Original Research Article

Generalized Friedrich Numbers

Abstract. In this paper, we introduce and investigate the generalized Friedrich sequences and we deal with, in detail, two special cases, namely, Friedrich and Friedrich-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Friedrich, Friedrich-Lucas and third order Jacobsthal, modified third-order Jacobsthal, third order Jacobsthal-Lucas numbers.

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1. Introduction

Third-order Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (OEIS: A077947, [8]), modified third-order Jacobsthal sequence $\{K_n\}_{n\geq 0}$ (OEIS: A186575, [8]) and third-order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ (OEIS: A226308, [8]) are defined, respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2} + J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, J_2 = 1, \tag{1.1}$$

$$K_{n+3} = K_{n+2} + K_{n+1} + 2K_n, \quad K_0 = 3, K_1 = 1, K_2 = 3. \tag{1.2}$$

$$j_{n+3} = j_{n+2} + j_{n+1} + 2j_n, \quad j_0 = 2, j_1 = 1, j_2 = 5, \tag{1.3}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ are defined in [2] and $K_n\}_{n\geq 0}$ is given in [1]. For more details on the generalized third-order Jacobsthal numbers and its special cases, see [15].

The sequences $\{J_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} + \frac{1}{2}J_{-(n-3)}, \\ K_{-n} &= -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)}, \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} + \frac{1}{2}j_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1)-(1.3) hold for all integer n.

Now, we define two sequences related to third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers. Friedrich and Friedrich-Lucas numbers are defined as

$$F_n = F_{n-1} + F_{n-2} + 2F_{n-3} + 1, \quad \text{with} \;\; F_0 = 0, F_1 = 1, F_2 = 2, \;\; n \geq 3,$$

and

$$C_n = C_{n-1} + C_{n-2} + 2C_{n-3} - 3, \quad \text{with} \;\; C_0 = 4, C_1 = 2, C_2 = 4, \;\; n \geq 3,$$

respectively. The first few values of Friedrich and Friedrich-Lucas numbers are

$$0, 1, 2, 4, 9, 18, 36, 73, 146, 292, 585, 1170, 2340, 4681, ...$$

and

$$4, 2, 4, 11, 16, 32, 67, 128, 256, 515, 1024, 2048, 4099, 8192, ...$$

respectively. The sequences $\{F_n\}$ and $\{C_n\}$ satisfy the following fourth order linear recurrences:

$$\begin{aligned} F_n &= 2F_{n-1} + F_{n-3} - 2F_{n-4}, \qquad F_0 = 0, F_1 = 1, F_2 = 2, F_3 = 4, \qquad n \geq 4, \\ C_n &= 2C_{n-1} + C_{n-3} - 2C_{n-4}, \qquad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 11, \qquad n \geq 4. \end{aligned}$$

There are close relations between Friedrich, Friedrich-Lucas and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} 3F_n &= J_{n+2} + 2J_n - 1, \\ 2C_n &= -J_{n+2} + 7J_{n+1} - 3J_n + 2, \\ 147F_n &= 17K_{n+2} + 10K_{n+1} - 4K_n - 49, \\ C_n &= K_n + 1, \\ 18F_n &= j_{n+2} + 3j_{n+1} - j_n - 6, \\ 24C_n &= 11j_{n+2} - 21j_{n+1} + 19j_n + 24, \end{aligned}$$

and

$$\begin{aligned} J_{n+1} &= F_{n+1} - F_n, \\ 147J_n &= 19C_{n+2} - 9C_{n+1} - 16C_n + 6, \\ 4K_n &= F_{n+2} + 13F_{n+1} - 23F_n - 3, \\ 3K_n &= C_{n+3} - C_{n+2} - C_{n+1} + C_n, \\ j_n &= -F_{n+3} + 3F_{n+2} - 2F_n, \\ 49j_n &= -5C_{n+2} + 23C_{n+1} + 30C_n - 48. \end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Friedrich, Friedrich-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized (r, s, t, u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \qquad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \tag{1.4}$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [3,5,6,7,10,12,13,16,17]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $u \neq 0$. Therefore, recurrence (1.4) holds for all integers n.

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \tag{1.5}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= r, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\ \alpha\beta\gamma\delta &= -u. \end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) For all integers n, Binet's formula of generalized Tetranacci numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{1.6}$$

where

$$\begin{aligned} p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\ p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0. \end{aligned}$$

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (1.5) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (1.5) for non-negative integers is valid for all integers n (see [4]).

Next, we consider two special cases of the generalized (r, s, t, u) sequence $\{W_n\}$ which we call them (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas sequences. (r, s, t, u)-Fibonacci sequence $\{G_n\}_{n\geq 0}$ and (r, s, t, u)-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3} + sG_{n+2} + tG_{n+1} + uG_n, & (1.7) \\ G_0 &= 0, G_1 = 1, G_2 = r, G_3 = r^2 + s, \\ H_{n+4} &= rH_{n+3} + sH_{n+2} + tH_{n+1} + uH_n, & (1.8) \\ H_0 &= 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)} - \frac{s}{u}G_{-(n-2)} - \frac{r}{u}G_{-(n-3)} + \frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)} - \frac{s}{u}H_{-(n-2)} - \frac{r}{u}H_{-(n-3)} + \frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.7) and (1.8) hold for all integers n.

