GENERALIZATION OF THE EXPANSION OF THE LAGRANGE MULTIPLIER IDENTIFIER INTEGRALS FOR THE VARIATIONAL ITERATION METHOD

ABSTRACT

This study presents a theorem for resolving certain integrals. These integrals usually arise in the Correction Functional of the Variational Iteration Method (VIM). Resolving these integrals in the Correction Functional of VIM enables the accurate determination of the Lagrange multiplier for the method, which in turn leads to faster convergence rate for the solution ofVIM. The study begins with a brief overview of the Lagrange Multiplier and VIM. This is followed by the theorem and a few experimental problems considered to demonstrate the usefulness of the theorem in the implementation of VIM. Comparison of results obtained shows that accurately determined Lagrange Multiplier for VIM improves the convergence rate of the method compared to using approximately determined Lagrange multiplier for the scheme. Evaluation ~~are~~is carried out using Maple 18 software.

1.0 INTRODUCTION

The Lagrange multiplier is usually employed inconstrained optimization problems where there is need to find the maximum or minimum of a given Objective Function subject to another function called the Constraint.Let $f(x_1, x_2, \dots, x_n)$ be a multi variable function subject to the constraint $g(x_1, x_2, \dots, x_n) = k$. Where$k$ is a constant. Then to determine the maximum or minimum of $f(x_1, x_2, \dots, x_n)$, by the method of Lagrange Multiplier, a new function $f(x_1, x_2, \dots x_n, \lambda)$ is constructed and written as:

$$f(x_1, x_2, \dots, x_n, \lambda) = f(x_1, x_2, \dots, x_n) \pm \lambda(g(x_1, x_2, \dots, x_n) - k) \quad (1)$$

The new parameter λ, is called the Lagrange multiplier. ~~Making (~~Making (1) stationary by setting the L.H.S of (1) to zero and taking partial derivatives of (1) w.r.t $x_1, x_2, \dots, x_n$& λwe get the following system of equations.

$$\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \cdots, \frac{\partial f}{\partial x_n} = 0, \frac{\partial f}{\partial \lambda} = 0 \quad (2)$$

Solving the system (2) for $x_1, x_2, \ldots, x_n$ & λ, enables the determination ofthe maximum or minimum of $f(x_1, x_2, \ldots, x_n)$. In the end, the parameter λ (the Lagrange multiplier) disappears. Its role is as an aid, but in the ~~end~~end,it has no part in the solution [8].

He's Variational Iteration Method (VIM) also employs the use of the Lagrange multiplier in its correction functional. The accurate determination of the Lagrange multiplier for VIM is of much interest to mathematician. The success of VIM mainly depends upon the accurate determination of the Lagrange multiplier [9]. But accurate determination of the Lagrange multiplier for VIM can sometimes be challenging [5], [7], [9]. Over the years, a number of authors have proposed various methods for the determination of the Lagrange multiplier for VIM [2], [3],[9],[10]. In [7], we proposed the use of Integrating Factor method and Operator D-Method in certain aspects of the determination of the Lagrange Multiplier for VIM. In this study, we present a theory for resolving integrals that often arise in the implementation of VIM. Resolving these integrals aids the accurate determination of the Lagrange multiplier for VIM.

2.0**He's Variational Iteration Method**

Given a nonlinear differential equation of the form

$$Ly + Ny = g(x) \qquad (3)$$

Where$L$ is a linear operator, N is a nonlinear operator and $g(x)$ is a known function. By the Variational Iteration Method, we construct a correction functional of the form:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)(Ly_n(s) + N\tilde{y}_n(s) - g(s))ds \qquad (4)$$

Where $\lambda(s)$ is a general Lagrange's multiplier, which may be a constant or a function, and may be identified optimally via variational theory [1].The subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered a restricted variation [12], i.e. $\delta\tilde{u}_n = 0$. The Lagrange multiplier can be determined using

$$\lambda(s) = \frac{(-1)^n}{(n-1)!}(s-x)^{n-1} \qquad (5)$$

Where n represent the highest derivative in the given function.

3.0**Theorem** 1.0

Given an integral expression of the form

$$\int \lambda(s) y_n^{(k)}(s) ds, \qquad (6)$$

the expansion resolves as

$$\int \lambda(s) y_n^{(k)}(s) ds = \lambda(s) y_n^{(K-1)}(s) - \lambda'(s) y_n^{(K-2)}(s) + \lambda''(s) y_n^{(K-3)}(s) - \lambda'''(s) y_n^{(K-4)}(s) + \cdots - \lambda^{(K-1)}(s) y_n(s) + \int \lambda^{(K)}(s) y_n(s) ds \quad (7)$$

for all $k = 1,2,3 \ldots n$; where k represents the order of the derivative.

