Original Research Article

~~On~~ Generalized Third-Order Jacobsthal Numbers

Abstract. In this paper, we investigate the generalized third order Jacobsthal sequences and we deal with, ~~in detail,~~ four special cases, namely, third order Jacobsthal, third order Jacobsthal-Lucas, modified third order Jacobsthal, third order Jacobsthal Perrin sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

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1. Introduction

In this paper, we introduce the generalized third order Jacobsthal sequences and we investigate, in detail, four special cases: third order Jacobsthal, third order Jacobsthal-Lucas, modified third order Jacobsthal and Jacobsthal Perrin sequences.

It is well-known that the Jacobsthal sequence (sequence A001045 in [34]) $\{J_n\}$ is defined recursively by the equation, for $n \geq 0$

$$J_{n+2} = J_{n+1} + 2J_n$$

in which $J_0 = 0$ and $J_1 = 1$. Then Jacobsthal sequence (second order Jacobsthal sequence) is

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, ...$$

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1,2,5,12,13,16,17,20,22,23,27,28,29].

For higher order Jacobsthal sequences, see [6,7,8,9,10,11,15,39].

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [3,4,14,18,19,30,31,32,33,35,38,41,42].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B \\ \beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B \\ \gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \quad B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3} \\ \Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \quad \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3) \end{aligned}$$

Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma &= r, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\ \alpha\beta\gamma &= t. \end{aligned}$$

If $\Delta(r,s,t) > 0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{1.3}$$

where

$$b_1 = W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \; b_2 = W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \; b_3 = W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0.$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [24]. This result of Howard and Saidak [24] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r = s = 1$, $t = 2$ and in this case we write $V_n = W_n$. A generalized third order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + 2V_{n-3} \tag{1.4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.4) holds for all integer n.

(1.3) can be used to obtain Binet formula of generalized third order Jacobsthal numbers. Binet formula of generalized third order Jacobsthal numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{1.5}$$

Here, α, β and $\gamma$are the roots of the cubic equation $x^3 - x^2 - x - 2 = 0$. Moreover

$$\begin{aligned} \alpha &= 2 \\ \beta &= \frac{-1+i\sqrt{3}}{2} \\ \gamma &= \frac{-1-i\sqrt{3}}{2}. \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta+\gamma &= 1, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\ \alpha\beta\gamma &= 2. \end{aligned}$$

The first few generalized third order Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized third order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$\frac{1}{2}V_2 - \frac{1}{2}V_1 - \frac{1}{2}V_0$
2	V_2	$-\frac{1}{4}V_2 + \frac{3}{4}V_1 - \frac{1}{4}V_0$
3	$V_2 + V_1 + 2V_0$	$-\frac{1}{8}V_2 - \frac{1}{8}V_1 + \frac{7}{8}V_0$
4	$2V_2 + 3V_1 + 2V_0$	$\frac{7}{16}V_2 - \frac{9}{16}V_1 - \frac{9}{16}V_0$
5	$5V_2 + 4V_1 + 4V_0$	$-\frac{9}{32}V_2 + \frac{23}{32}V_1 - \frac{9}{32}V_0$
6	$9V_2 + 9V_1 + 10V_0$	$-\frac{9}{64}V_2 - \frac{9}{64}V_1 + \frac{55}{64}V_0$
7	$18V_2 + 19V_1 + 18V_0$	$\frac{55}{128}V_2 - \frac{73}{128}V_1 - \frac{73}{128}V_0$
8	$37V_2 + 36V_1 + 36V_0$	$-\frac{73}{256}V_2 + \frac{183}{256}V_1 - \frac{73}{256}V_0$

