Original Research Article

Bipartite Domination in Some Classes of Graphs

Abstract

For a nontrivial connected graph G, a non-empty set $S \subseteq V(G)$ is a bipartite dominating set of graph G, if the subgraph $G[S]$ induced by S is bipartite and for every vertex not in S is dominated by any vertex in S. The bipartite domination number denoted by $\gamma_{bip}(G)$ of graph G is the minimum cardinality of a bipartite dominating set G. In this paper, we determine the exact bipartite domination number of path graph and cycle graph via modulus. Moreover, this study generates the possible exact values of the bipartite domination number of the complete graph, complete bipartite graph, join graph, fan graph and wheel graph.

Keywords: bipartite dominating set, bipartite domination number.

2020 Mathematics Subject Classification: 05C35

1 Introduction

The concept of domination is one of the most interesting researched topics in graph theory. In fact, numerous studies related to this topic have already been published, for instance, the bipartite domination in graphs published by Bachstein et al.,[2]. They introduced and defined the concept of bipartite dominating set and bipartite domination number, which motivates this study. Their concept of the bipartite domination was inspired by the study of Ko, C., and Shepherd, F., [13]. In line with this, they investigated the case that a dominating set must induce a bipartite subgraph.

In this paper, we extended the study of the bipartite dominating sets in some classes of graphs. We investigated the bipartite domination number of path graphs and cycle graphs via modulus. We also characterize the bipartite dominating sets in graphs resulting from join graph. Lastly, we generate the exact values of the bipartite domination number of complete graph, complete bipartite graph, join graph, fan graph and wheel graph.

All graphs considered in this paper are undirected and nontrivial connected graph.

2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definitions in [2], [5], [14], [17].

Definition 2.1. [5] A graph $G = (V(G), E(G))$ is ***bipartite*** if $V(G)$ can be partitioned into two sets U and W (called ***partite sets***) so that every edge of G joins a vertex of U and a vertex of W.

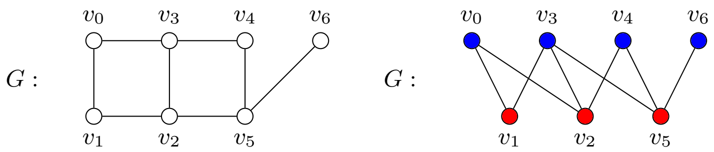


Figure 1: A bipartite graph G.

Definition 2.2. [2] **(Bipartite Dominating Set, Bipartite Domination number)** A dominating set S of a graph G is a bipartite dominating set if the induced subgraph S, $G[S]$, is bipartite graph. The minimum cardinality of a bipartite dominating set is called bipartite domination number of G, denoted by $\gamma_{bip}(G)$. A nonempty set $S \subseteq V(G)$ whose cardinality is the $\gamma_{bip}(G)$ is called γ_{bip}-set of G.

Example 2.1. *Consider the graph G in Fig. 2 below. The possible bipartite dominating sets for the graph G are $B_1 = \{v_1, v_2, v_5\}$, $B_2 = \{v_2, v_3, v_5\}$, and $B_3 = \{v_0, v_1, v_5, v_6\}$. B_1 can be partitioned into partite sets $U = \{v_2, v_5\}$ and $W = \{v_1\}$, B_2 can be partitioned into partite sets $U = \{v_2\}$ and $W = \{v_3, v_5\}$, and B_3 can be partitioned into partite sets $U = \{v_0, v_5\}$ and $W = \{v_1, v_6\}$. Notice that S_1 and S_2 consist the minimum cardinality of a bipartite dominating set. Thus, $\gamma_{bip}(G) = 3$.*

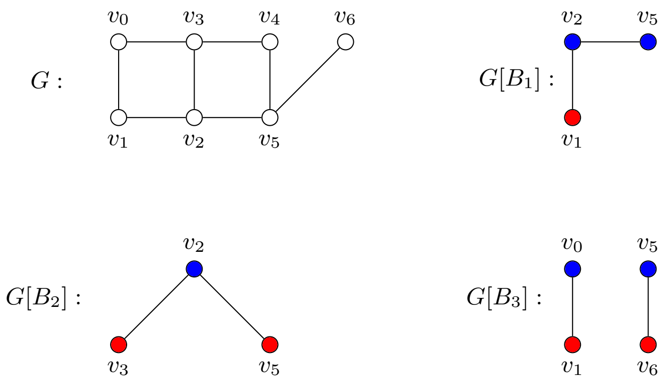


Figure 2: Bipartite dominating sets of graph G.