For all integers n, (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas numbers (using initial conditions in (1.7) or (1.8)) can be expressed using Binet's formulas as in the following corollary.

COROLLARY 2. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) Binet's formula of (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas numbers are*

$$G_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

respectively.

Proof. Take $W_n = G_n$ and $W_n = H_n$ in Theorem 1, respectively. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

LEMMA 3. *Suppose that* $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized* (r,s,t,u) *sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{1 - rz - sz^2 - tz^3 - uz^4}. \tag{1.9}$$

Proof. For a proof, see Soykan [10, Lemma 1]. □

The following theorem presents Simson's formula of generalized (r,s,t,u) sequence (generalized Tetranacci sequence) $\{W_n\}$.

THEOREM 4 (Simson's Formula of Generalized (r,s,t,u) Numbers). *For all integers* n, *we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{1.10}$$

Proof. (1.10) is given in Soykan [9]. □

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

THEOREM 5. *For* $n \in \mathbb{Z}$, *for the generalized Tetranacci sequence (or generalized* (r,s,t,u)*-sequence or 4-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_n W_{2n} - 3H_n^2 W_n + 3H_{2n} W_n + W_0 H_n^3 + 2W_0 H_{3n} - 3W_0 H_n H_{2n}) \\ &= (-1)^{-n-1} u^{-n}(W_{3n} - H_n W_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [11, Theorem 1.]. □

Using Theorem 5, we have the following corollary, see Soykan [11, Corollary 4].

COROLLARY 6. *For* $n \in \mathbb{Z}$, *we have*

(a): $2(-u)^{n+4} G_{-n} = -(3ru^2 + t^3 - 3stu)^2 G_n^3 - (2su - t^2)^2 G_{n+3}^2 G_n - (-rt^2 - tu + 2rsu)^2 G_{n+2}^2 G_n - (-st^2 + 2s^2u + 4u^2 + rtu)^2 G_{n+1}^2 G_n + 2(3ru^2 + t^3 - 3stu)((-2su + t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1})G_n^2 + 2(2su - t^2)(-rt^2 - tu + 2rsu)G_{n+3}G_{n+2}G_n + 2(2su - t^2)(-st^2 + 2s^2u + 4u^2 + rtu)G_{n+3}G_{n+1}G_n - 2(-st^2 + 2s^2u + 4u^2 + rtu)(-rt^2 - tu + 2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su + t^2)G_{2n+3}G_n + u^2(-rt^2 - tu + 2rsu)G_{2n+2}G_n + u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n+1}G_n - 2u^2(2su - t^2)G_{2n}G_{n+3} + 2u^2(-rt^2 - tu + 2rsu)G_{2n}G_{n+2} + 2u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n}G_{n+1} - 3u^2(3ru^2 + t^3 - 3stu)G_{2n}G_n.$

(b): $H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 4$ in Theorem 5,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n} + 6H_nG_{2n} - 3H_n^2G_n + 3H_{2n}G_n), \tag{1.11}$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right), \tag{1.12}$$

respectively.

If we define the square matrix A of order 4 as

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix}$$

then we get the following Theorem.

THEOREM 7. *For all integers m, n, we have*

(a): $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

(b): $U_1A^n = A^nU_1$.

(c): $U_{n+m} = U_nB_m = B_mU_n$.

Proof. For the proof, see Soykan [10, Theorem 19]. □

THEOREM 8. *For all integers m, n, we have*

$$W_{n+m} = W_nG_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \tag{1.13}$$

Proof. For the proof, see Soykan [10, Theorem 20]. □

In the next sections, we present new results.

2. Generalized Friedrich Sequence

In this paper, we consider the case $r = 2, s = 0, t = 1, u = -2$. A generalized Friedrich sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = 2W_{n-1} + W_{n-3} - 2W_{n-4} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{1}{2}W_{-(n-1)} + W_{-(n-3)} - \frac{1}{2}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integers n.

