formulas as follows:

**COROLLARY 10.** *(Four Distinct Roots Case:  $\alpha \neq \beta \neq \gamma \neq \delta = 1$ ) For all integers  $n$ , Binet's formulas of Friedrich and Friedrich-Lucas numbers are*

$$\begin{aligned} F_n &= \frac{(\alpha^2 + \alpha + 2)\alpha^n}{2\alpha^2 + 5\alpha - 4} + \frac{(\beta^2 + \beta + 2)\beta^n}{2\beta^2 + 5\beta - 4} + \frac{(\gamma^2 + \gamma + 2)\gamma^n}{2\gamma^2 + 5\gamma - 4} - \frac{1}{3} \\ &= \frac{1}{7} \times 2^{n+2} - \frac{1}{42}(5 + i\sqrt{3}) \left( \frac{-1 + i\sqrt{3}}{2} \right)^n - \frac{1}{42}(5 - i\sqrt{3}) \left( \frac{-1 - i\sqrt{3}}{2} \right)^n - \frac{1}{3}, \end{aligned} \quad (2.4)$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1 = 2^n + \left( \frac{-1 + i\sqrt{3}}{2} \right)^n + \left( \frac{-1 - i\sqrt{3}}{2} \right)^n + 1, \quad (2.5)$$

respectively.

Note that for all integers  $n$ , third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$J_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)} \quad (2.6)$$

$$K_n = \alpha^n + \beta^n + \gamma^n \quad (2.7)$$

$$j_n = \frac{(2\alpha^2 - \alpha + 2)\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(2\beta^2 - \beta + 2)\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{(2\gamma^2 - \gamma + 2)\gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \quad (2.8)$$

respectively, see Soykan [40] for more details. So, by using Binet's formulas of Friedrich, Friedrich-Lucas and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

**LEMMA 11.** *For all integers  $n$ , the following equalities (identities) are true:*

**(a):**

- $J_{n+1} = F_{n+1} - F_n.$

- $2J_n = F_{n+3} - 2F_{n+2} + F_n$ .
- $3F_{n+4} = 13J_{n+2} + 15J_{n+1} + 14J_n - 1$ .
- $3F_n = J_{n+2} + 2J_n - 1$ .
- $2J_n = -F_{n+2} + F_{n+1} + 3F_n + 1$ .

(b):

- $147J_{n+3} = 40C_{n+3} + 21C_{n+2} - 7C_{n+1} - 54C_n$ .
- $147J_n = -2C_{n+3} + 21C_{n+2} - 7C_{n+1} - 12C_n$ .
- $C_{n+4} = 10J_{n+2} + 5J_{n+1} + 6J_n + 1$ .
- $2C_n = -J_{n+2} + 7J_{n+1} - 3J_n + 2$ .
- $147J_n = 19C_{n+2} - 9C_{n+1} - 16C_n + 6$ .
- $C_{n+1} + 6C_n = 19J_{n+1} - 10J_n + 7$ .

(c):

- $K_{n+3} = F_{n+3} + F_{n+2} + 4F_{n+1} - 6F_n$ .
- $4K_n = -3F_{n+3} + 4F_{n+2} + 16F_{n+1} - 17F_n$ .
- $147F_{n+4} = 195K_{n+2} + 181K_{n+1} + 202K_n - 49$ .
- $147F_n = 17K_{n+2} + 10K_{n+1} - 4K_n - 49$ .
- $4K_n = F_{n+2} + 13F_{n+1} - 23F_n - 3$ .
- $3(17F_{n+1} - 27F_n) = 14K_n - K_{n+1} + 10$ .

(d):

- $3K_{n+3} = 4C_{n+3} - C_{n+2} - C_{n+1} - 2C_n$ .
- $3K_n = C_{n+3} - C_{n+2} - C_{n+1} + C_n$ .
- $C_{n+4} = 2K_{n+2} + 3K_{n+1} + 2K_n + 1$ .
- $C_n = K_n + 1$ .
- $K_n = C_n - 1$ .

(e):

- $j_{n+3} = F_{n+3} + 3F_{n+2} - 4F_n$ .
- $j_n = -F_{n+3} + 3F_{n+2} - 2F_n$ .
- $9F_{n+4} = 11j_{n+2} + 10j_n + 9j_{n+1} - 3$ .
- $18F_n = j_{n+2} + 3j_{n+1} - j_n - 6$ .
- $j_n = 2F_{n+2} - F_{n+1} - 4F_n - 1$ .
- $3(F_{n+1} - 4F_n) = -2j_{n+1} + j_n + 3$ .

(f):

- $49j_{n+3} = 72C_{n+3} - 21C_{n+2} + 7C_{n+1} - 58C_n$ .
- $49j_n = 16C_{n+3} - 21C_{n+2} + 7C_{n+1} - 2C_n$ .
- $3C_{n+4} = 4j_{n+2} + 8j_n + 9j_{n+1} + 3$ .
- $24C_n = 11j_{n+2} - 21j_{n+1} + 19j_n + 24$ .

- $49j_n = -5C_{n+2} + 23C_{n+1} + 30C_n - 48$ .
- $11C_{n+1} + 10C_n = 5j_{n+1} + 18j_n + 21$ .

Next, we give the ordinary generating function  $\sum_{n=0}^{\infty} W_n z^n$  of the sequence  $W_n$  (by setting  $r = 2, s = 0, t = 1, u = -2$  in Lemma 3).