Proof:

Recall the formula from basic calculus for integration by part which reads:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \quad (8)$$

Let $k = 1$

Then the integral (6) is written as

$$\int \lambda(s) y_n'(s) ds \quad (9)$$

By comparing (9) to (8), we note that

$$dv = y_n'(s), v = \int dv = \int y_n'(s) = y_n(s), u = \lambda(s), du = \lambda'(s) ds \quad (10)$$

So that we get

$$\int \lambda(s) y_n'(s) ds = \lambda(s) y_n(s) - \int \lambda'(s) y_n(s) ds \quad (11)$$

Similarly, for $k = 2$

Then the integral (6) is written as

$$\int \lambda(s) y_n''(s) ds \quad (12)$$

We note again that

$$dv = y_n''(s), v = y_n'(s), u = \lambda(s) du = \lambda'(s) ds \quad (13)$$

So that we get

$$\int \lambda(s) y_n''(s) ds = \lambda(s) y'_n(s) - \int \lambda'(s) y'_n(s) ds \quad (14)$$

Integrating the integral sign $\int \lambda'(s) y'_n(s)\, ds$ in (14), we get

$\int \lambda'(s)y'_n(s)\, ds = \lambda'(s)y_n(s) - \int \lambda''(s)y_n(s)\, ds$ (15)

Substituting (15) into (14), we get

$\int \lambda(s)y''_n(s)ds = \lambda(s)y'_n(s) - \lambda'(s)y_n(s) + \int \lambda''(s)y_n(s)\, ds$(16)

$\vdots \qquad \vdots \qquad \vdots$

By mathematical induction, it follows that

When$\boldsymbol{k = n - 1}$, the integral (6) is written as

$\int \lambda(s)y_n^{(k-1)}(s)ds$ (17)

We note that

$dv = y_n^{(k-1)}(s)$, $v = \int dv = \int y_n^{(k-1)}(s) = y_n^{(k-2)}(s)$, $u = \lambda(s)du = \lambda'(s)ds$

So that we get,

$\int \lambda(s)y_n^{(k-1)}(s)ds = \lambda(s)y_n^{(k-2)}(s) - \int \lambda'(s)y_n^{(k-2)}(s)ds$(18)

Again, integrating the integral sign $\int \lambda'(s)y_n^{(k-2)}(s)ds$ in (18), we get:

$\int \lambda'(s)y_n^{(k-2)}(s)ds = \lambda'(s)y_n^{(k-3)}(s) - \int \lambda''(s)y_n^{(k-3)}(s)ds$ (19)

Substituting (19) into (18) we get

$\int \lambda(s)y_n^{(k-1)}(s)ds = \lambda(s)y_n^{(k-2)}(s) - \lambda'^{(s)y_n^{(k-3)}(s)} + \int \lambda''(s)y_n^{(k-3)}(s)ds$ (20)

$\vdots \qquad \vdots \qquad \vdots$

$\int \lambda(s)y_n^{(k-1)}(s)ds = \lambda(s)y_n^{(K-2)}(s) - \lambda'(s)y_n^{(K-3)}(s) + \lambda''(s)y_n^{(K-4)}(s) - \lambda'''(s)y_n^{(K-5)}(s) + \cdots + \lambda^{(K-2)}(s)y_n(s) - \int \lambda^{(K-1)}(s)y_n(s)ds$ (21)

And

$\int \lambda(s)y_n^{(k)}(s)ds = \lambda(s)y_n^{(K-1)}(s) - \lambda'(s)y_n^{(K-2)}(s) + \lambda''(s)y_n^{(K-3)}(s) - \lambda'''(s)y_n^{(K-4)}(s) + \cdots - \lambda^{(K-1)}(s)y_n(s) + \int \lambda^{(K)}(s)y_n(s)ds\, for\ all\ k$ (22)

Which completes the proof.