Characteristic equation of $\{W_n\}$ is

$$z^4 - 2z^3 - z + 2 = (z^3 - z^2 - z - 2)(z - 1) = (z^2 + z + 1)(z - 2)(z - 1) = 0$$

whose roots are

$$\begin{aligned} \alpha &= 2, \\ \beta &= \frac{-1 + i\sqrt{3}}{2}, \\ \gamma &= \frac{-1 - i\sqrt{3}}{2}, \\ \delta &= 1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= 2, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 0, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\ \alpha\beta\gamma\delta &= 2. \end{aligned}$$

Note also that

$$\begin{aligned} \alpha + \beta + \gamma &= 1, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 2. \end{aligned}$$

The first few generalized Friedrich numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Friedrich numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{2}(W_0+2W_2-W_3)$
2	W_2	$\frac{1}{4}(W_0+4W_1-W_3)$
3	W_3	$\frac{1}{8}(9W_0-W_3)$
4	$W_1-2W_0+2W_3$	$\frac{1}{16}(9W_0+16W_2-9W_3)$
5	$W_2-4W_0+4W_3$	$\frac{1}{32}(9W_0+32W_1-9W_3)$
6	$9W_3-8W_0$	$\frac{1}{64}(73W_0-9W_3)$
7	$W_1-18W_0+18W_3$	$\frac{1}{128}(73W_0+128W_2-73W_3)$
8	$W_2-36W_0+36W_3$	$\frac{1}{256}(73W_0+256W_1-73W_3)$
9	$73W_3-72W_0$	$\frac{1}{512}(585W_0-73W_3)$
10	$W_1-146W_0+146W_3$	$\frac{1}{1024}(585W_0+1024W_2-585W_3)$
11	$W_2-292W_0+292W_3$	$\frac{1}{2048}(585W_0+2048W_1-585W_3)$
12	$585W_3-584W_0$	$\frac{1}{4096}(4681W_0-585W_3)$
13	$W_1-1170W_0+1170W_3$	$\frac{1}{8192}(4681W_0+8192W_2-4681W_3)$

Note that the sequences $\{F_n\}$ and $\{C_n\}$ which are defined in the section Introduction, are the special cases of the generalized Friedrich sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$ in this section as well. Friedrich sequence $\{F_n\}_{n\geq 0}$ and Friedrich-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} F_n &= 2F_{n-1}+F_{n-3}-2F_{n-4}, \qquad F_0=0, F_1=1, F_2=2, F_3=4, \qquad n\geq 4, \\ C_n &= 2C_{n-1}+C_{n-3}-2C_{n-4}, \qquad C_0=4, C_1=2, C_2=4, C_3=11, \qquad n\geq 4. \end{aligned}$$

The sequences $\{F_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} F_{-n} &= \frac{1}{2}F_{-(n-1)}+F_{-(n-3)}-\frac{1}{2}F_{-(n-4)}, \\ C_{-n} &= \frac{1}{2}C_{-(n-1)}+C_{-(n-3)}-\frac{1}{2}C_{-(n-4)}, \end{aligned}$$

for $n=1,2,3,...$ respectively.

Next, we present the first few values of the Friedrich and Friedrich-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	0	1	2	4	9	18	36	73	146	292	585	1170	2340	4681
F_{-n}	0	0	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{9}{16}$	$-\frac{9}{32}$	$-\frac{9}{64}$	$-\frac{73}{128}$	$-\frac{73}{256}$	$-\frac{73}{512}$	$-\frac{585}{1024}$	$-\frac{585}{2048}$
C_n	4	2	4	11	16	32	67	128	256	515	1024	2048	4099	8192
C_{-n}	4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{25}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{193}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1537}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{12289}{4096}$	$\frac{1}{8192}$

Theorem 1 can be used to obtain the Binet formula of generalized Friedrich numbers. Using these (the above) roots and the recurrence relation, Binet's formula of generalized Friedrich numbers can be given as follows:

THEOREM 9. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formula of generalized Friedrich numbers is*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+2)W_1 - 2W_0)\alpha^n}{2\alpha^2+5\alpha-4} \\ &+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+2)W_1 - 2W_0)\beta^n}{2\beta^2+5\beta-4} \\ &+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+2)W_1 - 2W_0)\gamma^n}{2\gamma^2+5\gamma-4} \\ &+ \frac{W_3 - W_2 - W_1 - 2W_0}{-3}. \end{aligned}$$

Friedrich and Friedrich-Lucas numbers can be expressed using Binet's formulas as follows:

COROLLARY 10. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formulas of Friedrich and Friedrich-Lucas numbers are*

$$\begin{aligned} F_n &= \frac{(\alpha^2+\alpha+2)\alpha^n}{2\alpha^2+5\alpha-4} + \frac{(\beta^2+\beta+2)\beta^n}{2\beta^2+5\beta-4} + \frac{(\gamma^2+\gamma+2)\gamma^n}{2\gamma^2+5\gamma-4} - \frac{1}{3} \\ &= \frac{1}{7} \times 2^{n+2} - \frac{1}{42}(5+i\sqrt{3})\left(\frac{-1+i\sqrt{3}}{2}\right)^n - \frac{1}{42}(5-i\sqrt{3})\left(\frac{-1-i\sqrt{3}}{2}\right)^n - \frac{1}{3}, \end{aligned}$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1 = 2^n + \left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n + 1,$$

respectively.