3 Main Results

In this section, the bipartite domination number of path graph, cycle graph, complete graph, complete bipartite graph, join graph, fan graph and wheel graph are shown. As well as, the characteristics of the bipartite dominating set of join graph.

By definition [12], every bipartite dominating set of a graph G is a total dominating set of G. Thus, the following result is immediate:

Remark 3.1. Let G be a nontrivial connected graph. Then,

$$\gamma_t(G) \leq \gamma_{bip}(G).$$

3.1 Bipartite Domination Number of the Path Graph, P_n and Cycle Graph, C_n

Theorem 3.1. *Given a path graph, P_n , $n \geq 2$ and a cycle graph, C_n , $n \geq 3$. Then,*

$$\gamma_{bip}(P_n) = \gamma_{bip}(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 4) \\ \frac{n+1}{2} & \text{if } n \equiv 1,3 (\text{mod } 4) \\ \frac{n+2}{2} & \text{if } n \equiv 2 (\text{mod } 4) \end{cases}$$

Proof. Let $S \subseteq V(P_n)$ and $S \subseteq V(C_n)$ be a γ_{bip} - set of P_n and C_n. Let $P_n = [x_1, x_2, ..., x_n]$ and $C_n = [x_1, x_2, ..., x_n]$ be the labeling of P_n and C_n . Consider the following cases.

Case 1: $n \equiv 0(\text{mod } 4)$.

Choose $B = \{x_2, x_3, ..., x_{n-2}, x_{n-1}\}$. Now, B can be partitioned into two sets B_1 and B_2 where $B_1 = \{x_2, x_6, ..., x_{n-2}\}$ and $B_2 = \{x_3, x_7, ..., x_{n-1}\}$. Clearly, $P_n[B] = P_n[B_1 \cup B_2]$ is a bipartite graph and that B is a dominating set. Thus, B is a bipartite dominating set. Thus, $|S| \leq |B| = |B_1| + |B_2| = \frac{n}{4} + \frac{n}{4} = \frac{n}{2}$. On the other hand, since every bipartite dominating set is a total dominating set, S must have at least $\frac{n}{2}$ vertices. Hence, $|S| \geq \frac{n}{2}$. Therefore, $\gamma_{bip}(P_n) = |S| = \frac{n}{2}$. Similarly, for the graph C_n by setting $B = \{x_2, x_3, ..., x_{n-2}, x_{n-1}\}$, B can be partitioned into two sets B_1 and B_2 where $B_1 = \{x_2, x_6, ..., x_{n-2}\}$ and $B_2 = \{x_3, x_7, ..., x_{n-1}\}$. Clearly, $C_n[B] = C_n[B_1 \cup B_2]$ is a bipartite graph and that B is a dominating set. Thus, B is a bipartite dominating set. Thus, $|S| \leq |B| = |B_1| + |B_2| = \frac{n}{4} + \frac{n}{4} = \frac{n}{2}$. On the other hand, since every bipartite dominating set is a total dominating set, S must have at least $\frac{n}{2}$ vertices. Hence, $|S| \geq \frac{n}{2}$. Therefore, $\gamma_{bip}(C_n) = |S| = \frac{n}{2}$.

Case 2: $n \equiv 1(\text{mod } 4)$.