We summarize result of the resolved integrals for , $k = 1, 2, 3, \ldots n$ in the Table 1 below:

Table 1

Table Showing Expansion of the Integrals Arising From the Correction Functional of VIM

$\boldsymbol{k}$	$\int \boldsymbol{\lambda(s) y_n^{(k)}(s), ds}$	**Expansion**
1	$\int \lambda(s) y_n'(s) ds$	$\lambda(s) y_n(s) - \int \lambda'(s) y_n(s)$
2	$\int \lambda(s) y_n''(s) ds$	$\lambda(s) y'_n(s) - \lambda'(s) y_n(s) + \int \lambda''(s) y_n(s)$
3	$\int \lambda(s) y_n'''(s) ds$	$\lambda(s) y''_n(s) - \lambda'(s) y'_n(s) + \lambda''(s) y_n(s) - \int \lambda'''(s) y_n(s)$
4	$\int \lambda(s) y_n^{(iv)}(s) ds$	$\lambda(s) y'''_n(s) - \lambda'(s) y''_n(s) + \lambda''(s) y'_n(s) - \lambda'''(s) y_n(s) + \int \lambda^{(iv)}(s) y_n(s) ds$
5	$\int \lambda(s) y_n^{(v)}(s) ds$	$\lambda(s) y_n^{(iv)}(s) - \lambda'(s) y_n'''(s) + \lambda''(s) y_n''(s) - \lambda'''(s) y_n'(s) + \lambda^{(iv)}(s) y_n(s) - \int \lambda^{(v)}(s) y_n(s) ds$
6	$\int \lambda(s) y_n^{(vi)}(s) ds$	$\lambda(s) y_n^{(v)}(s) - \lambda'(s) y_n^{(iv)}(s) + \lambda''(s) y_n'''(s) - \lambda'''(s) y_n''(s) + \lambda^{(iv)}(s) y'_n(s) - \lambda^{(v)}(s) y_n(s) + \int \lambda^{(vi)}(s) y_n(s) ds$
7	$\int \lambda(s) y_n^{(vii)}(s) ds$	$\lambda(s) y_n^{(vi)}(s) - \lambda'(s) y_n^{(v)}(s) + \lambda''(s) y_n^{(iv)}(s) - \lambda'''(s) y_n'''(s) + \lambda^{(iv)}(s) y''_n(s) - \lambda^{(v)}(s) y'_n(s) + \lambda^{(vi)}(s) y_n(s) - \int \lambda^{(vii)}(s) y_n(s) ds$
8	$\int \lambda(s) y_n^{(viii)}(s) ds$	$\lambda(s) y_n^{(vii)}(s) - \lambda'(s) y_n^{(vi)}(s) + \lambda''(s) y_n^{(v)}(s) - \lambda'''(s) y_n^{(iv)}(s) + \lambda^{(iv)}(s) y'''_n(s) - \lambda^{(v)}(s) y''_n(s) + \lambda^{(vi)}(s) y'_n(s) - \lambda^{(vii)}(s) y_n(s) + \int \lambda^{(viii)}(s) y_n(s) ds$

9	$\int \lambda(s)y_n^{(ix)}(s)ds$	$\lambda(s)y_n^{(viii)}(s) - \lambda'(s)y_n^{(vii)}(s) + \lambda''(s)y_n^{(vi)}(s) - \lambda'''(s)y_n^{(v)}(s) + \lambda^{(iv)}(s)y^{(iv)}{}_n(s) - \lambda^{(v)}(s)y'''{}_n(s) + \lambda^{(vi)}(s)y''{}_n(s) - \lambda^{(vii)}(s)y'{}_n(s) + \lambda^{(viii)}(s)y_n(s) - \int \lambda^{(ix)}(s)y_n(s)ds$
10	$\int \lambda(s)y_n^{(x)}(s)ds$	$\lambda(s)y_n^{(ix)}(s) - \lambda'(s)y_n^{(viii)}(s) + \lambda''(s)y_n^{(vii)}(s) - \lambda'''(s)y_n^{(vi)}(s) + \lambda^{(iv)}(s)y^{(v)}{}_n(s) - \lambda^{(v)}(s)y^{(iv)}{}_n(s) + \lambda^{(vi)}(s)y'''{}_n(s) - \lambda^{(vii)}(s)y''{}_n(s) + \lambda^{(viii)}(s)y'{}_n(s) - \lambda^{(ix)}(s)y_n(s) + \int \lambda^{(x)}(s)y_n(s)ds$
11	$\int \lambda(s)y_n^{(xi)}(s)ds$	$\lambda(s)y_n^{(x)}(s) - \lambda'(s)y_n^{(ix)}(s) + \lambda''(s)y_n^{(viii)}(s) - \lambda'''(s)y_n^{(vii)}(s) + \lambda^{(iv)}(s)y^{(vi)}{}_n(s) - \lambda^{(v)}(s)y^{(v)}{}_n(s) + \lambda^{(vi)}(s)y^{(iv)}{}_n(s) - \lambda^{(vii)}(s)y'''{}_n(s) + \lambda^{(viii)}(s)y''{}_n(s) - \lambda^{(ix)}(s)y'{}_n(s) + \lambda^{(x)}(s)y_n(s) - \int \lambda^{(xi)}(s)y_n(s)ds$
12	$\int \lambda(s)y_n^{(xii)}(s)ds$	$\lambda(s)y_n^{(xi)}(s) - \lambda'(s)y_n^{(x)}(s) + \lambda''(s)y_n^{(ix)}(s) - \lambda'''(s)y_n^{(viii)}(s) + \lambda^{(iv)}(s)y^{(vii)}{}_n(s) - \lambda^{(v)}(s)y^{(vi)}{}_n(s) + \lambda^{(vi)}(s)y^{(v)}{}_n(s) - \lambda^{(vii)}(s)y_n^{(iv)}(s) + \lambda^{(viii)}(s)y'''{}_n(s) - \lambda^{(ix)}(s)y''{}_n(s) + \lambda^{(x)}(s)y'{}_n(s) - \lambda^{(xi)}(s)y_n(s) + \int \lambda^{(xii)}(s)y_n(s)ds$