Note that for all integers n, third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned} J_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ K_n &= \alpha^n + \beta^n + \gamma^n \\ j_n &= \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \end{aligned}$$

respectively, see Soykan [15] for more details. So, by using Binet's formulas of Friedrich, Friedrich-Lucas and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

LEMMA 11. *For all integers n, the following equalities (identities) are true:*

(a):

- $J_{n+1} = F_{n+1} - F_n$.

- $2J_n = F_{n+3} - 2F_{n+2} + F_n$.
- $3F_{n+4} = 13J_{n+2} + 15J_{n+1} + 14J_n - 1$.
- $3F_n = J_{n+2} + 2J_n - 1$.
- $2J_n = -F_{n+2} + F_{n+1} + 3F_n + 1$.

(b):

- $147J_{n+3} = 40C_{n+3} + 21C_{n+2} - 7C_{n+1} - 54C_n$.
- $147J_n = -2C_{n+3} + 21C_{n+2} - 7C_{n+1} - 12C_n$.
- $C_{n+4} = 10J_{n+2} + 5J_{n+1} + 6J_n + 1$.
- $2C_n = -J_{n+2} + 7J_{n+1} - 3J_n + 2$.
- $147J_n = 19C_{n+2} - 9C_{n+1} - 16C_n + 6$.
- $C_{n+1} + 6C_n = 19J_{n+1} - 10J_n + 7$.

(c):

- $K_{n+3} = F_{n+3} + F_{n+2} + 4F_{n+1} - 6F_n$.
- $4K_n = -3F_{n+3} + 4F_{n+2} + 16F_{n+1} - 17F_n$.
- $147F_{n+4} = 195K_{n+2} + 181K_{n+1} + 202K_n - 49$.
- $147F_n = 17K_{n+2} + 10K_{n+1} - 4K_n - 49$.
- $4K_n = F_{n+2} + 13F_{n+1} - 23F_n - 3$.
- $3(17F_{n+1} - 27F_n) = 14K_n - K_{n+1} + 10$.

(d):

- $3K_{n+3} = 4C_{n+3} - C_{n+2} - C_{n+1} - 2C_n$.
- $3K_n = C_{n+3} - C_{n+2} - C_{n+1} + C_n$.
- $C_{n+4} = 2K_{n+2} + 3K_{n+1} + 2K_n + 1$.
- $C_n = K_n + 1$.
- $K_n = C_n - 1$.

(e):

- $j_{n+3} = F_{n+3} + 3F_{n+2} - 4F_n$.
- $j_n = -F_{n+3} + 3F_{n+2} - 2F_n$.
- $9F_{n+4} = 11j_{n+2} + 10j_n + 9j_{n+1} - 3$.
- $18F_n = j_{n+2} + 3j_{n+1} - j_n - 6$.
- $j_n = 2F_{n+2} - F_{n+1} - 4F_n - 1$.
- $3(F_{n+1} - 4F_n) = -2j_{n+1} + j_n + 3$.

(f):

- $49j_{n+3} = 72C_{n+3} - 21C_{n+2} + 7C_{n+1} - 58C_n$.
- $49j_n = 16C_{n+3} - 21C_{n+2} + 7C_{n+1} - 2C_n$.
- $3C_{n+4} = 4j_{n+2} + 8j_n + 9j_{n+1} + 3$.
- $24C_n = 11j_{n+2} - 21j_{n+1} + 19j_n + 24$.

- $49j_n = -5C_{n+2} + 23C_{n+1} + 30C_n - 48.$
- $11C_{n+1} + 10C_n = 5j_{n+1} + 18j_n + 21.$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

LEMMA 12. *Suppose that* $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized Friedrich sequence* $\{W_n\}$. *Then,* $\sum_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 2W_0)z + (W_2 - 2W_1)z^2 + (W_3 - 2W_2 - W_0)z^3}{1 - 2z - z^3 + 2z^4}.$$

Proof. Take $r = 2, s = 0, t = 1, u = -2$ in Lemma 3.

The previous lemma gives the following results as particular examples.

COROLLARY 13. *Generating functions of Friedrich and Friedrich-Lucas numbers are*

$$\sum_{n=0}^{\infty} F_n z^n = \frac{z}{1 - 2z - z^3 + 2z^4} = \frac{z}{(2z^3 + z^2 + z - 1)(z - 1)},$$
$$\sum_{n=0}^{\infty} C_n z^n = \frac{4 - 6z - z^3}{1 - 2z - z^3 + 2z^4} = \frac{4 - 6z - z^3}{(2z^3 + z^2 + z - 1)(z - 1)},$$

respectively.

3. Simson Formulas

Now, we present Simson's formula of generalized Friedrich numbers.