LEMMA 12. Suppose that  $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$  is the ordinary generating function of the generalized Friedrich sequence  $\{W_n\}$ . Then,  $\sum_{n=0}^{\infty} W_n z^n$  is given by

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 2W_0)z + (W_2 - 2W_1)z^2 + (W_3 - 2W_2 - W_0)z^3}{1 - 2z - z^3 + 2z^4}. \quad (2.9)$$

The previous lemma gives the following results as particular examples.

COROLLARY 13. Generating functions of Friedrich and Friedrich-Lucas numbers are

$$\sum_{n=0}^{\infty} F_n z^n = \frac{z}{1 - 2z - z^3 + 2z^4} = \frac{z}{(2z^3 + z^2 + z - 1)(z - 1)}, \quad (2.10)$$

$$\sum_{n=0}^{\infty} C_n z^n = \frac{4 - 6z - z^3}{1 - 2z - z^3 + 2z^4} = \frac{4 - 6z - z^3}{(2z^3 + z^2 + z - 1)(z - 1)}, \quad (2.11)$$

respectively.

### 3. Simson Formulas

Now, we present Simson's formula of generalized Friedrich numbers (by setting  $r = 2, s = 0, t = 1, u = -2$  in Theorem 4).

THEOREM 14 (Simson's Formula of Generalized Friedrich Numbers). For all integers  $n$ , we have

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = 2^{n-3} \times (W_3 - W_0)(W_3 - W_2 - W_1 - 2W_0)(W_3^2 + 7W_2^2 + 7W_1^2 + 4W_0^2 - 5W_2W_3 + W_1W_3 - 7W_1W_2 + 2W_0W_3 - 2W_0W_2 - 8W_0W_1).$$

The previous theorem gives the following results as particular examples.

COROLLARY 15. *For all integers  $n$ , the Simson's formulas of Friedrich and Friedrich-Lucas numbers are given as*

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+1} & F_n & F_{n-1} \\ F_{n+1} & F_n & F_{n-1} & F_{n-2} \\ F_n & F_{n-1} & F_{n-2} & F_{n-3} \end{vmatrix} = 2^{n-1},$$

$$\begin{vmatrix} C_{n+3} & C_{n+2} & C_{n+1} & C_n \\ C_{n+2} & C_{n+1} & C_n & C_{n-1} \\ C_{n+1} & C_n & C_{n-1} & C_{n-2} \\ C_n & C_{n-1} & C_{n-2} & C_{n-3} \end{vmatrix} = -1323 \times 2^{n-3},$$

respectively.

#### 4. Some Identities

In this section, we obtain some identities of Friedrich and Friedrich-Lucas numbers. First, we can give a few basic relations between  $\{W_n\}$  and  $\{F_n\}$ .

LEMMA 16. *The following equalities are true:*

- (a):  $16W_n = (9W_0 + 16W_2 - 9W_3)F_{n+5} - 16(2W_2 - W_3)F_{n+4} - 16(2W_0 - W_1)F_{n+3} - (9W_0 + 32W_1 - 9W_3)F_{n+2}$ .
- (b):  $8W_n = (9W_0 - W_3)F_{n+4} - 8(2W_0 - W_1)F_{n+3} - 8(2W_1 - W_2)F_{n+2} - (9W_0 + 16W_2 - 9W_3)F_{n+1}$ .
- (c):  $4W_n = (W_0 + 4W_1 - W_3)F_{n+3} - 4(2W_1 - W_2)F_{n+2} - 4(2W_2 - W_3)F_{n+1} - (9W_0 - W_3)F_n$ .
- (d):  $2W_n = (W_0 + 2W_2 - W_3)F_{n+2} - 2(2W_2 - W_3)F_{n+1} - 2(2W_0 - W_1)F_n - (W_0 + 4W_1 - W_3)F_{n-1}$ .
- (e):  $W_n = W_0F_{n+1} + (W_1 - 2W_0)F_n + (W_2 - 2W_1)F_{n-1} + (W_3 - 2W_2 - W_0)F_{n-2}$ .

Proof. Note that all the identities hold for all integers  $n$ . We prove (a). To show (a), writing

$$W_n = a \times F_{n+5} + b \times F_{n+4} + c \times F_{n+3} + d \times F_{n+2}$$

and solving the system of equations

$$W_0 = a \times F_5 + b \times F_4 + c \times F_3 + d \times F_2$$

$$W_1 = a \times F_6 + b \times F_5 + c \times F_4 + d \times F_3$$

$$W_2 = a \times F_7 + b \times F_6 + c \times F_5 + d \times F_4$$

$$W_3 = a \times F_8 + b \times F_7 + c \times F_6 + d \times F_5$$

we find that  $a = \frac{1}{16}(9W_0 + 16W_2 - 9W_3)$ ,  $b = W_3 - 2W_2$ ,  $c = W_1 - 2W_0$ ,  $d = \frac{1}{16}(9W_3 - 32W_1 - 9W_0)$ . The other equalities can be proved similarly.  $\square$

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between  $\{W_n\}$  and  $\{C_n\}$ .