13	$\int \lambda(s) y_n^{(xiii)}(s) ds$	$\lambda(s)y_n^{(xii)}(s) - \lambda'(s)y_n^{(xi)}(s) + \lambda''(s)y_n^{(x)}(s) - \lambda'''(s)y_n^{(ix)}(s) + \lambda^{(iv)}(s){y^{(viii)}}_n(s) - \lambda^{(v)}(s){y^{(vii)}}_n(s) + \lambda^{(vi)}(s){y^{(vi)}}_n(s) - \lambda^{(vii)}(s)y_n^{(v)}(s) + \lambda^{(viii)}(s){y^{(iv)}}_n(s) - \lambda^{(ix)}(s){y'''}_n(s) + \lambda^{(x)}(s){y''}_n(s) - \lambda^{(xi)}(s){y'}_n(s) + \lambda^{(xii)}(s)y_n(s) - \int \lambda^{(xii)}(s)y_n(s)ds$
14	$\int \lambda(s) y_n^{(xiv)}(s) ds$	$\lambda(s)y_n^{(xiii)}(s) - \lambda'(s)y_n^{(xii)}(s) + \lambda''(s)y_n^{(xi)}(s) - \lambda'''(s)y_n^{(x)}(s) + \lambda^{(iv)}(s){y^{(ix)}}_n(s) - \lambda^{(v)}(s){y^{(viii)}}_n(s) + \lambda^{(vi)}(s){y^{(vii)}}_n(s) - \lambda^{(vii)}(s)y_n^{(vi)}(s) + \lambda^{(viii)}(s){y^{(v)}}_n(s) - \lambda^{(ix)}(s){y^{(iv)}}_n(s) + \lambda^{(x)}(s){y'''}_n(s) - \lambda^{(xi)}(s){y''}_n(s) + \lambda^{(xii)}(s){y'}_n(s) - \lambda^{(xiii)}(s)y_n(s) + \int \lambda^{(xiv)}(s)y_n(s)ds$
15	$\int \lambda(s) y_n^{(xv)}(s) ds$	$\lambda(s)y_n^{(xiv)}(s) - \lambda'(s)y_n^{(xiii)}(s) + \lambda''(s)y_n^{(xii)}(s) - \lambda'''(s)y_n^{(xi)}(s) + \lambda^{(iv)}(s){y^{(x)}}_n(s) - \lambda^{(v)}(s){y^{(ix)}}_n(s) + \lambda^{(vi)}(s){y^{(viii)}}_n(s) - \lambda^{(vii)}(s)y_n^{(vii)}(s) + \lambda^{(viii)}(s){y^{(vi)}}_n(s) - \lambda^{(ix)}(s){y^{(v)}}_n(s) + \lambda^{(x)}(s){y^{(iv)}}_n(s) - \lambda^{(xi)}(s){y'''}_n(s) + \lambda^{(xii)}(s){y''}_n(s) - \lambda^{(xiii)}(s){y'}_n(s) + \lambda^{(xiv)}(s)y_n(s) - \int \lambda^{(xv)}(s)y_n(s)ds$
⋮	⋮	⋮

$k-1$	$\int \lambda(s) y_n^{(k-1)}(s) ds$	$\lambda(s) y_n^{(K-2)}(s) - \lambda'(s) y_n^{(K-3)}(s) + \lambda''(s) y_n^{(K-4)}(s) - \lambda'''(s) y_n^{(K-5)}(s) + \cdots + \lambda^{(K-2)}(s) y_n(s) - \int \lambda^{(K-1)}(s) y_n(s) ds$
k	$\int \lambda(s) y_n^{(k)}(s) ds$	$\lambda(s) y_n^{(K-1)}(s) - \lambda'(s) y_n^{(K-2)}(s) + \lambda''(s) y_n^{(K-3)}(s) - \lambda'''(s) y_n^{(K-4)}(s) + \cdots - \lambda^{(K-1)}(s) y_n(s) + \int \lambda^{(K)}(s) y_n(s) ds$

We remark that these expansions can be substituted directly where necessary,into the correction functional of VIM to aid the process of computing the exact Lagrange multiplier for VIM.