THEOREM 14 (Simson's Formula of Generalized Friedrich Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = 2^{n-3} \times (W_3 - W_0)(W_3 - W_2 - W_1 - 2W_0)(W_3^2 + 7W_2^2 + 7W_1^2 + 4W_0^2 - 5W_2W_3 + W_1W_3 - 7W_1W_2 + 2W_0W_3 - 2W_0W_2 - 8W_0W_1).$$

Proof. Take $r = 2, s = 0, t = 1, u = -2$ in Theorem 4. □

The previous theorem gives the following results as particular examples.

COROLLARY 15. *For all integers n, the Simson's formulas of Friedrich and Friedrich-Lucas numbers are given as*

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_{n} \\ F_{n+2} & F_{n+1} & F_{n} & F_{n-1} \\ F_{n+1} & F_{n} & F_{n-1} & F_{n-2} \\ F_{n} & F_{n-1} & F_{n-2} & F_{n-3} \end{vmatrix} = 2^{n-1},$$

$$\begin{vmatrix} C_{n+3} & C_{n+2} & C_{n+1} & C_{n} \\ C_{n+2} & C_{n+1} & C_{n} & C_{n-1} \\ C_{n+1} & C_{n} & C_{n-1} & C_{n-2} \\ C_{n} & C_{n-1} & C_{n-2} & C_{n-3} \end{vmatrix} = -1323 \times 2^{n-3},$$

respectively.

4. Some Identities

In this section, we obtain some identities of Friedrich and Friedrich-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{F_n\}$.

LEMMA 16. *The following equalities are true:*

- **(a):** $16W_n = (9W_0 + 16W_2 - 9W_3)F_{n+5} - 16(2W_2 - W_3)F_{n+4} - 16(2W_0 - W_1)F_{n+3} - (9W_0 + 32W_1 - 9W_3)F_{n+2}$.
- **(b):** $8W_n = (9W_0 - W_3)F_{n+4} - 8(2W_0 - W_1)F_{n+3} - 8(2W_1 - W_2)F_{n+2} - (9W_0 + 16W_2 - 9W_3)F_{n+1}$.
- **(c):** $4W_n = (W_0 + 4W_1 - W_3)F_{n+3} - 4(2W_1 - W_2)F_{n+2} - 4(2W_2 - W_3)F_{n+1} - (9W_0 - W_3)F_n$.
- **(d):** $2W_n = (W_0 + 2W_2 - W_3)F_{n+2} - 2(2W_2 - W_3)F_{n+1} - 2(2W_0 - W_1)F_n - (W_0 + 4W_1 - W_3)F_{n-1}$.
- **(e):** $W_n = W_0F_{n+1} + (W_1 - 2W_0)F_n + (W_2 - 2W_1)F_{n-1} + (W_3 - 2W_2 - W_0)F_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times F_{n+5} + b \times F_{n+4} + c \times F_{n+3} + d \times F_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times F_5 + b \times F_4 + c \times F_3 + d \times F_2 \\ W_1 &= a \times F_6 + b \times F_5 + c \times F_4 + d \times F_3 \\ W_2 &= a \times F_7 + b \times F_6 + c \times F_5 + d \times F_4 \\ W_3 &= a \times F_8 + b \times F_7 + c \times F_6 + d \times F_5 \end{aligned}$$

we find that $a = \frac{1}{16}(9W_0 + 16W_2 - 9W_3), b = W_3 - 2W_2, c = W_1 - 2W_0, d = \frac{1}{16}(9W_3 - 32W_1 - 9W_0)$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{C_n\}$.

Lemma 17. *The following equalities are true:*

- **(a):** $588W_n = -(139W_0 + 28W_1 + 112W_2 - 83W_3)C_{n+5} + 28(4W_0 + 7W_2 - 4W_3)C_{n+4} + 28(8W_0 - W_3)C_{n+3} + (195W_0 + 224W_1 + 112W_2 - 139W_3)C_{n+2}$.
- **(b):** $294W_n = -(83W_0 + 28W_1 + 14W_2 - 27W_3)C_{n+4} + 14(8W_0 - W_3)C_{n+3} + 14(2W_0 + 7W_1 - 2W_3)C_{n+2} + (139W_0 + 28W_1 + 112W_2 - 83W_3)C_{n+1}$.
- **(c):** $147W_n = -(27W_0 + 28W_1 + 14W_2 - 20W_3)C_{n+3} + 7(2W_0 + 7W_1 - 2W_3)C_{n+2} + 7(4W_0 + 7W_2 - 4W_3)C_{n+1} + (83W_0 + 28W_1 + 14W_2 - 27W_3)C_n$.
- **(d):** $147W_n = -(40W_0 + 7W_1 + 28W_2 - 26W_3)C_{n+2} + 7(4W_0 + 7W_2 - 4W_3)C_{n+1} + 7(8W_0 - W_3)C_n + 2(27W_0 + 28W_1 + 14W_2 - 20W_3)C_{n-1}$.
- **(e):** $147W_n = -(52W_0 + 14W_1 + 7W_2 - 24W_3)C_{n+1} + 7(8W_0 - W_3)C_n + 7(2W_0 + 7W_1 - 2W_3)C_{n-1} + 2(40W_0 + 7W_1 + 28W_2 - 26W_3)C_{n-2}$.