Choose $B = \{x_2, x_3, ..., x_{n-2}, x_{n-1}\}$. Now, B can be partitioned into two sets B_1 and B_2 where $B_1 = \{x_2, x_4, ..., x_{n-2}\}$ and $B_2 = \{x_3, x_7, ..., x_{n-1}\}$. Clearly, $P_n[B] = P_n[B_1 \cup B_2]$ is a bipartite graph and that B is a dominating set. Thus, B is a bipartite dominating set. Thus, $|S| \leq |B| = |B_1| + |B_2| = \frac{n+1}{4} + \frac{n+1}{4} = \frac{n+1}{2}$. On the other hand, since every bipartite dominating set is a total dominating set, S must have at least $\frac{n+1}{2}$ vertices. Hence, $|S| \geq \frac{n+1}{2}$. Therefore, $\gamma_{bip}(P_n) = |S| = \frac{n+1}{2}$. Similarly, for the graph C_n by setting $B = \{x_2, x_3, ..., x_{n-2}, x_{n-1}\}$, B can be partitioned into two sets B_1 and B_2 where $B_1 = \{x_2, x_4, ..., x_{n-2}\}$ and $B_2 = \{x_3, x_7, ..., x_{n-1}\}$. Clearly, $C_n[B] = C_n[B_1 \cup B_2]$ is a bipartite graph and that B is a dominating set. Thus, B is a bipartite dominating set. Thus, $|S| \leq |B| = |B_1| + |B_2| = \frac{n+1}{4} + \frac{n+1}{4} = \frac{n+1}{2}$. On the other hand, since every bipartite dominating set is a total dominating set, S must have at least $\frac{n+1}{2}$ vertices. Hence, $|S| \geq \frac{n+1}{2}$. Therefore, $\gamma_{bip}(C_n) = |S| = \frac{n+1}{2}$.

Case 3: $n \equiv 3(\text{mod } 4)$.

Choose $B = \{x_2, x_3, ..., x_{n-2}, x_{n-1}\}$. Now, B can be partitioned into two sets B_1 and B_2 where $B_1 = \{x_2, x_6, ..., x_n - 2\}$ and $B_2 = \{x_3, x_7, ..., x_{n-1}\}$. Clearly, $P_n[B] = P_n[B_1 \cup B_2]$ is a bipartite graph and that B is a dominating set. Thus, B is a bipartite dominating set. Thus, $|S| \leq |B| = |B_1| + |B_2| = \frac{n+1}{4} + \frac{n+1}{4} = \frac{n+1}{2}$. On the other hand, since every bipartite dominating set is a total dominating set, S must have at least $\frac{n+1}{2}$ vertices. Hence, $|S| \geq \frac{n+1}{2}$. Therefore, $\gamma_{bip}(P_n) = |S| = \frac{n+1}{2}$. Similarly, for the graph C_n by setting $B = \{x_2, x_3, ..., x_{n-2}, x_{n-1}\}$, B can be partitioned into two sets B_1 and B_2 where $B_1 = \{x_2, x_6, ..., x_n - 2\}$ and $B_2 = \{x_3, x_7, ..., x_{n-1}\}$. Clearly, $C_n[B] = C_n[B_1 \cup B_2]$ is a bipartite graph and that B is a dominating set. Thus, B is a bipartite dominating set. Thus, $|S| \leq |B| = |B_1| + |B_2| = \frac{n+1}{4} + \frac{n+1}{4} = \frac{n+1}{2}$. On the other hand, since every bipartite dominating set is a total dominating set, S must have at least $\frac{n+1}{2}$ vertices. Hence, $|S| \geq \frac{n+1}{2}$. Therefore, $\gamma_{bip}(C_n) = |S| = \frac{n+1}{2}$.

Case 4: $n \equiv 2 (\mathrm{mod}\ 4)$.

Choose $B = \{x_1, x_2, ..., x_{n-2}, x_{n-1}\}$. Now, B can be partitioned into two sets B_1 and B_2 where $B_1 = \{x_1, x_5, ..., x_{n-2}\}$ and $B_2 = \{x_2, x_6, ..., x_{n-1}\}$. Clearly, $P_n[B] = P_n[B_1 \cup B_2]$ is a bipartite graph and that B is a dominating set. Thus, B is a bipartite dominating set. Thus, $|S| \leq |B| = |B_1| + |B_2| = \frac{n+2}{4} + \frac{n+2}{4} = \frac{n+2}{2}$. On the other hand, since every bipartite dominating set is a total dominating set, S must have at least $\frac{n+2}{2}$ vertices. Hence, $|S| \geq \frac{n+2}{2}$. Therefore, $\gamma_{bip}(P_n) = |S| = \frac{n+2}{2}$. Similarly, for the graph C_n by setting $B = \{x_2, x_3, ..., x_{n-1}, x_{n-2}\}$, B can be partitioned into two sets B_1 and B_2 where $B_1 = \{x_2, x_6, ..., x_{n-2}\}$ and $B_2 = \{x_3, x_7, ..., x_{n-1}\}$. Clearly, $C_n[B] = C_n[B_1 \cup B_2]$ is a bipartite graph and that B is a dominating set. Thus, B is a bipartite dominating set. Thus, $|S| \leq |B| = |B_1| + |B_2| = \frac{n+2}{4} + \frac{n+2}{4} = \frac{n+2}{2}$. On the other hand, since every bipartite dominating set is a total dominating set, S must have at least $\frac{n+2}{2}$ vertices. Hence, $|S| \geq \frac{n+2}{2}$. Therefore, $\gamma_{bip}(C_n) = |S| = \frac{n+2}{2}$.