4.0 NUMERICAL EXAMPLES

Problem 1 [11]

$y''' = -4y' + x,\ x \in [0, 4\pi]$ (23)

$y(0) = 0,\ y'(0) = 0,\quad y''(0) = 1$ (24)

$y(x) = \left(\frac{3}{16}\right)(1 - \cos 2x) + \frac{1}{8}x^2$ (25)

A. Problem solved using an approximate Lagrange Multiplier

To solve the above problem using the Variational Iteration Method, we construct a correction functional of the form:

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda(s) \left(\frac{d^3 y_n}{ds^3} + 4\frac{dy_n}{ds} - s\right) ds \quad (26)$$

Using the formula (5), we determine the Lagrange multiplier as $(s) = -\frac{(s-x)^2}{2}$, so that we get the iteration formula as:

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(s-x)^2}{2} \left(\frac{d^3 y_n}{ds^3} + 4\frac{dy_n}{ds} - s\right) ds \quad (27)$$

The initial approximation is determined using

$$y_0(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2} \quad (28)$$

$$\Rightarrow y_0(x) = \frac{1}{2}x^2 \quad (29)$$

Successive iterations are determined using the initial approximation (29) and the iteration formula (27) as:

$$y_0(x) = \frac{1}{2}x^2$$

$$y_1(x) = \frac{1}{2}x^2 - \frac{1}{8}x^4$$

$$y_2(x) = \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{60}x^6$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_5(x) = \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{60}x^6 - \frac{1}{840}x^8 + \frac{1}{18900}x^{10} - \frac{1}{623700}x^{12}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_{38}(x) = \frac{1}{2}x^2 + 1/ \; 199834151289843357697126200236765184533242737906204200794977428790720843 \; 402148933410644531 25\,x^{78} - \ldots$$

…and so on.

B. Problem solved using a more exact Lagrange Multiplier

We proceed by taking variation of the correction functional (26) to get

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(s)\left(\frac{d^3 y_n}{ds^3} + 4\frac{dy_n}{ds} - s\right) ds \quad (30)$$

Using theorem (1.0) on the integral in (30), (30) expands as:

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\, \lambda(s) y''_n(s) - \delta\lambda'(s) y'_n(s) + \delta\lambda''(s) y_n(s) - \delta \int \lambda'''(s) y_n(s) + 4\delta\lambda(s) y_n(s) - 4\delta \int \lambda'(s) y_n(s) \quad (31)$$

Making (31) stationary by setting the ~~Left hand~~Left-hand side of (31) to zero, we get the following system.

$\delta y_n(x): 1 + 4\lambda(s) + \lambda''(s) = 0$ (a)

$\delta y'_n(s): -\lambda'(s) = 0$ (b)

$\delta y''_n(s): \lambda(s) = 0$ (c)

$\delta y_n^*(s): -\lambda'''(s) - 4\lambda'(s) = 0$ (d)

Solving the system (a)-(d), we determine a general solution of the form.

$$\lambda(s) = A + B\cos(2s) + C\sin(2s)$$

The coefficients A, B and C are determined as

$$A = -\frac{1}{4},\ B = \frac{1}{4}\frac{\cos(2x)}{\cos(2x)^2 + \sin(2x)^2},\ and\ C = \frac{1}{4}\frac{\sin(2x)}{\cos(2x)^2 + \sin(2x)^2} \quad (32)$$

So that we determine the Lagrange multiplier as

$$\lambda(s) = -\frac{1}{4} + \frac{1}{4}\frac{\cos(2x)\cos(2s)}{\cos(2x)^2 + \sin(2x)^2} + \frac{1}{4}\frac{\sin(2x)\sin(2s)}{\cos(2x)^2 + \sin(2x)^2} \quad (33)$$

And the iteration formula as

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(-\frac{1}{4} + \frac{1}{4}\frac{\cos(2x)\cos(2s)}{\cos(2x)^2 + \sin(2x)^2} + \frac{1}{4}\frac{\sin(2x)\sin(2s)}{\cos(2x)^2 + \sin(2x)^2}\right)\left(\frac{d^3 y_n}{ds^3} + 4\frac{dy_n}{ds} - s\right) ds \quad (34)$$