LEMMA 17. *The following equalities are true:*

- (a):  $588W_n = -(139W_0 + 28W_1 + 112W_2 - 83W_3)C_{n+5} + 28(4W_0 + 7W_2 - 4W_3)C_{n+4} + 28(8W_0 - W_3)C_{n+3} + (195W_0 + 224W_1 + 112W_2 - 139W_3)C_{n+2}$ .
- (b):  $294W_n = -(83W_0 + 28W_1 + 14W_2 - 27W_3)C_{n+4} + 14(8W_0 - W_3)C_{n+3} + 14(2W_0 + 7W_1 - 2W_3)C_{n+2} + (139W_0 + 28W_1 + 112W_2 - 83W_3)C_{n+1}$ .
- (c):  $147W_n = -(27W_0 + 28W_1 + 14W_2 - 20W_3)C_{n+3} + 7(2W_0 + 7W_1 - 2W_3)C_{n+2} + 7(4W_0 + 7W_2 - 4W_3)C_{n+1} + (83W_0 + 28W_1 + 14W_2 - 27W_3)C_n$ .
- (d):  $147W_n = -(40W_0 + 7W_1 + 28W_2 - 26W_3)C_{n+2} + 7(4W_0 + 7W_2 - 4W_3)C_{n+1} + 7(8W_0 - W_3)C_n + 2(27W_0 + 28W_1 + 14W_2 - 20W_3)C_{n-1}$ .
- (e):  $147W_n = -(52W_0 + 14W_1 + 7W_2 - 24W_3)C_{n+1} + 7(8W_0 - W_3)C_n + 7(2W_0 + 7W_1 - 2W_3)C_{n-1} + 2(40W_0 + 7W_1 + 28W_2 - 26W_3)C_{n-2}$ .

Now, we give a few basic relations between  $\{F_n\}$  and  $\{C_n\}$ .

LEMMA 18. *The following equalities are true:*

$$\begin{aligned}
147F_n &= 20C_{n+5} - 14C_{n+4} - 28C_{n+3} - 27C_{n+2}, \\
147F_n &= 26C_{n+4} - 28C_{n+3} - 7C_{n+2} - 40C_{n+1}, \\
147F_n &= 24C_{n+3} - 7C_{n+2} - 14C_{n+1} - 52C_n, \\
147F_n &= 41C_{n+2} - 14C_{n+1} - 28C_n - 48C_{n-1}, \\
147F_n &= 68C_{n+1} - 28C_n - 7C_{n-1} - 82C_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
16C_n &= F_{n+5} + 48F_{n+4} - 96F_{n+3} - F_{n+2}, \\
8C_n &= 25F_{n+4} - 48F_{n+3} - F_{n+1}, \\
4C_n &= F_{n+3} + 12F_{n+1} - 25F_n, \\
2C_n &= F_{n+2} + 6F_{n+1} - 12F_n - F_{n-1}, \\
C_n &= 4F_{n+1} - 6F_n - F_{n-2}.
\end{aligned}$$

## 5. Relations Between Special Numbers

In this section, we present identities on Friedrich, Friedrich-Lucas numbers and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers. We know from Lemma 11 that

$$\begin{aligned}
3F_n &= J_{n+2} + 2J_n - 1, \\
C_n &= K_n + 1.
\end{aligned}$$

Note also that from Lemma 16 and Lemma 17, we have the formulas of  $W_n$  as

$$\begin{aligned} 4W_n &= (W_0 + 4W_1 - W_3)F_{n+3} - 4(2W_1 - W_2)F_{n+2} - 4(2W_2 - W_3)F_{n+1} - (9W_0 - W_3)F_n, \\ 147W_n &= -(27W_0 + 28W_1 + 14W_2 - 20W_3)C_{n+3} + 7(2W_0 + 7W_1 - 2W_3)C_{n+2} \\ &\quad + 7(4W_0 + 7W_2 - 4W_3)C_{n+1} + (83W_0 + 28W_1 + 14W_2 - 27W_3)C_n. \end{aligned}$$

Using the above identities, we obtain relation of generalized Friedrich numbers in the following forms (in terms of third-order Jacobsthal and modified third-order Jacobsthal numbers):

LEMMA 19. *For all integers  $n$ , we have the following identities:*

$$\begin{aligned} \text{(a): } 6W_n &= (-W_3 + 4W_2 - 2W_1 - W_0)J_{n+2} + 3(W_3 - 2W_2 + W_0)J_{n+1} + (W_3 - 4W_2 + 8W_1 - 5W_0)J_n - \\ &\quad 2W_3 + 2W_2 + 2W_1 + 4W_0. \\ \text{(b): } 147W_n &= (6W_3 - 14W_2 + 21W_1 - 13W_0)K_{n+2} + (-8W_3 + 35W_2 - 28W_1 + W_0)K_{n+1} + (13W_3 - \\ &\quad 14W_2 - 28W_1 + 29W_0)K_n - 49W_3 + 49W_2 + 49W_1 + 98W_0. \end{aligned}$$

## 6. On the Recurrence Properties of Generalized Friedrich Sequence

Taking  $r = 2, s = 0, t = 1, u = -2$  in Theorem 5, we obtain the following Proposition.

PROPOSITION 20. *For  $n \in \mathbb{Z}$ , generalized Friedrich numbers (the case  $r = 2, s = 0, t = 1, u = -2$ ) have the following identity:*

$$W_{-n} = \frac{2^{-n-1}}{3} (-6W_{3n} + 6C_n W_{2n} - 3C_n^2 W_n + 3C_{2n} W_n + W_0 C_n^3 + 2W_0 C_{3n} - 3W_0 C_n C_{2n}).$$

From the above Proposition 20 (or by taking  $G_n = F_n$  and  $H_n = C_n$  in (1.15) and (1.16) respectively), we have the following corollary which gives the connection between the special cases of generalized Friedrich sequence at the positive index and the negative index: for Friedrich and Friedrich-Lucas numbers: take  $W_n = F_n$  with  $F_0 = 0, F_1 = 1, F_2 = 2, F_3 = 4$  and take  $W_n = C_n$  with  $C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 11$ , respectively. Note that in this case  $H_n = C_n$ .