□

By [12], we have the following remarks:

Remark 3.2. For Path graph, P_n and Cycle graph C_n,

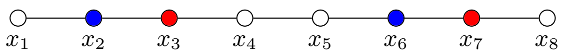
$$\gamma_t(P_n) = \gamma_t(C_n) = \gamma_{bip}(P_n) = \gamma_{bip}(C_n).$$

Remark 3.2 shows the equality of Remark 3.1. Hence, the bound in Remark 3.1 is sharp.

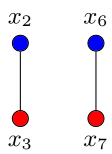
Example 3.2. *Consider the following graphs for $P_n, n \geq 2$.*

For $n \equiv 0 (mod\ 4)$, choose P_8, illustrated below. The set $A = \{x_2, x_3, x_6, x_7\} \subseteq V(P_8)$ is the bipartite dominating set and also the minimum bipartite dominating set.

P_8 :



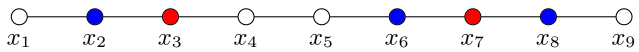
$P_8[A]$:



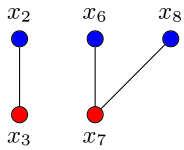
Clearly, $P_8[A] = P_8[A_1 \cup A_2]$ is a bipartite dominating set of P_8, where $A_1 = \{x_2, x_6\}$ and $A_2 = \{x_3, x_7\}$. Thus, $|A| = |A_1| + |A_2| = \frac{8}{4} + \frac{8}{4} = \frac{8}{2}$. Hence, $\gamma_{bip}(P_8) = \frac{8}{2} = 4$.

For $n \equiv 1 (mod\ 4)$, choose P_9, the set $B = \{x_2, x_3, x_6, x_7, x_8\} \subseteq V(P_9)$ is the bipartite dominating set and also the minimum bipartite dominating set.

P_9 :



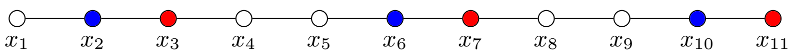
$P_9[B]$:



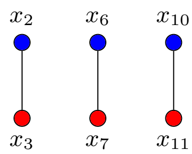
Clearly, $P_9[B] = P_9[B_1 \cup B_2]$ *is a bipartite dominating set of* P_9*, where* $B_1 = \{x_2, x_6, x_8\}$ *and* $B_2 = \{x_3, x_7\}$*. Thus,* $|B| = |B_1| + |B_2| = \frac{9+1}{4} + \frac{9+1}{4} = \frac{9+1}{2}$*. Hence,* $\gamma_{bip}(P_9) = \frac{9+1}{2} = 5$*.*

For $n \equiv 3(mod\ 4)$*, choose* P_{11}*. As we can see, the set* $C = \{x_2, x_3, x_6, x_7, x_{10}, x_{11}\} \subseteq V(P_{11})$ *is the bipartite dominating set and also the minimum bipartite dominating set.*

P_{11} :



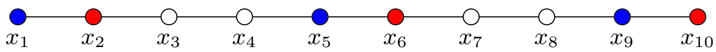
$P_{11}[C]$:



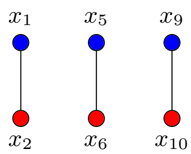
Clearly, $P_{11}[C] = P_{11}[C_1 \cup C_2]$ *is a bipartite dominating set of* P_{11}*, where* $C_1 = \{x_2, x_6, x_{10}\}$ *and* $C_2 = \{x_3, x_7, x_{11}\}$*. Thus,* $|C| = |C_1| + |C_2| = \frac{11+1}{4} + \frac{11+1}{4} = \frac{11+1}{2}$*. Hence,* $\gamma_{bip}(P_{11}) = \frac{11+1}{2} = 6$*.*

For $n \equiv 2(mod\ 4)$*, choose* P_{10}*. As we can see in* P_{10}*, the set* $D = \{x_1, x_2, x_5, x_6, x_9, x_{10}\} \subseteq V(P_{12})$ *is the bipartite dominating set and also the minimum bipartite dominating set.*

P_{10} :



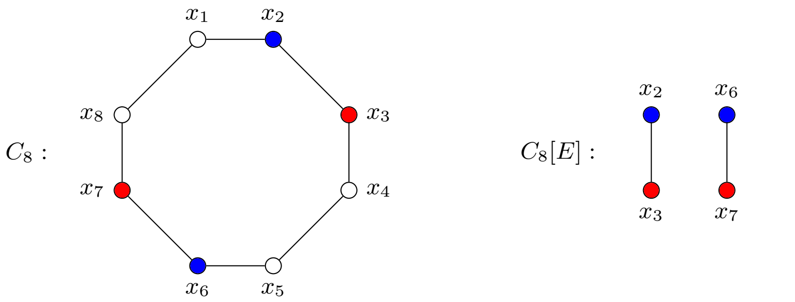
$P_{10}[D]$:



Clearly, $P_{10}[D] = P_{10}[D_1 \cup D_2]$ *is a bipartite dominating set of* P_{10}*, where* $D_1 = \{x_1, x_5, x_9\}$ *and* $D_2 = \{x_2, x_6, x_{10}\}$*. Thus,* $|D| = |D_1| + |D_2| = \frac{10+2}{4} + \frac{10+2}{4} = \frac{10+2}{2}$*. Hence,* $\gamma_{bip}(P_{10}) = \frac{10+2}{2} = 6$*.*

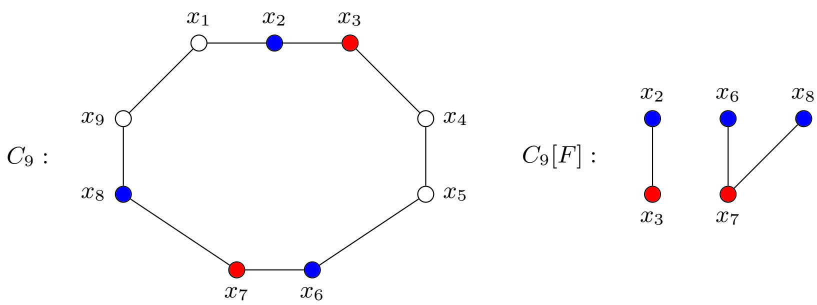
Example 3.3. *Consider the following graphs for* $C_n, n \geq 3$*.*

For $n \equiv 0(mod\ 4)$*, choose* C_8*, the set* $E = \{x_2, x_3, x_6, x_7\} \subseteq V(C_8)$ *is the bipartite dominating set and also the minimum bipartite dominating set.*



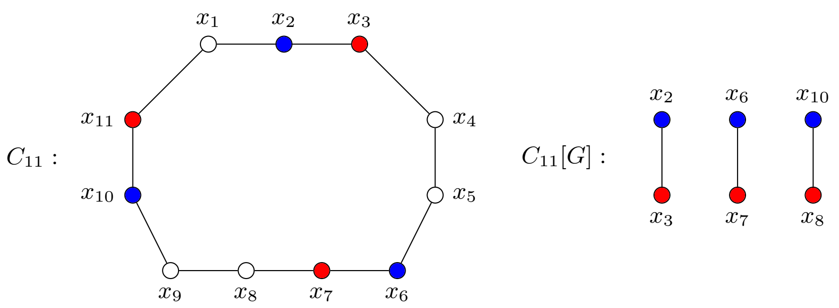
Clearly, $C_8[E] = C_8[E_1 \cup E_2]$ *is a bipartite dominating set of* C_8*, where* $E_1 = \{x_2, x_6\}$ *and* $E_2 = \{x_3, x_7\}$*. Thus,* $|E| = |E_1| + |E_2| = \frac{8}{4} + \frac{8}{4} = \frac{8}{2}$*. Hence,* $\gamma_{bip}(C_8) = \frac{8}{2} = 4$*.*