The initial approximation is chosen using as $y_0(x) = \frac{1}{2}x^2$ (35)

Using the initial approximation (35) and the iteration formula (34), we obtain the following successive

$$y_1(x) = \frac{1}{8}x^2 - \frac{9}{8}\cos(x)^2 + \frac{3}{8} + \frac{3}{4}\cos(x)^4 + \frac{3}{4}\sin(x)^2\cos(x)^2$$

$$y_2(x) = \frac{1}{8}x^2 - \frac{9}{8}\cos(x)^2 + \frac{3}{8} + \frac{3}{4}\cos(x)^4 + \frac{3}{4}\sin(x)^2\cos(x)^2$$

We note that only one iteration was required to attain convergence using a more accurate Lagrange Multiplier. We compare results in Table 2.

Table 2.

Table comparing exact solution, solution of the 37th iteration using approximate Lagrange Multiplier, and 1st Iteration using a more accurate Lagrange multiplier for problem 1.

x	Exact	$y_{38}(x)$ with Approx. $\lambda(s)$	Error with Approx. $\lambda(s)$	$y_1(x)$ with More Exact $\lambda(s)$	Error with More Exact. $\lambda(s)$
0.00	0.0000000000	0.0000000000	0.0000e+00	0.0000000000	0.0000e+00
1.00	0.3905275318	0.3905275319	1.0000e-10	0.3905275319	1.0000e-10
2.00	0.8100581789	0.8100581789	0.0000e+00	0.8100581790	0.0000e+00
3.00	1.1324680710	1.1324680710	0.0000e+00	1.1324680710	0.0000e+00
4.00	2.2147812560	2.2147812560	0.0000e+00	2.2147812560	0.0000e+00
5.00	3.4698259120	3.4698259120	0.0000e+00	3.4698259120	0.0000e+00
6.00	4.5292773830	4.5292773830	0.0000e+00	4.5292773830	0.0000e+00
7.00	6.2868617720	6.2868617720	0.0000e+00	6.2868617720	0.0000e+00
8.00	8.3670611530	8.3670611530	0.0000e+00	8.3670611530	0.0000e+00
9.00	10.1886906200	10.1886906200	0.0000e+00	10.1886906200	0.0000e+00
10.00	12.6109846100	12.6109846100	0.0000e+00	12.6109846100	0.0000e+00
11.00	15.4999926600	15.4999926500	1.0000e-08	15.4999926500	1.0000e-08
12.00	18.1079664400	18.1079664400	0.0000e+00	18.1079664400	0.0000e+00

Problem 2

Consider the non-linear PDE

$y_{ttt} + ty_{xx} - 4y_t = -te^{-2t}\cos(x)$(36)

With the initial conditions

$y(x,0) = \cos(x)\,,\, y_t(x,0) = -2\cos(x)\,,\, y_{tt}(x,0) = 4\cos(x)$ (37)

$y(x,t) = e^{-2t}\cos(x)$ (38)

A. Problem solved using an approximate Lagrange Multiplier

To solve the above problem using the VIM, we construct a correction functional of the form:

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \lambda(s)\left(\frac{\partial y_n^3}{\partial s^3} + s\frac{\partial y_n^2}{\partial x^2} - 4\frac{\partial y}{\partial s} + se^{-2s}\cos(x)\right)ds \quad (39)$$

Using the formula (5) we determine the Lagrange multiplier as,$(s) = -\frac{(s-x)^2}{2}$, so that we get the iteration formula as:

$$y_{n+1}(x,t) = y_n(x,t) - \int_0^t \lambda\frac{(s-x)^2}{2})\left(\frac{\partial y_n^3}{\partial s^3} + s\frac{\partial y_n^2}{\partial x^2} - 4\frac{\partial y}{\partial s} + se^{-2s}\cos(x)\right)ds \quad (40)$$