COROLLARY 21. *For  $n \in \mathbb{Z}$ , we have the following recurrence relations:*

(a): *Friedrich sequence:*

$$F_{-n} = \frac{2^{-n-1}}{3} (-6F_{3n} + 6C_n F_{2n} - 3C_n^2 F_n + 3C_{2n} F_n).$$

(b): *Friedrich-Lucas sequence:*

$$C_{-n} = \frac{2^{-n-1}}{3} (C_n^3 + 2C_{3n} - 3C_{2n} C_n).$$

We can also present the formulas of  $F_{-n}$  and  $C_{-n}$  in the following forms.

COROLLARY 22. *For  $n \in \mathbb{Z}$ , we have the following recurrence relations:*

- (a):  $F_{-n} = \frac{2^{-n-5}}{3}(-96F_{3n} + 24(F_{n+3} + 12F_{n+1} - 25F_n)F_{2n} - 3(F_{n+3} + 12F_{n+1} - 25F_n)^2F_n + 12(F_{2n+3} + 12F_{2n+1} - 25F_{2n})F_n)$ .
- (b):  $3F_{-n} = \frac{1}{2^n}(3J_n^2 + 6J_{n-2}^2 + (J_{n+2} - 7J_{n+1} + 2J_{n-2})J_n - 14J_{n-1}J_{n-2} + 2J_{2n} + 4J_{2n-4} - 2^n)$ .
- (c):  $147F_{-n} = \frac{1}{2^n}(-2K_n^2 + 10K_{n-1}^2 + 34K_{n-2}^2 + 2K_{2n} - 10K_{2n-2} - 34K_{2n-4} - 49 \times 2^n)$ .
- (d):  $2^{n+2}C_{-n} = -9J_n^2 + 42J_{n-1}^2 - 12J_{n-2}^2 - (3J_{n+2} - 21J_{n+1} + 98J_{n-1} + 4J_{n-2})J_n + 14(J_{n+1} + 2J_{n-2})J_{n-1} - 6J_{2n} + 28J_{2n-2} - 8J_{2n-4} + 2^{n+2}$ .
- (e):  $C_{-n} = \frac{1}{2^{n+1}}(K_n^2 - K_{2n} + 2^{n+1})$ .

Proof. We use the identities, see Soykan [39],

$$J_{-n} = \frac{1}{2^{n+1}}(3J_n^2 + 2J_{2n} + J_{n+2}J_n - 7J_{n+1}J_n),$$

$$K_{-n} = \frac{1}{2^{n+1}}(K_n^2 - K_{2n}).$$

(a): By using the identity  $4C_n = F_{n+3} + 12F_{n+1} - 25F_n$  and Corollary 21, (or by using Corollary 6 (a)), we obtain (a).

(b): Since

$$3F_n = J_{n+2} + 2J_n - 1,$$

and

$$J_{-n} = \frac{1}{2^{n+1}}(3J_n^2 + 2J_{2n} + J_{n+2}J_n - 7J_{n+1}J_n),$$

we get (b)

(c): Since  $147F_n = 17K_{n+2} + 10K_{n+1} - 4K_n - 49$  and  $K_{-n} = \frac{1}{2^{n+1}}(K_n^2 - K_{2n})$ , we obtain (c).

(d): Since  $2C_n = -J_{n+2} + 7J_{n+1} - 3J_n + 2$  and  $J_{-n} = \frac{1}{2^{n+1}}(3J_n^2 + 2J_{2n} + J_{n+2}J_n - 7J_{n+1}J_n)$ , we get (d).

(e): Since  $C_n = K_n + 1$  and  $K_{-n} = \frac{1}{2^{n+1}}(K_n^2 - K_{2n})$ , we obtain (e).  $\square$

## 7. Sum Formulas

The following Corollary gives sum formulas of third-order Jacobsthal numbers.

COROLLARY 23. [40] For  $n \geq 0$ , third-order Jacobsthal numbers have the following properties:

- (a):  $\sum_{k=0}^n J_k = \frac{1}{3}(J_{n+3} - J_{n+1} - 1)$ .
- (b):  $\sum_{k=0}^n J_{2k} = \frac{1}{3}(J_{2n+1} + 2J_{2n} - 1)$ .
- (c):  $\sum_{k=0}^n J_{2k+1} = \frac{1}{3}(J_{2n+2} + 2J_{2n+1})$ .

The following Corollary presents sum formulas of Friedrich and Friedrich-Lucas numbers.

COROLLARY 24. For  $n \geq 0$ , Friedrich and Friedrich-Lucas numbers have the following properties (in terms of third-order Jacobsthal numbers):

- (a):

- (i):  $\sum_{k=0}^n F_k = \frac{1}{3}(2J_{n+2} + J_{n+1} + 2J_n - n - 3)$ .  
(ii):  $\sum_{k=0}^n F_{2k} = \frac{1}{3}(J_{2n+2} + J_{2n+1} + 2J_{2n} - n - 2)$ .  
(iii):  $\sum_{k=0}^n F_{2k+1} = \frac{1}{3}(2J_{2n+2} + 3J_{2n+1} + 2J_{2n} - n - 2)$ .