For $n \equiv 1(mod\ 4)$*, choose* C_9*, the set* $F = \{x_2, x_3, x_6, x_7, x_8\} \subseteq V(C_9)$ *is the bipartite dominating set and also the minimum bipartite dominating set.*



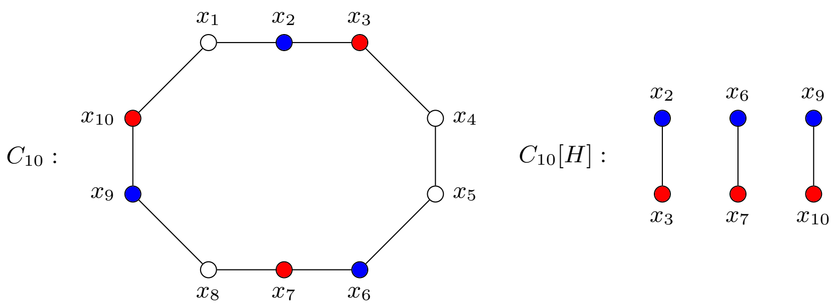
Clearly, $C_9[F] = C_9[F_1 \cup F_2]$ *is a bipartite dominating set of* C_9*, where* $F_1 = \{x_2, x_6, x_8\}$ *and* $F_2 = \{x_3, x_7\}$*. Thus,* $|F| = |F_1| + |F_2| = \frac{9+1}{4} + \frac{9+1}{4} = \frac{9+1}{2}$*. Hence,* $\gamma_{bip}(C_9) = \frac{9+1}{2} = 5$*.*

For $n \equiv 3(mod\ 4)$*, choose* C_{11}*, the set* $G = \{x_2, x_3, x_6, x_7, x_{10}, x_{11}\} \subseteq V(C_{11})$ *is the bipartite dominating set and also the minimum bipartite dominating set.*



Clearly, $C_{11}[G] = C_{11}[G_1 \cup G_2]$ *is a bipartite dominating set of* C_{11}*, where* $G_1 = \{x_2, x_6, x_{10}\}$ *and* $G_2 = \{x_3, x_7, x_{11}\}$*. Thus,* $|G| = |G_1| + |G_2| = \frac{11+1}{4} + \frac{11+1}{4} = \frac{11+1}{2}$*. Hence,* $\gamma_{bip}(C_{11}) = \frac{11+1}{2} = 6$*.*

For $n \equiv 2(mod\ 4)$, choose C_{10}, the set $H = \{x_2, x_3, x_6, x_7, x_9, x_{10}\} \subseteq V(C_{10})$ is the bipartite dominating set and also the minimum bipartite dominating set.



Clearly, $C_{10}[H] = C_{10}[H_1 \cup H_2]$ is a bipartite dominating set of C_{10}, where $H_1 = \{x_2, x_6, x_9\}$ and $H_2 = \{x_3, x_7, x_{10}\}$. Thus, $|H| = |H_1| + |H_2| = \frac{10+2}{4} + \frac{10+2}{4} = \frac{10+2}{2}$. Hence, $\gamma_{bip}(C_{10}) = \frac{10+2}{2} = 6$.

The next results can be easily determined.

Remark 3.3. For complete graph, K_n, $n \geq 2$, and complete bipartite graph, $K_{m,n}$, $m, n \geq 2$,

$$\gamma_{bip}(K_n) = 2 = \gamma_{bip}(K_{m,n}).$$

3.2 Bipartite Domination Number of a Join Graph, $K_{m,n}$

Theorem 3.4. *Let G be a nontrivial connected graph and H be a trivial graph. Then, $\emptyset \neq S \subseteq V(G \vee H)$ is a bipartite dominating set if and only if one of the following holds:*

1. *$S \subseteq V(G)$ such that S is a bipartite dominating set of G.*
2. *$S = B_1 \cup B_2$ such that $B_1 \subseteq V(G)$ and $B_2 = H$ for all $u, v \in V(G), u \neq v, u \notin N_G(v)$ and $v \notin N_G(u)$.*

