A variant of chess with an upper bound for the number of reachable positions obtained from various initial configurations of pieces

Short Research Article

Abstract

We assume: only checkmate or stalemate ends the chess game, there are no castlings and en passant captures, no pawn can advance two squares in one move, the initial configuration of pieces is not fixed and satisfies some general conditions. Let $\mathcal{I}$ denote the set of all these configurations. By our assumptions, for every $\mathcal{C} \in \mathcal{I}$, after 0 or more moves, the configuration obtained from $\mathcal{C}$ and the information who has a move determine the set of all ways of continuing the game i.e. the reachable position. For $\mathcal{C} \in \mathcal{I}$, let $\mathcal{R}(\mathcal{C})$ denote the set of all reachable positions obtained from $\mathcal{C}$. A *MuPAD* program shows that $\mathrm{card}\left(\bigcup_{\mathcal{C}\in\mathcal{I}} \mathcal{R}(\mathcal{C})\right) < 4295923212088255192398899494807384799479217319544.$

Keywords: chess in which the initial configuration of pieces is not fixed, chess with simplified rules, upper bound for the number of reachable positions obtained from various initial configurations of pieces.

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We assume: only checkmate or stalemate ends the chess game, there are no castlings and en passant captures, no pawn can advance two squares in one move, the initial configuration of pieces is not fixed and satisfies the following conditions:

(1) For every colour, the number of pawns belongs to the set $\{0,\dots,8\}$.
(2) Pawns are in rows $2-7$.
(3) For every colour, the number of queens belongs to the set $\{0,1\}$.
(4) For every colour, the number of bishops (knights, rooks) belongs to the set $\{0,1,2\}$.
(5) The kings are not in adjacent squares.
(6) Black king is not checked.

Let $\mathcal{I}$ denote the set of all initial configurations of pieces satisfying conditions (1)-(6). By our assumptions, for every $\mathcal{C} \in \mathcal{I}$, after 0 or more moves, the configuration obtained from $\mathcal{C}$ and the information who has a move determine the set of all ways of continuing the game i.e. the reachable position. For $\mathcal{C} \in \mathcal{I}$, let $\mathcal{R}(\mathcal{C})$ denote the set of all reachable positions obtained from $\mathcal{C}$.

For the classical chess, a non-trivial upper bound for the number of reachable positions is computed in [1]. For every $\mathcal{C} \in \mathcal{I}$, the bound from Theorem 1 applies to Theorem 3.

Theorem 1.

$$\mathrm{card}\left(\bigcup_{\mathcal{C}\in\mathcal{I}} \mathcal{R}(\mathcal{C})\right) \leqslant 4295923212088255192398899494807384879947921731954 4$$

Proof. The execution of the following *MuPAD* program returns the above number.

```
kings0:=0:
/* kings0 equals the number of legal configurations of kings */
/* in which none king is in rows 2-7 */
kings1:=0:
/* kings1 equals the number of legal configurations of kings */
/* in which exactly one king is in rows 2-7 */
kings2:=0:
/* kings2 equals the number of legal configurations of kings */
/* in which both kings are in rows 2-7 */
for king1x from 1 to 8 do
for king1y from 1 to 8 do
for king2x from 1 to 8 do
for king2y from 1 to 8 do
m:=0:
if king1y>1 and king1y<8 then m:=m+1 end_if:
if king2y>1 and king2y<8 then m:=m+1 end_if:
if abs(king1x-king2x)<2 and abs(king1y-king2y)<2 then m:=3 end_if:
if m=0 then kings0:=kings0+1 end_if:
if m=1 then kings1:=kings1+1 end_if:
if m=2 then kings2:=kings2+1 end_if:
end_for:
end_for:
end_for:
end_for:
bound:=0:
for n from 0 to 2 do
if n=0 then ki:=kings0 end_if:
if n=1 then ki:=kings1 end_if:
if n=2 then ki:=kings2 end_if:
for p1 from 0 to 8 do
/* p1 denotes the number of white pawns */
pa1:=binomial(64-16-n,p1):
for p2 from 0 to 8 do
/* p2 denotes the number of black pawns */
pa2:=binomial(64-16-n-p1,p2):
```

```
for q1 from 0 to 1+8-p1 do
/* q1 denotes the number of white queens */
qu1:=binomial(64-2-p1-p2,q1):
for q2 from 0 to 1+8-p2 do
/* q2 denotes the number of black queens */
qu2:=binomial(64-2-p1-p2-q1,q2):
for b1 from 0 to 2+8-p1-max(q1-1,0) do
/* b1 denotes the number of white bishops */
bi1:=binomial(64-2-p1-p2-q1-q2,b1):
for b2 from 0 to 2+8-p2-max(q2-1,0) do
/* b2 denotes the number of black bishops */
bi2:=binomial(64-2-p1-p2-q1-q2-b1,b2):
for k1 from 0 to 2+8-p1-max(q1-1,0)-max(b1-2,0) do
/* k1 denotes the number of white knights */
kn1:=binomial(64-2-p1-p2-q1-q2-b1-b2,k1):
for k2 from 0 to 2+8-p2-max(q2-1,0)-max(b2-2,0) do
/* k2 denotes the number of black knights */
kn2:=binomial(64-2-p1-p2-q1-q2-b1-b2-k1,k2):
for r1 from 0 to 2+8-p1-max(q1-1,0)-max(b1-2,0)-max(k1-2,0) do
/* r1 denotes the number of white rooks */
ro1:=binomial(64-2-p1-p2-q1-q2-b1-b2-k1-k2,r1):
for r2 from 0 to 2+8-p2-max(q2-1,0)-max(b2-2,0)-max(k2-2,0) do
/* r2 denotes the number of black rooks */
ro2:=binomial(64-2-p1-p2-q1-q2-b1-b2-k1-k2-r1,r2):
d:=ki*pa1*pa2*qu1*qu2*bi1*bi2*kn1*kn2*ro1*ro2:
bound:=bound+d:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
print(2*bound):
/* the factor 2 corresponds to the alternative */
/* (White has a move) or (Black has a move) */
```