(b):

- (i):  $\sum_{k=0}^n C_k = 3J_{n+1} + J_n + n + 1$ .  
(ii):  $\sum_{k=0}^n C_{2k} = \frac{1}{3}(2J_{2n+2} + 5J_{2n+1} - 4J_{2n} + 3n + 5)$ .  
(iii):  $\sum_{k=0}^n C_{2k+1} = \frac{1}{3}(7J_{2n+2} - 2J_{2n+1} + 4J_{2n} + 3n + 1)$ .

Proof. The proof follows from Corollary 23 and the identities

$$\begin{aligned} 3F_n &= J_{n+2} + 2J_n - 1, \\ 2C_n &= -J_{n+2} + 7J_{n+1} - 3J_n + 2. \quad \square \end{aligned}$$

## 8. Matrices and Identities Related With Generalized Friedrich Numbers

If we define the square matrix  $A$  of order 4 as

$$A = \begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (8.1)$$

and also define

$$B_n = \begin{pmatrix} F_{n+1} & F_{n-1} - 2F_{n-2} & F_n - 2F_{n-1} & -2F_n \\ F_n & F_{n-2} - 2F_{n-3} & F_{n-1} - 2F_{n-2} & -2F_{n-1} \\ F_{n-1} & F_{n-3} - 2F_{n-4} & F_{n-2} - 2F_{n-3} & -2F_{n-2} \\ F_{n-2} & F_{n-4} - 2F_{n-5} & F_{n-3} - 2F_{n-4} & -2F_{n-3} \end{pmatrix} \quad (8.2)$$

and

$$U_n = \begin{pmatrix} W_{n+1} & W_{n-1} - 2W_{n-2} & W_n - 2W_{n-1} & -2W_n \\ W_n & W_{n-2} - 2W_{n-3} & W_{n-1} - 2W_{n-2} & -2W_{n-1} \\ W_{n-1} & W_{n-3} - 2W_{n-4} & W_{n-2} - 2W_{n-3} & -2W_{n-2} \\ W_{n-2} & W_{n-4} - 2W_{n-5} & W_{n-3} - 2W_{n-4} & -2W_{n-3} \end{pmatrix}. \quad (8.3)$$

then we get the following Theorem (by setting  $r = 2, s = 0, t = 1, u = -2$  in Theorem 7).

**THEOREM 25.** *For all integers  $m, n$ , we have*

(a):  $B_n = A^n$ , i.e.,

$$\begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_{n-1} - 2F_{n-2} & F_n - 2F_{n-1} & -2F_n \\ F_n & F_{n-2} - 2F_{n-3} & F_{n-1} - 2F_{n-2} & -2F_{n-1} \\ F_{n-1} & F_{n-3} - 2F_{n-4} & F_{n-2} - 2F_{n-3} & -2F_{n-2} \\ F_{n-2} & F_{n-4} - 2F_{n-5} & F_{n-3} - 2F_{n-4} & -2F_{n-3} \end{pmatrix}. \quad (8.4)$$

$$(b): U_1 A^n = A^n U_1.$$

$$(c): U_{n+m} = U_n B_m = B_m U_n.$$

Using the above last Theorem and the identity

$$3F_n = J_{n+2} + 2J_n - 1,$$

we obtain the following identity for third-order Jacobsthal numbers.

COROLLARY 26. *For all integers  $n$ , we have the following formula for third-order Jacobsthal numbers:*

$$A^n = \begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{3} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$a_{11} = J_{n+3} + 2J_{n+1} - 1$$

$$a_{21} = J_{n+2} + 2J_n - 1$$

$$a_{31} = J_{n+1} + 2J_{n-1} - 1$$

$$a_{41} = J_n + 2J_{n-2} - 1$$

$$a_{12} = J_{n+1} - 2J_n + 2J_{n-1} - 4J_{n-2} + 1$$

$$a_{22} = J_n - 2J_{n-1} + 2J_{n-2} - 4J_{n-3} + 1$$

$$a_{32} = J_{n-1} - 2J_{n-2} + 2J_{n-3} - 4J_{n-4} + 1$$

$$a_{42} = J_{n-2} - 2J_{n-3} + 2J_{n-4} - 4J_{n-5} + 1$$

$$a_{13} = J_{n+2} - 2J_{n+1} + 2J_n - 4J_{n-1} + 1$$

$$a_{23} = J_{n+1} - 2J_n + 2J_{n-1} - 4J_{n-2} + 1$$

$$a_{33} = J_n - 2J_{n-1} + 2J_{n-2} - 4J_{n-3} + 1$$

$$a_{43} = J_{n-1} - 2J_{n-2} + 2J_{n-3} - 4J_{n-4} + 1$$

$$a_{14} = -2(J_{n+2} + 2J_n - 1)$$

$$a_{24} = -2(J_{n+1} + 2J_{n-1} - 1)$$

$$a_{34} = -2(J_n + 2J_{n-2} - 1)$$

$$a_{44} = -2(J_{n-1} + 2J_{n-3} - 1)$$

Next, we present an identity for  $W_{n+m}$  (by setting  $r = 2, s = 0, t = 1, u = -2$  in Theorem 8).

THEOREM 27. *For all integers  $m, n$ , we have*

$$W_{n+m} = W_n F_{m+1} + W_{n-1} (F_{m-1} - 2F_{m-2}) + W_{n-2} (F_m - 2F_{m-1}) - 2W_{n-3} F_m \quad (8.5)$$

As particular cases of the above theorem, we give identities for  $F_{n+m}$  and  $C_{n+m}$ .