Proof. Let $\emptyset \neq S \subseteq V(G \vee H)$ be a bipartite dominating set of $G \vee H$. Clearly, $S \nsubseteq V(H)$. Now, suppose $S \subseteq V(G)$, since S is a bipartite dominating set in $G \vee H$. S must be a bipartite dominating set in G. On the other hand, suppose $S = B_1 \cup B_2$ where $B_1 \subseteq V(G)$ and $B_2 = H$. Again, since S is a bipartite dominating set in $G \vee H$ then for all $u, v \in B_1 \subseteq V(G), u \notin N_G(v)$ and $v \notin N_G(u)$.

Conversely, suppose S satisfies property (*1*). Then, clearly S is a bipartite dominating set in $G \vee H$. Now, suppose S satisfies property (*2*). S is clearly a bipartite dominating set. Since for each $u, v \in V(G), u \neq v$ such that $u \notin N_G(v)$ and $v \notin N_G(u)$, $G \vee H[S]$ is a bipartite graph. Thus, S is a bipartite dominating set in $G \vee H$.

□

The following results immediately follow from Theorem 3.4.

Corollary 3.5. *For the fan graph, F_n, $n \geq 2$ and wheel graph, W_n, $n \geq 3$,*

$$\gamma_{bip}(F_n) = 2 = \gamma_{bip}(W_n).$$

Proof. This is immediate from Theorem 3.4.

□

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Theorem 3.6. *Let G and H be two nontrivial connected graph. Then, $\emptyset \neq S \subseteq V(G \vee H)$ is a bipartite dominating set if and only if one of the following holds:*

1. *$S \subseteq V(G)$ such that S is a bipartite dominating set of G.*
2. *$S \subseteq V(H)$ such that S is a bipartite dominating set of H.*
3. *$S = B_1 \cup B_2$ such that $B_1 \subseteq V(G)$ and $B_2 \subseteq V(H)$ for all $x, y \in B_1$, $x \notin N_G(y)$ and $y \notin N_G(x)$ and for all $u, v \in B_2$, $u \notin N_H(v)$ and $v \notin N_H(u)$.*

Proof. Let $\emptyset \neq S \subseteq V(G \vee H)$ be a bipartite dominating set of $G \vee H$. Clearly, $S \nsubseteq V(H)$. Now, suppose $S \subseteq V(G)$, since S is a bipartite dominating set in $G \vee H$. S must be a bipartite dominating set in G. Similarly, $S \nsubseteq V(G)$. Now, suppose $S \subseteq V(H)$, since S is a bipartite dominating set in $G \vee H$. S must be a bipartite dominating set in H. On the other hand, suppose $S = B_1 \cup B_2$ where $B_1 \subseteq V(G)$ and $B_2 \subseteq V(H)$. Again, since S is a bipartite dominating set in $G \vee H$ then for all $x, y \in B_1 \subseteq V(G), x \notin N_G(y)$ and $y \notin N_G(x)$. Similarly, for all $u, v \in B_2 \subseteq V(H), u \notin N_H(v)$ and $v \notin N_H(u)$.

Conversely, suppose S satisfies property (*1*). Then, clearly S is a bipartite dominating set in $G \vee H$. Similarly for S satisfying property (*2*). Now, suppose S satisfies property (*3*). S is clearly a bipartite dominating set. Since for each $x, y \in V(G), x \neq y$ such that $x \notin N_G(y)$ and $y \notin N_G(x)$. Similarly, for each $u, v \in V(H), u \neq v$ such that $u \notin N_G(v)$ and $v \notin N_G(u)$, $G \vee H[S]$ is a bipartite graph. Thus, S is a bipartite dominating set in $G \vee H$.

□

Corollary 3.7. *Let G and H be two nontrivial connected graph. Then,*

$$\gamma_{bip}(G \vee H) = 2.$$

Proof. This is immediate from Theorem 3.6. □

4 Conclusion

In this article, the bipartite domination number resulting from path, cycle, and complete graph are studied. As future line of research, it would be interesting to determine further results on some graph clustering.

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