Now we present four special case of the sequence $\{V_n\}$. Third-order Jacobsthal sequence$\{J_n\}_{n\geq 0}$, third-order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$, modified third-order Jacobsthal sequence $\{K_n\}_{n\geq 0}$ and third-order Jacobsthal Perrin sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2} + J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, J_2 = 1, \tag{1.6}$$

$$j_{n+3} = j_{n+2} + j_{n+1} + 2j_n, \quad j_0 = 2, j_1 = 1, j_2 = 5 \tag{1.7}$$

and

$$K_{n+3} = K_{n+2} + K_{n+1} + 2K_n, \quad K_0 = 3, K_1 = 1, K_2 = 3. \tag{1.8}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ are defined in [15] and $K_n\}_{n\geq 0}$ is given in [6]. In this paper we introduce another third order sequence namely: third-order Jacobsthal Perrin sequence which is given by the third order reccurence relations

$$Q_{n+3} = Q_{n+2} + Q_{n+1} + 2Q_n, \quad Q_0 = 3, Q_1 = 0, Q_2 = 2. \tag{1.9}$$

The sequences $\{J_n\}_{n\geq 0}$, $\{j_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n} = -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} + \frac{1}{2}J_{-(n-3)} \tag{1.10}$$

and

$$j_{-n} = -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} + \frac{1}{2}j_{-(n-3)} \tag{1.11}$$

and

$$K_{-n} = -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)} \tag{1.12}$$

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and

$$Q_{-n} = -\frac{1}{2}Q_{-(n-1)} - \frac{1}{2}Q_{-(n-2)} + \frac{1}{2}Q_{-(n-3)} \tag{1.13}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.10), (1.11) (1.12) and (1.13) hold for all integer n.

In the rest of the paper, for easy writing, we drop the superscripts and write J_n, j_n, K_n and Q_n for $J_n^{(3)}, j_n^{(3)}, K_n^{(3)}$ and $Q_n^{(3)}$ respectively.

Note that J_n is the sequence A077947 in [34] associated with the expansion of $1/(1-x-x^2-2x^3)$, j_n is the sequence A226308 in [34] and K_n is the sequence A186575 in [34] associated with the expansion of $(1+2x+6x^2)/(1-x-x^2-2x^3)$ in powers of x. Q_n is not indexed in [34] yet.

Next, we present the first few values of the third-order Jacobsthal, third-order Jacobsthal-Lucas, modified third-order Jacobsthal and Jacobsthal Perrin numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
J_n	0	1	1	2	5	9	18	37	73	146	293	585	1170	2341
J_{-n}		0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{7}{16}$	$-\frac{9}{32}$	$-\frac{9}{64}$	$\frac{55}{128}$	$-\frac{73}{256}$	$-\frac{73}{512}$	$\frac{439}{1024}$	$-\frac{585}{2048}$	$-\frac{585}{4096}$
j_n	2	1	5	10	17	37	74	145	293	586	1169	2341	4682	9361
j_{-n}		1	-1	1	$\frac{1}{2}$	$-\frac{5}{4}$	$\frac{7}{8}$	$\frac{7}{16}$	$-\frac{41}{32}$	$\frac{55}{64}$	$\frac{55}{128}$	$-\frac{329}{256}$	$\frac{439}{512}$	$\frac{439}{1024}$
K_n	3	1	3	10	15	31	66	127	255	514	1023	2047	4098	8191
K_{-n}		$-\frac{1}{2}$	$-\frac{3}{4}$	$\frac{17}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$\frac{129}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$\frac{1025}{512}$	$-\frac{1023}{1024}$	$-\frac{2047}{2048}$	$\frac{8193}{4096}$	$-\frac{8191}{8192}$
Q_n	3	0	2	8	10	22	48	90	182	368	730	1462	2928	5850
Q_{-n}		$-\frac{1}{2}$	$-\frac{5}{4}$	$\frac{19}{8}$	$-\frac{13}{16}$	$-\frac{45}{32}$	$\frac{147}{64}$	$-\frac{109}{128}$	$-\frac{365}{256}$	$\frac{1171}{512}$	$-\frac{877}{1024}$	$-\frac{2925}{2048}$	$\frac{9363}{4096}$	$-\frac{7021}{8192}$

For all integers n, third-order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal and Jacobsthal Perrin numbers(using initial conditions in (1.5)) can be expressed using Binet's formulas as

$$J_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$j_n = \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n,$$

and

$$Q_n = \frac{(3\alpha^2-3\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(3\beta^2-3\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(3\gamma^2-3\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

respectively.

2. Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1. *Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized third-order Jacobsthal sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - 2x^3}. \tag{2.1}$$

Proof. Using the definition of generalized third-order Jacobsthal numbers, and substracting $x\sum_{n=0}^{\infty} V_n x^n$, $x^2\sum_{n=0}^{\infty} V_n x^n$ and $2x^3\sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned}
(1 - x - x^2 - 2x^3)\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - x\sum_{n=0}^{\infty} V_n x^n - x^2\sum_{n=0}^{\infty} V_n x^n - 2x^3\sum_{n=0}^{\infty} V_n x^n \\
&= \sum_{n=0}^{\infty} V_n x^n - \sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+2} - 2\sum_{n=0}^{\infty} V_n x^{n+3} \\
&= \sum_{n=0}^{\infty} V_n x^n - \sum_{n=1}^{\infty} V_{n-1} x^n - \sum_{n=2}^{\infty} V_{n-2} x^n - 2\sum_{n=3}^{\infty} V_{n-3} x^n \\
&= (V_0 + V_1 x + V_2 x^2) - (V_0 x + V_1 x^2) - V_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (V_n - V_{n-1} - V_{n-2} - 2V_{n-3}) x^n \\
&= V_0 + V_1 x + V_2 x^2 - V_0 x - V_1 x^2 - V_0 x^2 \\
&= V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - 2x^3}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2. *Generated functions of third-order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal and Jacobsthal Perrin numbers are*

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x}{1 - x - x^2 - 2x^3},$$

and

$$\sum_{n=0}^{\infty} j_n x^n = \frac{2 - x + 2x^2}{1 - x - x^2 - 2x^3},$$

and

$$\sum_{n=0}^{\infty} K_n x^n = \frac{3 - 2x - x^2}{1 - x - x^2 - 2x^3},$$

$$\sum_{n=0}^{\infty} Q_n x^n = \frac{3-3x-x^2}{1-x-x^2-2x^3},$$

respectively.

3. Obtaining Binet Formula From Generating Function

We next find Binet formula of generalized third order Jacobsthal numbers $\{V_n\}$ by the use of generating function for V_n.

THEOREM 3. *(Binet formula of generalized third order Jacobsthal numbers)*

$$V_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{3.1}$$

where

$$\begin{aligned} d_1 &= V_0\alpha^2 + (V_1-V_0)\alpha + (V_2-V_1-V_0), \\ d_2 &= V_0\beta^2 + (V_1-V_0)\beta + (V_2-V_1-V_0), \\ d_3 &= V_0\gamma^2 + (V_1-V_0)\gamma + (V_2-V_1-V_0). \end{aligned}$$

Proof. Let

$$h(x) = 1-x-x^2-2x^3.$$

Then for some α, β and γ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

i.e.,

$$1-x-x^2-2x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x) \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$, ve $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives α, β, and γ as the roots of

$$h(\frac{1}{x}) = 1-\frac{1}{x}-\frac{1}{x^2}-\frac{2}{x^3} = 0.$$

This implies $x^3-x^2-x-2=0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)}. \tag{3.3}$$

So

$$V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2 = A_1(1-\beta x)(1-\gamma x) + A_2(1-\alpha x)(1-\gamma x) + A_3(1-\alpha x)(1-\beta x).$$

If we consider $x = \frac{1}{\alpha}$, we get $V_0 + (V_1 - V_0)\frac{1}{\alpha} + (V_2 - V_1 - V_0)\frac{1}{\alpha^2} = A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})$. This gives

$$A_1 = \frac{\alpha^2(V_0 + (V_1 - V_0)\frac{1}{\alpha} + (V_2 - V_1 - V_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)} = \frac{V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1 - V_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$A_2 = \frac{V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1 - V_0)}{(\beta-\alpha)(\beta-\gamma)}, A_3 = \frac{V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1 - V_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$$

where

$$\begin{aligned} A_1 &= \frac{V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1 - V_0)}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1 - V_0)}{(\beta-\alpha)(\beta-\gamma)} \\ A_3 &= \frac{V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1 - V_0)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

and then we get (3.1).