Pawns cannot be in rows 1 and 8. It applies to the instructions with the numbers $64-16-n$ and $64-16-n-p1$. Non-negative integers $p1$, $p2$, $q1$, $q2$, $b1$, $b2$, $k1$, $k2$, $r1$, $r2$ satisfy:

$p1, p2 \in \{0, \dots, 8\}$

$q1 \in \{0, \dots, 1+8-p1\}$

$q2 \in \{0, \dots, 1+8-p2\}$

$b1 \in \{0, \ldots, 2 + 8 - p1 - \max(q1 - 1, 0)\}$
$b2 \in \{0, \ldots, 2 + 8 - p2 - \max(q2 - 1, 0)\}$
$k1 \in \{0, \ldots, 2 + 8 - p1 - \max(q1 - 1, 0) - \max(b1 - 2, 0)\}$
$k2 \in \{0, \ldots, 2 + 8 - p2 - \max(q2 - 1, 0) - \max(b2 - 2, 0)\}$
$r1 \in \{0, \ldots, 2 + 8 - p1 - \max(q1 - 1, 0) - \max(b1 - 2, 0) - \max(k1 - 2, 0)\}$
$r2 \in \{0, \ldots, 2 + 8 - p2 - \max(q2 - 1, 0) - \max(b2 - 2, 0) - \max(k2 - 2, 0)\}$

We have: `bound =`

$$\sum_{\substack{n \in \{0,1,2\} \\ p1,p2 \in \{0,\ldots,8\} \\ q1,q2 \in \{0,\ldots,1+8\} \\ b1,b2,k1,k2,r1,r2 \in \{0,\ldots,2+8\} \\ p1+\max(q1-1,0)+\max(b1-2,0)+ \\ \max(k1-2,0)+\max(r1-2,0) \leqslant 8 \\ p2+\max(q2-1,0)+\max(b2-2,0)+ \\ \max(k2-2,0)+\max(r2-2,0) \leqslant 8}} ki \cdot pa1 \cdot pa2 \cdot qu1 \cdot qu2 \cdot bi1 \cdot bi2 \cdot kn1 \cdot kn2 \cdot ro1 \cdot ro2$$

□

Theorem 2. *Theorem 1 holds with strong inequality.*

Proof. The bound in Theorem 1 takes into account many cases in which black king is in check and `White` has a move. The bound in Theorem 1 takes into account many cases in which white king is in check and `Black` has a move. □

Theorem 3. *([2, p. 128]). A player who is in a winning position is always able to enforce a win in a number of moves that is less than the number of positions in the game.*

References

[1] S. Chinchalkar, *An upper bound for the number of reachable positions,* J. Int. Comput. Games Assoc. 19 (1996), no 3, 181–183, `http://doi.org/10.3233/ICG-1996-19305`.

[2] U. Schwalbe, P. Waker, *Zermelo and the early history of game theory,* Games Econom. Behav. 34 (2001), no. 1, 123–137, `http://doi.org/10.1006/game.2000.0794`.