Now, we give a few basic relations between $\{F_n\}$ and $\{C_n\}$.

Lemma 18. *The following equalities are true:*

$$\begin{aligned}
147F_n &= 20C_{n+5} - 14C_{n+4} - 28C_{n+3} - 27C_{n+2}, \\
147F_n &= 26C_{n+4} - 28C_{n+3} - 7C_{n+2} - 40C_{n+1}, \\
147F_n &= 24C_{n+3} - 7C_{n+2} - 14C_{n+1} - 52C_n, \\
147F_n &= 41C_{n+2} - 14C_{n+1} - 28C_n - 48C_{n-1}, \\
147F_n &= 68C_{n+1} - 28C_n - 7C_{n-1} - 82C_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
16C_n &= F_{n+5} + 48F_{n+4} - 96F_{n+3} - F_{n+2}, \\
8C_n &= 25F_{n+4} - 48F_{n+3} - F_{n+1}, \\
4C_n &= F_{n+3} + 12F_{n+1} - 25F_n, \\
2C_n &= F_{n+2} + 6F_{n+1} - 12F_n - F_{n-1}, \\
C_n &= 4F_{n+1} - 6F_n - F_{n-2}.
\end{aligned}$$

5. Relations Between Special Numbers

In this section, we present identities on Friedrich, Friedrich-Lucas numbers and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers. We know from Lemma 11 that

$$\begin{aligned}
3F_n &= J_{n+2} + 2J_n - 1, \\
C_n &= K_n + 1.
\end{aligned}$$

Note also that from Lemma 16 and Lemma 17, we have the formulas of W_n as

$$\begin{aligned} 4W_n &= (W_0+4W_1-W_3)F_{n+3}-4(2W_1-W_2)F_{n+2}-4(2W_2-W_3)F_{n+1}-(9W_0-W_3)F_n, \\ 147W_n &= -(27W_0+28W_1+14W_2-20W_3)C_{n+3}+7(2W_0+7W_1-2W_3)C_{n+2} \\ & \quad +7(4W_0+7W_2-4W_3)C_{n+1}+(83W_0+28W_1+14W_2-27W_3)C_n. \end{aligned}$$

Using the above identities, we obtain relation of generalized Friedrich numbers in the following forms (in terms of third-order Jacobsthal and modified third-order Jacobsthal numbers):

LEMMA 19. *For all integers n, we have the following identities:*

(a): $6W_n = (-W_3+4W_2-2W_1-W_0)J_{n+2}+3(W_3-2W_2+W_0)J_{n+1}+(W_3-4W_2+8W_1-5W_0)J_n-2W_3+2W_2+2W_1+4W_0.$

(b): $147W_n = (6W_3-14W_2+21W_1-13W_0)K_{n+2}+(-8W_3+35W_2-28W_1+W_0)K_{n+1}+(13W_3-14W_2-28W_1+29W_0)K_n-49W_3+49W_2+49W_1+98W_0.$

6. On the Recurrence Properties of Generalized Friedrich Sequence

Taking $r=2, s=0, t=1, u=-2$ in Theorem 5, we obtain the following Proposition.

PROPOSITION 20. *For $n \in \mathbb{Z}$, generalized Friedrich numbers (the case $r=2, s=0, t=1, u=-2$) have the following identity:*

$$W_{-n} = \frac{2^{-n-1}}{3}(-6W_{3n}+6C_nW_{2n}-3C_n^2W_n+3C_{2n}W_n+W_0C_n^3+2W_0C_{3n}-3W_0C_nC_{2n}).$$

From the above Proposition 20 (or by taking $G_n=F_n$ and $H_n=C_n$ in (1.11) and (1.12) respectively), we have the following corollary which gives the connection between the special cases of generalized Friedrich sequence at the positive index and the negative index: for Friedrich and Friedrich-Lucas numbers: take $W_n=F_n$ with $F_0=0, F_1=1, F_2=2, F_3=4$ and take $W_n=C_n$ with $C_0=4, C_1=2, C_2=4, C_3=11$, respectively. Note that in this case $H_n=C_n$.

COROLLARY 21. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *Friedrich sequence:*

$$F_{-n} = \frac{2^{-n-1}}{3}(-6F_{3n}+6C_nF_{2n}-3C_n^2F_n+3C_{2n}F_n).$$

(b): *Friedrich-Lucas sequence:*

$$C_{-n} = \frac{2^{-n-1}}{3}\left(C_n^3+2C_{3n}-3C_{2n}C_n\right).$$

We can also present the formulas of F_{-n} and C_{-n} in the following forms.

COROLLARY 22. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): $F_{-n} = \dfrac{2^{-n-5}}{3}(-96F_{3n}+24(F_{n+3}+12F_{n+1}-25F_n)F_{2n}-3(F_{n+3}+12F_{n+1}-25F_n)^2F_n+12(F_{2n+3}+12F_{2n+1}-25F_{2n})F_n)$.