Note that from (1.5) and (3.1) we have

$$\begin{aligned} V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0 &= V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1 - V_0), \\ V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0 &= V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1 - V_0), \\ V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0 &= V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1 - V_0). \end{aligned}$$

Next, using Theorem 3, we present the Binet formulas of third-order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal and Jacobsthal Perrin sequences.

Corollary 4. *Binet formulas of third-order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal and Jacobsthal Perrin numbers sequences are*

$$J_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$j_n = \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n$$

and

$$Q_n = \frac{(3\alpha^2 - 3\alpha - 1)\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(3\beta^2 - 3\beta - 1)\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{(3\gamma^2 - 3\gamma - 1)\gamma^n}{(\gamma - \alpha)(\gamma - \beta)}$$

respectively.

Matrix method which is given in [26] for Pell numbers can be adjusted to third order Jacobsthall numbers. Take $k = i = 3$ in Corollary 3.1 in [26]. Let

$$\begin{aligned}
\Lambda &= \begin{pmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{pmatrix}, \\
\Lambda_2 &= \begin{pmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{pmatrix}.
\end{aligned}$$

Then the Binet formula for third-order Jacobsthal numbers is

$$\begin{aligned}
J_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} J_{4-j} \det(\Lambda_j) = \frac{1}{\Lambda}(J_3 \det(\Lambda_1) + J_2 \det(\Lambda_2) + J_1 \det(\Lambda_3)) \\
&= \frac{1}{\det(\Lambda)}(2\det(\Lambda_1) + \det(\Lambda_2) + \det(\Lambda_3)) \\
&= \left(2\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \\
&= \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)}.
\end{aligned}$$

Similarly, we obtain the Binet formula for third-order Jacobsthal-Lucas, modified third-order Jacobsthal and Jacobsthal Perrin numbers as

$$\begin{aligned}
j_n &= \frac{1}{\Lambda}(j_3 \det(\Lambda_1) + j_2 \det(\Lambda_2) + j_1 \det(\Lambda_3)) \\
&= \left(10\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 5\begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \\
&= \frac{(2\alpha^2 - \alpha + 2)\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(2\beta^2 - \beta + 2)\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{(2\gamma^2 - \gamma + 2)\gamma^n}{(\gamma - \alpha)(\gamma - \beta)}
\end{aligned}$$

and

$$\begin{aligned} K_n &= \frac{1}{\Lambda}(K_3 \det(\Lambda_1) + K_2 \det(\Lambda_2) + K_1 \det(\Lambda_3)) \\ &= \left(10 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 3 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix}\right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \\ &= \alpha^n + \beta^n + \gamma^n \end{aligned}$$

and

$$\begin{aligned} Q_n &= \frac{1}{\Lambda}(Q_3 \det(\Lambda_1) + Q_2 \det(\Lambda_2) + Q_1 \det(\Lambda_3)) \\ &= \left(8 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 2 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix}\right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \\ &= \frac{(3\alpha^2 - 3\alpha - 1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(3\beta^2 - 3\beta - 1)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(3\gamma^2 - 3\gamma - 1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \end{aligned}$$

respectively.

4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized third-order Jacobsthal sequence $\{V_n\}_{n\geq 0}$.

Theorem 5 (Simson Formula of Generalized Third-Order Pell Numbers). *For all integers n, we have*

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = 2^n \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. (4.1) is given in Soykan [36].

The previous Theorem gives the following results as particular examples.