COROLLARY 28. *For all integers  $m, n$ , we have*

$$F_{n+m} = F_n F_{m+1} + F_{n-1}(F_{m-1} - 2F_{m-2}) + F_{n-2}(F_m - 2F_{m-1}) - 2F_{n-3}F_m, \quad (8.6)$$

$$C_{n+m} = C_n F_{m+1} + C_{n-1}(F_{m-1} - 2F_{m-2}) + C_{n-2}(F_m - 2F_{m-1}) - 2C_{n-3}F_m. \quad (8.7)$$

## 9. Conclusions

Sequences have been a fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are very well-known examples of second order recurrence sequences. For rich applications of these second order sequences in science and nature, one can see the citations in [20]. The generalization of Fibonacci sequence leads to several nice and interesting sequences.

As a fourth order sequence, we introduce the generalized Friedrich sequence (and its two special cases, namely, Friedrich and Friedrich-Lucas sequences) and we present Binet's formulas, generating functions, Simson formulas, the sum formulas, some identities, recurrence properties and matrices for these sequences.

We have shown that there are close relations between Friedrich, Friedrich-Lucas numbers (which are fourth order linear recurrences) and special third order linear recurrences (numbers), namely third order Jacobsthal, modified third-order Jacobsthal, third order Jacobsthal-Lucas numbers.

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [1].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [4].
- For the application of Jacobsthal numbers to special matrices, see [61].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [15].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [59].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [60].
- For the applications of generalized Fibonacci numbers to binomial sums, see [57].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [41].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [42].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [10].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [28].
- For the application of higher order Jacobsthal numbers to quaternions, see [25].

- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [13].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [3].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [18].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [19].

We now present some applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [7] and [6], respectively.
- For the application of Tribonacci numbers to special matrices, see [58].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [29] and [2], respectively.
- For the application of Pell-Padovan numbers to groups, see [11].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [14].
- For the application of Gaussian Tribonacci numbers to various graphs, see [55].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [12].
- For the application of Narayan numbers to finite groups see [21].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [43].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [44].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [45].
- For the application of generalized Tribonacci numbers to Sedenions, see [46].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [47].
- For the application of generalized Tribonacci numbers to circulant matrix, see [48].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [56].

Next, we now list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [49].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [50].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [51].
- For the application of generalized Tetranacci numbers to binomial transform, see [52].

We now present some applications of fifth order sequences.

- For the application of Pentanacci numbers to matrices, see [31].
- For the application of generalized Pentanacci numbers to quaternions, see [53].
- For the application of generalized Pentanacci numbers to binomial transform, see [54].

## References

- [1] Azak, A.Z., Pauli Gaussian Fibonacci and Pauli Gaussian Lucas Quaternions. *Mathematics*, 2022, 10, 4655. <https://doi.org/10.3390/math10244655>
- [2] Basu, M., Das, M., Tribonacci Matrices and a New Coding Theory, *Discrete Mathematics, Algorithms and Applications*, 6 (1), 1450008, (17 pages), 2014.
- [3] Bilgin, N.G., Fibonacci Lacunary Statistical Convergence of Order  $\gamma$  in IFNLS, *International Journal of Advances in Applied Mathematics and Mechanics*, 8(4), 28-36, 2021.
- [4] Büyük, H., Taşkara, N., On The Solutions of Three-Dimensional Difference Equation Systems Via Pell Numbers, *European Journal of Science and Technology, Special Issue 34*, 433-440, 2022.
- [5] Cerda-Morales, G., A Note On Modified Third-Order Jacobsthal Numbers, *Proyecciones Journal of Mathematics*, 39(2), 409-420, 2020. DOI: <https://doi.org/10.22199/issn.0717-6279-2020-02-0025>
- [6] Cerda-Morales, G., On a Generalization of Tribonacci Quaternions, *Mediterranean Journal of Mathematics* 14:239, 1-12, 2017.
- [7] Cerda-Morales, G., Identities for Third Order Jacobsthal Quaternions, *Advances in Applied Clifford Algebras* 27(2), 1043–1053, 2017.
- [8] Cook C. K., Bacon, M. R., Some Identities for Jacobsthal and Jacobsthal-Lucas Numbers Satisfying Higher Order Recurrence Relations, *Annales Mathematicae et Informaticae*, 41, 27–39, 2013.
- [9] Deveci, Ö., The Jacobsthal-Padovan p-Sequences and Their Applications, *Proceedings of Romanian Academy, Series A*, 20(3), 215–224, 2019.
- [10] Deveci, Ö., Shannon, A.G., On Recurrence Results From Matrix Transforms, *Notes on Number Theory and Discrete Mathematics*, 28(4), 589–592, 2022. DOI: 10.7546/nntdm.2022.28.4.589-592
- [11] Deveci, Ö., Shannon, A.G., Pell–Padovan-Circulant Sequences and Their Applications, *Notes on Number Theory and Discrete Mathematics*, 23(3), 100–114, 2017.
- [12] Dikmen, C.M., Altınsoy, M., On Third Order Hyperbolic Jacobsthal Numbers, *Konuralp Journal of Mathematics*, 10(1), 118-126, 2022.
- [13] Goy, T., Shattuck, M., Fibonacci and Lucas Identities from Toeplitz-Hessenberg Matrices, *Appl. Appl. Math.*, 14(2), 699–715, 2019.
- [14] Göcen, M., The Exact Solutions of Some Difference Equations Associated with Adjusted Jacobsthal-Padovan Numbers, *Kırklareli University Journal of Engineering and Science* 8(1), 1-14, 2022. DOI: 10.34186/klujes.1078836
- [15] Gül, K., Generalized k-Order Fibonacci Hybrid Quaternions, *Erzincan University Journal of Science and Technology*, 15(2), 670-683, 2022. DOI: 10.18185/erzifbed.1132164
- [16] G. S. Hathiwala, D. V. Shah, Binet–Type Formula For The Sequence of Tetranacci Numbers by Alternate Methods, *Mathematical Journal of Interdisciplinary Sciences* 6(1) (2017), 37–48.
- [17] Howard F.T., Saidak, F., Zhou’s Theory of Constructing Identities, *Congress Numer.* 200 (2010), 225-237.
- [18] Kişi, Ö., Tuzcuoglu, I., Fibonacci Lacunary Statistical Convergence in Intuitionistic Fuzzy Normed Linear Spaces, *Journal of Progressive Research in Mathematics* 16(3), 3001-3007, 2020.
- [19] Kişi, Ö., Debnath, P., Fibonacci Ideal Convergence on Intuitionistic Fuzzy Normed Linear Spaces, *Fuzzy Information and Engineering*, 1-13, 2022. <https://doi.org/10.1080/16168658.2022.2160226>
- [20] Koshy, T., *Fibonacci and Lucas Numbers with Applications*. Wiley-Interscience. New York, 2001.
- [21] Kuloğlu, B., Özkan, E., Shannon, A.G., The Narayana Sequence in Finite Groups, *Fibonacci Quarterly*. 60(5), 212–221, 2022.
- [22] Melham, R.S., Some Analogs of the Identity  $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$ , *Fibonacci Quarterly*, (1999), 305-311.