The initial approximation is determined using

$$y_0(x) = y(0) + y'(0)t + y''(0)\frac{t^2}{2} \quad (41)$$

$$\Rightarrow y_0(x) = \cos(x) - 2t\cos(x) + \frac{4t^2}{2}\cos(x)$$

Using the initial approximation and the iteration formula, we determine the following iterations:

$$y_0(x) = \cos(x) - 2t\cos(x) + \frac{4t^2}{2}\cos(x)$$

$$y_1(x,t) = \frac{13}{16}\cos(x) - \frac{7}{4}t\cos(x) + \frac{15}{8}t^2\cos(x) - \frac{4}{3}t^3\cos(x) + \frac{17}{24}\cos(x)\,t^4 - \frac{1}{30}\cos(x)\,t^5 + \frac{1}{60}\cos(x)\,t^6 + \frac{3}{16}e^{-2t}\cos(x) + \frac{1}{8}e^{-2t}\cos(x)\,t$$

$$y_2(x,t) = \frac{149}{256}\cos(x) - \frac{43}{32}t\cos(x) + \frac{197}{128}t^2\cos(x) - \frac{7}{6}t^3\cos(x) + \frac{253}{384}\cos(x)\,t^4 - \frac{71}{240}\cos(x)\,t^5 + \frac{317}{2880}\cos(x)\,t^6 + \frac{107}{256}e^{-2t}\cos(x) + \frac{23}{128}e^{-2t}\cos(x)\,t - \frac{1}{64}t^2e^{-2t}\cos(x) + \frac{1}{43200}\cos(x)\,t^{10} - \frac{1}{105}\cos(x)\,t^7 + \frac{19}{5760}\cos(x)\,t^8 - \frac{1}{15120}\cos(x)\,t^9$$

…and so on.

B. Problem solved Using a More exact Lagrange Multiplier

We proceed by constructing correction functional of the form

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \lambda(s)\left(\frac{\partial y_n^3}{\partial s^3} + s\frac{\partial \tilde{y}_n^2}{\partial x^2} - 4\frac{\partial y}{\partial s} + se^{-2s}\cos(x)\right)ds \quad (42)$$

Taking variation of the correction functional (42) we get

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \delta\int_0^t \lambda(s)\left(\frac{\partial y_n^3}{\partial s^3} + s\frac{\partial \tilde{y}_n^2}{\partial x^2} - 4\frac{\partial y}{\partial s} + se^{-2s}\cos(x)\right)ds \quad (43)$$

Using theorem (1.0) on the integral sign in (43), (43) expands as:

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \delta\,\lambda(s)y''_n(x,s) - \delta\lambda'(s)y'_n(x,s) + \delta\lambda''(s)y_n(x,s) - \delta\int\lambda'''(s)y_n(x,s) - 4\delta\lambda(s)y_n(x,s) + 4\delta\int\lambda'(s)y_n(x,s) \quad (44)$$

Making (44) stationary by setting the ~~Left hand~~Left-hand side of (44) to zero, we get the following system

$\delta y_n(x): 1 - 4\lambda(s) + \lambda''(s) = 0$ (a)

$\delta y'_n(s): -\lambda'(s) = 0$ (b)

$\delta y''_n(s): \lambda(s) = 0$ (c)

$\delta y_n^*(s): -\lambda'''(s) + 4\lambda'(s) = 0$ (d)

Solving the system (a)-(d), we determine a general solution of the form.

$$\lambda(s) = A + Be^{2s} + Ce^{-2s} \quad (45)$$

The coefficients $A, B\ and\ C$ are determined as

$$A = \frac{1}{4}, B = -\frac{1}{8e^{2t}} \text{ and } C = -\frac{1}{8e^{-2t}} \quad (46)$$

And the Lagrange multiplier as

$$\lambda(s) = \frac{1}{4} - \frac{1}{8}\frac{e^{2s}}{e^{2t}} - \frac{1}{8}\frac{e^{-2s}}{e^{-2t}} \quad (47)$$

The iteration formula is determined as

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \left(\frac{1}{4} - \frac{1}{8}\frac{e^{2s}}{e^{2t}} - \frac{1}{8}\frac{e^{-2s}}{e^{-2t}}\right)\left(\frac{\partial y_n^3}{\partial s^3} + s\frac{\partial y_n^2}{\partial x^2} - 4\frac{\partial y}{\partial s} + se^{-2s}\cos(x)\right)ds \quad (48)$$

Using $y_0(x) = \cos(x) - 2t\cos(x) + \frac{4t^2}{2}\cos(x)$ as the initial approximation, and the Iteration formula (48), we obtain the following successive iterations

$$y_0(x) = \cos(x) - 2t\cos(x) + \frac{4t^2}{2}\cos(x)$$

$$y_1(x,t) = \cos(x) - 2t\cos(x) + 2t^2\cos(x) + \frac{1}{384}\cos(x)\left(-48t^4e^{2t} + 64t^3e^{2t} + 21e^{4t} - 960t^2e^{2t} + 864te^{2t} - 456e^{2t} - 24t^2 - 36t + 435\right)e^{-2t}$$