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Corollary 6. *Simson formula of third-order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal, Jacobsthal Perrin numbers are given as*

$$\begin{vmatrix} J_{n+2} & J_{n+1} & J_n \\ J_{n+1} & J_n & J_{n-1} \\ J_n & J_{n-1} & J_{n-2} \end{vmatrix} = -2^{n-1},$$

and

$$\begin{vmatrix} j_{n+2} & j_{n+1} & j_n \\ j_{n+1} & j_n & j_{n-1} \\ j_n & j_{n-1} & j_{n-2} \end{vmatrix} = -9 \times 2^{n+1},$$

and

$$\begin{vmatrix} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{vmatrix} = -147 \times 2^{n-2},$$

and

$$\begin{vmatrix} Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+1} & Q_n & Q_{n-1} \\ Q_n & Q_{n-1} & Q_{n-2} \end{vmatrix} = -35 \times 2^{n}$$

respectively.

5. Some Identities

In this section, we obtain some identities of third order Jacobsthal, third order Jacobsthal-Lucas, modified third order Jacobsthal and third order Jacobsthal Perrin numbers. First, we can give a few basic relations between $\{J_n\}$ and $\{j_n\}$.

Lemma 7. *The following equalities are true:*

(5.1) $$j_n = J_{n+4} - 2J_{n+3} + J_{n+2},$$

(5.2) $$j_n = -J_{n+3} + 2J_{n+2} + 2J_{n+1},$$

(5.3) $$j_n = J_{n+2} + J_{n+1} - 2J_n,$$

(5.4) $$j_n = 2J_{n+1} - J_n + 2J_{n-1},$$

(5.5) $$j_n = J_n + 4J_{n-1} + 4J_{n-2},$$

and

$$48J_n = 5j_{n+4} - 11j_{n+3} + 5j_{n+2}, \tag{5.6}$$

$$24J_n = -3j_{n+3} + 5j_{n+2} + 5j_{n+1}, \tag{5.7}$$

$$12J_n = j_{n+2} + j_{n+1} - 3j_n, \tag{5.8}$$

$$6J_n = j_{n+1} - j_n + j_{n-1}, \tag{5.9}$$

$$3J_n = j_{n-1} + j_{n-2}, \tag{5.10}$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$j_n = a \times J_{n+4} + b \times J_{n+3} + c \times J_{n+2}$$

and solving the system of equations

$$\begin{aligned} j_0 &= a \times J_4 + b \times J_3 + c \times J_2 \\ j_1 &= a \times J_5 + b \times J_4 + c \times J_3 \\ j_2 &= a \times J_6 + b \times J_5 + c \times J_4 \end{aligned}$$

we find that $a = 1, b = -2, c = 1$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{J_n\}$ and $\{K_n\}$.

Lemma 8. *The following equalities are true:*

$$\begin{aligned} 8K_n &= 17J_{n+4} - 23J_{n+3} - 15J_{n+2}, \\ 4K_n &= -3J_{n+3} + J_{n+2} + 17J_{n+1}, \\ 2K_n &= -J_{n+2} + 7J_{n+1} - 3J_n, \\ K_n &= 3J_{n+1} - 2J_n - J_{n-1}, \\ K_n &= J_n + 2J_{n-1} + 6J_{n-2} \end{aligned}$$

and

$$\begin{aligned} 294J_n &= -K_{n+4} - 15K_{n+3} + 55K_{n+2}, \\ 147J_n &= -8K_{n+3} + 27K_{n+2} - K_{n+1} \\ 147J_n &= 19K_{n+2} - 9K_{n+1} - 16K_n, \\ 147J_n &= 10K_{n+1} + 3K_n + 38K_{n-1}. \\ 147J_n &= 13K_n + 48K_{n-1} + 20K_{n-2}. \end{aligned}$$

Thirdly, we give a few basic relations between $\{j_n\}$ and $\{K_n\}$.