- [23] Marohnić, L., Strmečki T., Plastic Number: Construction and Applications, *Advanced Research in Scientific Areas*, 3(7), 1523-1528, 2012.
- [24] Natividad, L.R., On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, *International Journal of Mathematics and Computing*, 3(2) (2013), 38-40.
- [25] Özkan, E., Uysal, M., On Quaternions with Higher Order Jacobsthal Numbers Components, *Gazi University Journal of Science*, 36(1), 336-347, 2023. DOI: 10.35378/gujs.1002454
- [26] Padovan, R., Dom Hans van Der Laan and the Plastic Number. In: Williams K., Ostwald M. (eds) *Architecture and Mathematics from Antiquity to the Future*. Birkhäuser, Cham 407-419, 2015, [https://doi.org/10.1007/978-3-319-00143-2\\_27](https://doi.org/10.1007/978-3-319-00143-2_27). (First published as: Richard Padovan , “Dom Hans van Der Laan and the Plastic Number”, pp. 181–193 in *Nexus IV: Architecture and Mathematics*, Kim Williams and Jose Francisco Rodrigues, eds. Fucecchio (Florence): Kim Williams Books, 2002.
- [27] Padovan, R., Dom Hans van der Laan: Modern Primitive, *Architectura and Natura Press*, 1994. <http://www.nexusjournal.com/conferences/N2002-Padovan.html>
- [28] Prasad, K., Mahato, H., Cryptography Using Generalized Fibonacci Matrices with Affine-Hill Cipher, *Journal of Discrete Mathematical Sciences & Cryptography*, 25(8-A), 2341–2352, 2022. DOI : 10.1080/09720529.2020.1838744
- [29] Shtayat, J., Al-Kateeb, A., An Encoding-Decoding algorithm based on Padovan numbers, arXiv:1907.02007, 2019.
- [30] B. Singh, P. Bhadouria, O. Sikhwal, K. Sisodiya, A Formula for Tetranacci-Like Sequence, *Gen. Math. Notes*, 20(2) (2014), 136-141.
- [31] Sivakumar, B., James, V., A Notes on Matrix Sequence of Pentanacci Numbers and Pentanacci Cubes, *Communications in Mathematics and Applications*, 13(2), 603–611, 2022. DOI: 10.26713/cma.v13i2.1725
- [32] Sloane, N.J.A., The on-line encyclopedia of integer sequences, <http://oeis.org/>
- [33] Soykan, Y., A Study on Generalized Jacobsthal-Padovan Numbers, *Earthline Journal of Mathematical Sciences*, 4(2), 227-251, 2020. <https://doi.org/10.34198/ejms.4220.227251>
- [34] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, *International Journal of Advances in Applied Mathematics and Mechanics*, 7(2), 45-56, 2019.
- [35] Soykan, Y., Properties of Generalized (r,s,t,u)-Numbers, *Earthline Journal of Mathematical Sciences*, 5(2), 297-327, 2021. <https://doi.org/10.34198/ejms.5221.297327>
- [36] Soykan, Y., A Study On the Recurrence Properties of Generalized Tetranacci Sequence, *International Journal of Mathematics Trends and Technology*, 67(8), 185-192, 2021. doi:10.14445/22315373/IJMTT-V67I8P522
- [37] Soykan, Y., A Study on Generalized Fibonacci Polynomials: Sum Formulas, *International Journal of Advances in Applied Mathematics and Mechanics*, 10(1), 39-118, 2022. (ISSN: 2347-2529)
- [38] Soykan, Y., On Generalized Fibonacci Polynomials: Horadam Polynomials, *Earthline Journal of Mathematical Sciences*, 11(1), 23-114, 2023. E-ISSN: 2581-8147. <https://doi.org/10.34198/ejms.11123.23114>
- [39] Soykan, Y. On the Recurrence Properties of Generalized Tribonacci Sequence, *Earthline Journal of Mathematical Sciences*, 6(2), 253-269, 2021. <https://doi.org/10.34198/ejms.6221.253269>
- [40] Polath, E.E., Soykan, Y., On Generalized Third-Order Jacobsthal Numbers, *Asian Research Journal of Mathematics*, 17(2), 1-19, 2021. DOI: 10.9734/ARJOM/2021/v17i230270
- [41] Soykan, Y., Taşdemir, E., A Study On Hyperbolic Numbers With Generalized Jacobsthal Numbers Components, *International Journal of Nonlinear Analysis and Applications*, 13(2), 1965–1981, 2022. <http://dx.doi.org/10.22075/ijnaa.2021.22113.2328>
- [42] Soykan, Y., On Dual Hyperbolic Generalized Fibonacci Numbers, *Indian J Pure Appl Math*, 2021. <https://doi.org/10.1007/s13226-021-00128-2>