$$y_2(x,t) = \cos(x) - 2t\cos(x) + 2t^2\cos(x) + \frac{1}{384}\cos(x)\left(-48t^4e^{2t} + 64t^3e^{2t} + 21e^{4t} - 960t^2e^{2t} + 864te^{2t} - 456e^{2t} - 24t^2 - 36t + 435\right)e^{-2t} + \frac{1}{245760}\cos(x)\left(1280t^6e^{2t} - 2048t^5e^{2t} + 840e^{4t}t^2 + 48000t^4e^{2t} - 1260te^{4t} - 56320t^3e^{2t} - 21225e^{4t} + 180480t^2e^{2t} - 480t^4 - 84480te^{2t} - 2400t^3 + 74880e^2 + 12720t^2 + 20880t - 53655\right)e^{-2t}$$

…and so on.

We present a comparison of results in Table 3 and Table 4.

Table 3

Table comparing exact solution, solution of the 9th iteration using an approximate Lagrange Multiplier, and 4th Iteration using a more exact Lagrange multiplier for problem 2 for $0 \le x \le 10$, *and* $t = 3$

x	Exact	$y_{10}(x,t)$ with Approx. $\lambda(s)$	Error with Approx. $\lambda(s)$	$y_5(x,t)$ with More Exact $\lambda(s)$	Error with More Exact. $\lambda(s)$
0.00	0.0024787522	0.0025177847	3.9033e-05	0.0024824882	3.7361e-06
1.00	0.0013392755	0.0013603649	2.1089e-05	0.0013412933	2.0178e-06
2.00	-0.0010315249	-0.0010477682	1.6243e-05	-0.0010330788	1.5539e-06
3.00	-0.0024539461	-0.0024925880	3.8642e-05	-0.0024576491	3.7030e-06
4.00	-0.0016202205	-0.0016457339	2.5513e-05	-0.0016226611	2.4406e-06
5.00	0.0007031283	0.0007142003	1.1072e-05	0.0007041880	1.0597e-06
6.00	0.0023800242	0.0024175021	3.7478e-05	0.0023836157	3.5915e-06
7.00	0.0018687369	0.0018981636	2.9427e-05	0.0018715557	2.8188e-06
8.00	-0.0003606585	-0.0003663378	5.6792e-06	-0.0003612016	5.4305e-07
9.00	-0.0022584661	-0.0022940299	3.5564e-05	-0.0022618691	3.4030e-06
10.00	-0.0020798504	-0.0021126015	3.2751e-05	-0.0020829908	3.1404e-06

Table 4

Table comparing exact solution, solution of the 9th iteration using approximate Lagrange Multiplier, and 2nd ~~Iteration~~nd Iteration using a more accurate Lagrange multiplier for problem 2 for $0 \le x \le 10$, *and* $t = 0.8$

x	**Exact**	$y_{10}(x,t)$ **with Approx.** $\lambda(s)$	**Error with Approx.** $\lambda(s)$	$y_3(x,t)$ **with More Exact** $\lambda(s)$	**Error with More Exact.** $\lambda(s)$
0.00	0.2018965180	0.2018965180	0.0000e+00	0.2018965180	2.0000e-10
1.00	0.1090851542	0.1090851542	1.0000e-10	0.1090851542	1.0000e-10
2.00	-0.0840185973	-0.0840185973	0.0000e+00	-0.0840185973	0.0000e+00
3.00	-0.1998760379	-0.1998760379	0.0000e+00	-0.1998760379	0.0000e+00
4.00	-0.1319683711	-0.1319683711	0.0000e+00	-0.1319683711	0.0000e+00
5.00	0.0572704075	0.0572704075	0.0000e+00	0.0572704075	0.0000e+00
6.00	0.1938550376	0.1938550376	0.0000e+00	0.1938550376	0.0000e+00
7.00	0.1522102401	0.1522102401	0.0000e+00	0.1522102401	0.0000e+00

8.00	-0.0293759502	-0.0293759502	0.0000e+00	-0.0293759502	0.0000e+00
9.00	-0.1839540273	-0.1839540273	0.0000e+00	-0.1839540273	0.0000e+00
10.00	-0.1694056201	-0.1694056201	0.0000e+00	-0.1694056201	0.0000e+00

Conclusion

This study has presented a theorem for resolving integrals that often arise in the correction functional of VIM which aids the accurate determination of the Lagrange multiplier for VIM. Resolving these integrals can be tricky and time consuming, but the proposed theorem 1.0 gives a straight forward handling. The study also further demonstrated the importance of using accurately determined Lagrange multiplier as against using approximate Lagrange multiplier in the implementation of VIM.

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