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LEMMA 9. *The following equalities are true:*

$$49j_n = 15K_{n+3} - 20K_{n+2} + 8K_{n+1},$$
$$49j_n = -5K_{n+2} + 23K_{n+1} + 30K_n,$$
$$49j_n = 18K_{n+1} + 25K_n - 10K_{n-1},$$

and

$$48K_n = 19j_{n+3} + 3j_{n+2} - 61j_{n+1},$$
$$24K_n = 11j_{n+2} - 21j_{n+1} + 19j_n,$$
$$12K_n = -5j_{n+1} + 15j_n + 11j_{n-1}.$$

Fourthly, we give a few basic relations between $\{j_n\}$ and $\{Q_n\}$.

LEMMA 10. *The following equalities are true:*

$$70j_n = 4Q_{n+4} + 19Q_{n+3} - 26Q_{n+2},$$
$$70j_n = 23Q_{n+3} - 22Q_{n+2} + 8Q_{n+1},$$
$$70j_n = Q_{n+2} + 31Q_{n+1} + 46Q_n,$$
$$70j_n = 32Q_{n+1} + 47Q_n + 2Q_{n-1},$$
$$70j_n = 79Q_n + 34Q_{n-1} + 64Q_{n-2},$$

and

$$96Q_n = -71j_{n+4} + 121j_{n+3} + 57j_{n+2},$$
$$48Q_n = 25j_{n+3} - 7j_{n+2} - 71j_{n+1},$$
$$24Q_n = 9j_{n+2} - 23j_{n+1} + 25j_n,$$
$$12Q_n = -7j_{n+1} + 17j_n + 9j_{n-1},$$
$$6Q_n = 5j_n + j_{n-1} - 7j_{n-2}.$$

Fifthly, we give a few basic relations between $\{K_n\}$ and $\{Q_n\}$

LEMMA 11. *The following equalities are true:*

$$140K_n = -33Q_{n+4} + 97Q_{n+3} - 13Q_{n+2},$$
$$70K_n = 32Q_{n+3} - 23Q_{n+2} - 33Q_{n+1},$$
$$70K_n = 9Q_{n+2} - Q_{n+1} + 64Q_n,$$
$$70K_n = 8Q_{n+1} + 73Q_n + 18Q_{n-1},$$
$$70K_n = 81Q_n + 26Q_{n-1} + 16Q_{n-2},$$

and

$$
\begin{aligned}
588Q_n &= -145K_{n+4} + 471K_{n+3} - 257K_{n+2}, \\
294Q_n &= 163K_{n+3} - 201K_{n+2} - 145K_{n+1}, \\
147Q_n &= -19K_{n+2} + 9K_{n+1} + 163K_n, \\
147Q_n &= -10K_{n+1} + 144K_n - 38K_{n-1}, \\
147Q_n &= 134K_n - 48K_{n-1} - 20K_{n-2}.
\end{aligned}
$$

Sixthly, we give a few basic relations between $\{J_n\}$ and $\{Q_n\}$

LEMMA 12. *The following equalities are true:*

$$
\begin{aligned}
70J_n &= Q_{n+4} - 4Q_{n+3} + 11Q_{n+2}, \\
70J_n &= -3Q_{n+3} + 12Q_{n+2} + 2Q_{n+1}, \\
70J_n &= 9Q_{n+2} - Q_{n+1} - 6Q_n, \\
70J_n &= 8Q_{n+1} + 3Q_n + 18Q_{n-1}, \\
70J_n &= 11Q_n + 26Q_{n-1} + 16Q_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
8Q_n &= 19J_{n+4} - 29J_{n+3} - 13J_{n+2}, \\
4Q_n &= -5J_{n+3} + 3J_{n+2} + 19J_{n+1}, \\
2Q_n &= -J_{n+2} + 7J_{n+1} - 5J_n, \\
Q_n &= 3J_{n+1} - 3J_n - J_{n-1}, \\
Q_n &= 2J_{n-1} + 6J_{n-2}.
\end{aligned}
$$

6. Linear Sums

The following proposition presents some formulas of generalized third order Jacobsthal numbers with positive subscripts.