(b): $3F_{-n} = \dfrac{1}{2^n}(3J_n^2+6J_{n-2}^2+(J_{n+2}-7J_{n+1}+2J_{n-2})J_n-14J_{n-1}J_{n-2}+2J_{2n}+4J_{2n-4}-2^n)$.

(c): $147F_{-n} = \dfrac{1}{2^n}(-2K_n^2+10K_{n-1}^2+34K_{n-2}^2+2K_{2n}-10K_{2n-2}-34K_{2n-4}-49\times 2^n)$.

(d): $2^{n+2}C_{-n} = -9J_n^2+42J_{n-1}^2-12J_{n-2}^2-(3J_{n+2}-21J_{n+1}+98J_{n-1}+4J_{n-2})J_n+14(J_{n+1}+2J_{n-2})J_{n-1}-6J_{2n}+28J_{2n-2}-8J_{2n-4}+2^{n+2}$.

(e): $C_{-n} = \dfrac{1}{2^{n+1}}(K_n^2-K_{2n}+2^{n+1})$.

Proof. We use the identities, see Soykan [14],

$$\begin{aligned} J_{-n} &= \frac{1}{2^{n+1}}(3J_n^2+2J_{2n}+J_{n+2}J_n-7J_{n+1}J_n), \\ K_{-n} &= \frac{1}{2^{n+1}}(K_n^2-K_{2n}). \end{aligned}$$

(a): By using the identity $4C_n = F_{n+3}+12F_{n+1}-25F_n$ and Corollary 21, (or by using Corollary 6 (a)), we obtain (a).

(b): Since

$$3F_n = J_{n+2}+2J_n-1,$$

and

$$J_{-n} = \frac{1}{2^{n+1}}(3J_n^2+2J_{2n}+J_{n+2}J_n-7J_{n+1}J_n),$$

we get (b)

(c): Since $147F_n = 17K_{n+2}+10K_{n+1}-4K_n-49$ and $K_{-n} = \frac{1}{2^{n+1}}(K_n^2-K_{2n})$, we obtain (c).

(d): Since $2C_n = -J_{n+2}+7J_{n+1}-3J_n+2$ and $J_{-n} = \frac{1}{2^{n+1}}(3J_n^2+2J_{2n}+J_{n+2}J_n-7J_{n+1}J_n)$, , we get (d).

(e): Since $C_n = K_n+1$ and $K_{-n} = \frac{1}{2^{n+1}}(K_n^2-K_{2n})$, we obtain (e). □

7. Sum Formulas

The following Corollary gives sum formulas of third-order Jacobsthal numbers.

Corollary 23. *For $n \geq 0$, third-order Jacobsthal numbers have the following properties:*

(a): $\sum_{k=0}^{n} J_k = \frac{1}{3}(J_{n+3}-J_{n+1}-1)$.

(b): $\sum_{k=0}^{n} J_{2k} = \frac{1}{3}(J_{2n+1}+2J_{2n}-1)$.

(c): $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{3}(J_{2n+2}+2J_{2n+1})$.

Proof. It is given in Soykan [15]. □

The following Corollary presents sum formulas of Friedrich and Friedrich-Lucas numbers.

Corollary 24. *For $n \geq 0$, Friedrich and Friedrich-Lucas numbers have the following properties (in terms of third-order Jacobsthal numbers):*

(a):

(i): $\sum_{k=0}^{n} F_k = \frac{1}{3}(2J_{n+2} + J_{n+1} + 2J_n - n - 3)$.

(ii): $\sum_{k=0}^{n} F_{2k} = \frac{1}{3}(J_{2n+2} + J_{2n+1} + 2J_{2n} - n - 2)$.

(iii): $\sum_{k=0}^{n} F_{2k+1} = \frac{1}{3}(2J_{2n+2} + 3J_{2n+1} + 2J_{2n} - n - 2)$.

(b):

(i): $\sum_{k=0}^{n} C_k = 3J_{n+1} + J_n + n + 1$.

(ii): $\sum_{k=0}^{n} C_{2k} = \frac{1}{3}(2J_{2n+2} + 5J_{2n+1} - 4J_{2n} + 3n + 5)$.

(iii): $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{3}(7J_{2n+2} - 2J_{2n+1} + 4J_{2n} + 3n + 1)$.