- [43] Soykan, Y., Taşdemir, E., Göcen, M., Binomial Transform of the Generalized Third-Order Jacobsthal Sequence, *Asian-European Journal of Mathematics*, 15(12), 2022. <https://doi.org/10.1142/S1793557122502242>.
- [44] Soykan, Y., Taşdemir, E., Okumuş, İ., A Study on Binomial Transform of the Generalized Padovan Sequence, *Journal of Science and Arts*, 22(1), 63-90, 2022. <https://doi.org/10.46939/J.Sci.Arts-22.1-a06>
- [45] Soykan, Y., Taşdemir, E., Okumuş, İ., Göcen, M., Gaussian Generalized Tribonacci Numbers, *Journal of Progressive Research in Mathematics(JPRM)*, 14 (2), 2373-2387, 2018.
- [46] Soykan, Y., Okumuş, İ., Taşdemir, E., On Generalized Tribonacci Sedenions, *Sarajevo Journal of Mathematics*, 16(1), 103-122, 2020. ISSN 2233-1964, DOI: 10.5644/SJM.16.01.08
- [47] Soykan, Y., Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers, *Communications in Mathematics and Applications*, 11(2), 281-295, 2020. DOI: 10.26713/cma.v11i2.1102
- [48] Soykan, Y. Explicit Euclidean Norm, Eigenvalues, Spectral Norm and Determinant of Circulant Matrix with the Generalized Tribonacci Numbers, *Earthline Journal of Mathematical Sciences*, 6(1), 131-151, 2021. <https://doi.org/10.34198/ejms.6121.131151>
- [49] Soykan Y, Tetranacci and Tetranacci-Lucas Quaternions, *Asian Research Journal of Mathematics*, 15(1): 1-24, 2019; Article no.ARJOM.50749.
- [50] Soykan, Y., Gaussian Generalized Tetranacci Numbers, *Journal of Advances in Mathematics and Computer Science*, 31(3): 1-21, Article no.JAMCS.48063, 2019.
- [51] Soykan, Y., Matrix Sequences of Tetranacci and Tetranacci-Lucas Numbers, *Int. J. Adv. Appl. Math. and Mech.* 7(2), 57-69, 2019, (ISSN: 2347-2529).
- [52] Soykan, Y., On Binomial Transform of the Generalized Tetranacci Sequence, *International Journal of Advances in Applied Mathematics and Mechanics*, 9(2), 8-27, 2021.
- [53] Soykan, Y., Özmen, N., Göcen, M., On Generalized Pentanacci Quaternions, *Tbilisi Mathematical Journal*, 13(4), 169-181, 2020.
- [54] Soykan, Y., Binomial Transform of the Generalized Pentanacci Sequence, *Asian Research Journal of Current Science*, 3(1), 209-231, 2021.
- [55] Sunitha, K., Sheriba. M., Gaussian Tribonacci R-Graceful Labeling of Some Tree Related Graphs, *Ratio Mathematica*, 44, 188-196, 2022.
- [56] Taşyurdu, Y., Polat, Y.E., Tribonacci and Tribonacci-Lucas Hybrinomials, *Journal of Mathematics Research*; 13(5), 2021.
- [57] Ulutaş, Y.T., Toy, D., Some Equalities and Binomial Sums about the Generalized Fibonacci Number  $u_n$ , *Notes on Number Theory and Discrete Mathematics*, 28(2), 252–260, 2022. DOI: 10.7546/nntdm.2022.28.2.252-260
- [58] Yilmaz, N., Taskara, N., Tribonacci and Tribonacci-Lucas Numbers via the Determinants of Special Matrices, *Applied Mathematical Sciences*, 8(39), 1947-1955, 2014.
- [59] Yilmaz, N., Split Complex Bi-Periodic Fibonacci and Lucas Numbers, *Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.* 71(1), 153–164, 2022. DOI:10.31801/cfsuasmas.704435
- [60] Yilmaz, N., The Generalized Bivariate Fibonacci and Lucas Matrix Polynomials, *Mathematica Montisnigri*, Vol LIII, 33-44, 2022. DOI: 10.20948/mathmontis-2022-53-5
- [61] Vasanthi, S., Sivakumar, B., Jacobsthal Matrices and their Properties. *Indian Journal of Science and Technology* 15(5): 207-215, 2022, <https://doi.org/10.17485/IJST/v15i5.1948>
- [62] M. E. Waddill, Another Generalized Fibonacci Sequence, *M. E., Fibonacci Quarterly*, 5(3) (1967), 209-227.
- [63] M. E. Waddill, The Tetranacci Sequence and Generalizations, *Fibonacci Quarterly*, (1992), 9-20.