PROPOSITION 13. *If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} V_k = \frac{1}{3}(V_{n+3} - V_{n+1} - V_2 + V_0)$.
- **(b):** $\sum_{k=0}^{n} V_{2k} = \frac{1}{3}(V_{2n+1} + 2V_{2n} - V_1 + V_0)$.
- **(c):** $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{3}(V_{2n+2} + 2V_{2n+1} - V_2 + V_1)$.

Proof. This is given in [37].

As special cases of above proposition, we have the following four Corollaries. First one presents some summing formulas of third order Jacobsthal numbers. (take $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

Corollary 14. *For $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} J_k = \frac{1}{3}(J_{n+3} - J_{n+1} - 1)$.

(b): $\sum_{k=0}^{n} J_{2k} = \frac{1}{3}(J_{2n+1} + 2J_{2n} - 1)$.

(c): $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{3}(J_{2n+2} + 2J_{2n+1})$.

Second one presents some summing formulas of third order Jacobsthal-Lucas numbers. (take $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$).

Corollary 15. *For $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} j_k = \frac{1}{3}(j_{n+3} - j_{n+1} - 3)$.

(b): $\sum_{k=0}^{n} j_{2k} = \frac{1}{3}(j_{2n+1} + 2j_{2n} + 1)$.

(c): $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{3}(j_{2n+2} + 2j_{2n+1} - 4)$.

Third one presents some summing formulas of modified third order Jacobsthal numbers. (take $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$).

Corollary 16. *For $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} K_k = \frac{1}{3}(K_{n+3} - K_{n+1})$.

(b): $\sum_{k=0}^{n} K_{2k} = \frac{1}{3}(K_{2n+1} + 2K_{2n} + 2)$.

(c): $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{3}(K_{2n+2} + 2K_{2n+1} - 2)$.

Fourth one presents some summing formulas of third order Jacobsthal Perrin numbers. (take $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$).

Corollary 17. *For $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} Q_k = \frac{1}{3}(Q_{n+3} - Q_{n+1} + 1)$.

(b): $\sum_{k=0}^{n} Q_{2k} = \frac{1}{3}(Q_{2n+1} + 2Q_{2n} + 3)$.

(c): $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{3}(Q_{2n+2} + 2Q_{2n+1} - 2)$.

The following proposition presents some formulas of generalized third order Jacobsthal numbers with negative subscripts.

Proposition 18. *If $r = 1, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n} V_{-k} = \frac{1}{3}(-4V_{-n-1} - 3V_{-n-2} - 2V_{-n-3} + V_2 - V_0)$.

(b): $\sum_{k=1}^{n} V_{-2k} = \frac{1}{3}\left(-V_{-2n+1} + V_{-2n} + V_1 - V_0\right)$.

(c): $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{3}\left(-V_{-2n} - 2V_{-2n-1} + V_2 - V_1\right)$.

Proof. This is given in [37].

Taking $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of third order Jacobsthal numbers.

COROLLARY 19. *For $n \geq 1$, third order Jacobsthal numbers have the following properties.*

(a): $\sum_{k=1}^{n} J_{-k} = \frac{1}{3}(-4J_{-n-1} - 3J_{-n-2} - 2J_{-n-3} + 1)$.

(b): $\sum_{k=1}^{n} J_{-2k} = \frac{1}{3}(-J_{-2n+1} + J_{-2n} + 1)$.

(c): $\sum_{k=1}^{n} J_{-2k+1} = \frac{1}{3}(-J_{-2n} - 2J_{-2n-1})$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of third order Jacobsthal-Lucas numbers (take $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$).

COROLLARY 20. *For $n \geq 1$, third order Jacobsthal-Lucas numbers have the following properties.*

(a): $\sum_{k=1}^{n} j_{-k} = \frac{1}{3}(-4j_{-n-1} - 3j_{-n-2} - 2j_{-n-3} + 3)$.