Proof. The proof follows from Corollary 23 and the identities

$$\begin{aligned} 3F_n &= J_{n+2} + 2J_n - 1, \\ 2C_n &= -J_{n+2} + 7J_{n+1} - 3J_n + 2. \end{aligned} \qquad \square$$

8. Matrices and Identities Related With Generalized Friedrich Numbers

If we define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} F_{n+1} & F_{n-1} - 2F_{n-2} & F_n - 2F_{n-1} & -2F_n \\ F_n & F_{n-2} - 2F_{n-3} & F_{n-1} - 2F_{n-2} & -2F_{n-1} \\ F_{n-1} & F_{n-3} - 2F_{n-4} & F_{n-2} - 2F_{n-3} & -2F_{n-2} \\ F_{n-2} & F_{n-4} - 2F_{n-5} & F_{n-3} - 2F_{n-4} & -2F_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & W_{n-1} - 2W_{n-2} & W_n - 2W_{n-1} & -2W_n \\ W_n & W_{n-2} - 2W_{n-3} & W_{n-1} - 2W_{n-2} & -2W_{n-1} \\ W_{n-1} & W_{n-3} - 2W_{n-4} & W_{n-2} - 2W_{n-3} & -2W_{n-2} \\ W_{n-2} & W_{n-4} - 2W_{n-5} & W_{n-3} - 2W_{n-4} & -2W_{n-3} \end{pmatrix}.$$

then we get the following Theorem.

THEOREM 25. *For all integers m, n, we have*

(a): $B_n = A^n$, *i.e.*,

$$\begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_{n-1} - 2F_{n-2} & F_n - 2F_{n-1} & -2F_n \\ F_n & F_{n-2} - 2F_{n-3} & F_{n-1} - 2F_{n-2} & -2F_{n-1} \\ F_{n-1} & F_{n-3} - 2F_{n-4} & F_{n-2} - 2F_{n-3} & -2F_{n-2} \\ F_{n-2} & F_{n-4} - 2F_{n-5} & F_{n-3} - 2F_{n-4} & -2F_{n-3} \end{pmatrix}.$$

(b): $U_1 A^n = A^n U_1$.

(c): $U_{n+m} = U_n B_m = B_m U_n$.

Proof. Take $r = 2, s = 0, t = 1, u = -2$ in Theorem 7. □

Using the above last Theorem and the identity

$$3F_n = J_{n+2} + 2J_n - 1,$$

we obtain the following identity for third-order Jacobsthal numbers.

COROLLARY 26. *For all integers n, we have the following formula for third-order Jacobsthal numbers:*

$$A^n = \begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{3}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= J_{n+3} + 2J_{n+1} - 1 \\ a_{21} &= J_{n+2} + 2J_n - 1 \\ a_{31} &= J_{n+1} + 2J_{n-1} - 1 \\ a_{41} &= J_n + 2J_{n-2} - 1 \end{aligned}$$

$$\begin{aligned} a_{12} &= J_{n+1} - 2J_n + 2J_{n-1} - 4J_{n-2} + 1 \\ a_{22} &= J_n - 2J_{n-1} + 2J_{n-2} - 4J_{n-3} + 1 \\ a_{32} &= J_{n-1} - 2J_{n-2} + 2J_{n-3} - 4J_{n-4} + 1 \\ a_{42} &= J_{n-2} - 2J_{n-3} + 2J_{n-4} - 4J_{n-5} + 1 \end{aligned}$$

$$\begin{aligned} a_{13} &= J_{n+2} - 2J_{n+1} + 2J_n - 4J_{n-1} + 1 \\ a_{23} &= J_{n+1} - 2J_n + 2J_{n-1} - 4J_{n-2} + 1 \\ a_{33} &= J_n - 2J_{n-1} + 2J_{n-2} - 4J_{n-3} + 1 \\ a_{43} &= J_{n-1} - 2J_{n-2} + 2J_{n-3} - 4J_{n-4} + 1 \end{aligned}$$

$$\begin{aligned} a_{14} &= -2(J_{n+2} + 2J_n - 1) \\ a_{24} &= -2(J_{n+1} + 2J_{n-1} - 1) \\ a_{34} &= -2(J_n + 2J_{n-2} - 1) \\ a_{44} &= -2(J_{n-1} + 2J_{n-3} - 1) \end{aligned}$$

Next, we present an identity for W_{n+m}.

THEOREM 27. *For all integers m, n, we have*

$$W_{n+m} = W_n F_{m+1} + W_{n-1}(F_{m-1} - 2F_{m-2}) + W_{n-2}(F_m - 2F_{m-1}) - 2W_{n-3}F_m$$

Proof. Take $r = 2, s = 0, t = 1, u = -2$ in Theorem 8. $\square$

As particular cases of the above theorem, we give identities for F_{n+m} and C_{n+m}.

Corollary 28. *For all integers m, n, we have*

$$\begin{aligned} F_{n+m} &= F_n F_{m+1} + F_{n-1}(F_{m-1} - 2F_{m-2}) + F_{n-2}(F_m - 2F_{m-1}) - 2F_{n-3}F_m, \\ C_{n+m} &= C_n F_{m+1} + C_{n-1}(F_{m-1} - 2F_{m-2}) + C_{n-2}(F_m - 2F_{m-1}) - 2C_{n-3}F_m. \end{aligned}$$

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