(b): $\sum_{k=1}^{n} j_{-2k} = \frac{1}{3}(-j_{-2n+1} + j_{-2n} - 1)$.

(c): $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{3}(-j_{-2n} - 2j_{-2n-1} + 4)$.

Third one presents some summing formulas of modified third order Jacobsthal numbers. (take $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$).

COROLLARY 21. *For $n \geq 1$, third order modified Jacobsthal numbers have the following properties*

(a): $\sum_{k=1}^{n} K_{-k} = \frac{1}{3}(-4K_{-n-1} - 3K_{-n-2} - 2K_{-n-3})$.

(b): $\sum_{k=1}^{n} K_{-2k} = \frac{1}{3}(-K_{-2n+1} + K_{-2n} - 2)$.

(c): $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{3}(-K_{-2n} - 2K_{-2n-1} + 2)$.

Fourth one presents some summing formulas of third order Jacobsthal Perrin numbers. (take $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$).

COROLLARY 22. *For $n \geq 1$, third order Jacobsthal Perrin numbers have the following properties*

(a): $\sum_{k=1}^{n} Q_{-k} = \frac{1}{3}(-4Q_{-n-1} - 3Q_{-n-2} - 2Q_{-n-3} - 1)$.

(b): $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{3}(-Q_{-2n+1} + Q_{-2n} - 3)$.

(c): $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{3}(-Q_{-2n} - 2Q_{-2n-1} + 2)$.

7. Matrices related with Generalized Third-Order Jacobsthal numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

For matrix formulation (7.1), see [25]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. From (1.4) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = J$ in (7.2) we have

$$\begin{pmatrix} J_{n+2} \\ J_{n+1} \\ J_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} J_{n+1} \\ J_n \\ J_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n = \begin{pmatrix} J_{n+1} & J_n + 2J_{n-1} & 2J_n \\ J_n & J_{n-1} + 2J_{n-2} & 2J_{n-1} \\ J_{n-1} & J_{n-2} + 2J_{n-3} & 2J_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}$$

THEOREM 23. *For all integer $m, n \geq 0$, we have*

- **(a):** $B_n = A^n$
- **(b):** $C_1 A^n = A^n C_1$
- **(c):** $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

- **(a):** By expanding the vectors on the both sides of (7.3) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b): Using (a) and definition of C_1, (b) follows.

(c): We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & V_{n-1}+2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2}+2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3}+2V_{n-4} & 2V_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} V_{n+1} & V_n+2V_{n-1} & 2V_n \\ V_n & V_{n-1}+2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2}+2V_{n-3} & 2V_{n-2} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

Some properties of matrix A^n can be given as

$$A^n = A^{n-1} + A^{n-2} + 2A^{n-3}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

for all integer m and n.

THEOREM 24. *For $m, n \geq 0$ we have*

$$\begin{aligned} V_{n+m} &= V_nJ_{m+1} + 2J_{m-1}V_{n-1} + J_m(V_{n-1}+2V_{n-2}) && (7.4) \\ &= V_nJ_{m+1} + (V_{n-1}+2V_{n-2})J_m + 2J_{m-1}V_{n-1} && (7.5) \end{aligned}$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof.

REMARK 25. *By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So for all integers m, n, (7.4) is true.*

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Corollary 26. *For all integers m, n, we have*

$$J_{n+m} = J_n J_{m+1} + (J_{n-1} + 2J_{n-2})J_m + 2J_{m-1}J_{n-1} \quad (7.6)$$

$$j_{n+m} = j_n J_{m+1} + (j_{n-1} + 2j_{n-2})J_m + 2J_{m-1}j_{n-1} \quad (7.7)$$

$$K_{n+m} = K_n J_{m+1} + (K_{n-1} + 2K_{n-2})J_m + 2J_{m-1}K_{n-1} \quad (7.8)$$

$$Q_{n+m} = Q_n J_{m+1} + (Q_{n-1} + 2Q_{n-2})J_m + 2J_{m-1}Q_{n-1} \quad (7.9)$$

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