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# Clique Centrality and Global Clique Centrality of Graphs

**Original Research Article**

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## Abstract

We formally introduce in this paper two parameters in graph theory, namely, clique centrality and global clique centrality. Let  $G$  be a finite, simple and undirected graph of order  $n$ . A clique in  $G$  is a nonempty subset  $W \subseteq V(G)$  such that the subgraph  $\langle W \rangle_G$  induced by  $W$  is complete. The maximum size of any clique containing vertex  $u \in V(G)$  is called the clique centrality of  $u$  in  $G$ . Normalizing the sum of the clique centralities of all the vertices of  $G$  will lead us to the global clique centrality of  $G$ , whose value ranges from  $\frac{1}{m}$  to 1. In this paper, we study some general properties of the global clique centrality and then evaluate it for some parameterized families of graphs.

*Keywords: clique, centrality, global clique centrality, social network*

*2020 Mathematics Subject Classification: 05C69, 05C35*

## 1 Introduction

A clique in a graph  $G$  is a subset of the vertex set of  $G$  where any two vertices in the subset are connected by an edge in  $G$ . A clique's size is determined by how many vertices that make up the clique. Every node in a graph may be a member of one or more cliques of various sizes [1]. The largest size of a clique in  $G$  containing node  $u \in G$  is referred to in this paper as the clique centrality of  $u$ . Note that the maximum of the clique centrality values for all the nodes of  $G$  is widely known as the clique number  $\omega(G)$  of  $G$ .

Clique centrality may be added to the list of basic centrality indices at the node or vertex level of a graph, together with the more classical degree centrality, closeness centrality, betweenness centrality, harmonic centrality, and eigenvector centrality. This vertex level clique centrality can be upgraded

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to the corresponding graph level global clique centrality, which will be done in the next section, or to the clique centralization parameter following Freeman in 1979 [2], an exploration of which is being done separately.

In a social network  $G$  where the nodes or vertices represent people and the links or edges represent mutual acquaintances of the concerned people, a clique represents a particular subset of the people who all know each other. It is known that for an arbitrary network or graph  $G$ , the problem of finding a clique of maximum size (the maximum clique problem), and subsequently that of  $\omega(G)$ , is NP-hard (see for instance [3]). As a consequence, finding the global clique centrality of an arbitrary graph  $G$  could be very hard; nevertheless, its equivalent expression or simplified formula would be explored in some specifically structured graphs like those carried out in [5], [8], [9], [10], [11], [12], [13], and [14].

In this paper, we aim to show some general properties of the global clique centrality and then investigate it for some parameterized families of graphs such as path, cycle, complete graph, star, complete bipartite graph, crown, and complete split graph with the hope that the generated results in this study would be of use and help when one considers to study more complex graphs.

For basic graph theoretic terminologies not given here, please refer to [4]. Throughout this paper, all graphs are considered nonempty, finite, undirected, and simple with vertex set  $V(G)$  and edge set  $E(G)$ .

## 2 Basic Concepts and Elementary Properties

For emphasis, two of the main concepts in this paper are formally defined below.

**Definition 2.1.** Let  $G$  be a nontrivial, connected, and simple graph. A clique in a graph  $G$  is a subset of  $V(G)$  such that every pair of vertices in the subset are adjacent in  $G$ , that is, its induced subgraph is complete. A maximal clique is a clique that cannot be extended to a bigger clique by including one more adjacent vertex. The maximum cardinality of a clique containing vertex  $u \in V(G)$  is called the clique centrality of  $u$  and is denoted by  $\omega_G(u)$ .

**Example 2.2** In Figure 1 below, graph  $G$  has 3 maximal cliques, namely,  $W_1 = \{u_2, u_3, u_4\}$ ,  $W_2 = \{u_2, u_4, u_5\}$ , and  $W_3 = \{u_1, u_2, u_5, u_6\}$ . Observe that the clique centralities of the 6 vertices are given as follows:  $\omega_G(u_3) = \omega_G(u_4) = 3$ ,  $\omega_G(u_1) = \omega_G(u_2) = \omega_G(u_5) = \omega_G(u_6) = 4$ . From these values, we have  $\omega(G) = 4$ .

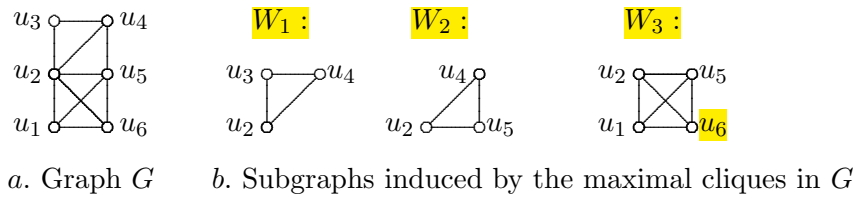


Figure 1: Graph  $G$  and its subgraphs induced by its maximal cliques

**Definition 2.3** The global clique centrality  $\hat{\omega}(G)$  of a graph  $G$  of order  $m$  is the ratio of the sum of the clique centralities of the vertices of  $G$  to the square of the order of  $G$ . In symbol,

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$$\widehat{\omega}(G) = \frac{\sum_{u \in V(G)} \omega_G(u)}{m^2}.$$

The following are some general properties of the global clique centrality of a graph.

**Theorem 2.4** Let  $G$  be any graph of order  $m$ . Then

$$\widehat{\omega}(G) \leq \frac{\omega(G)}{m} \leq 1.$$

Moreover,  $\widehat{\omega}(G) = 1$  iff  $G = K_m$ .

*Proof.* Note that

$$\widehat{\omega}(G) = \frac{\sum_{u \in V(G)} \omega_G(u)}{m^2} \leq \frac{\sum_{u \in V(G)} \omega(G)}{m^2} = \frac{m \cdot \omega(G)}{m^2} = \frac{\omega(G)}{m}$$

Clearly,

$$\widehat{\omega}(G) \leq \frac{\omega(G)}{m}$$

Since  $\omega(G) \leq m$ , it follows that

$$\widehat{\omega}(G) \leq \frac{\omega(G)}{m} \leq \frac{m}{m} = 1.$$

Moreover,  $\widehat{\omega}(G) = 1$  iff  $\omega_G(u) = m \forall u \in V(G)$ , that is,  $\widehat{\omega}(G) = 1$  iff  $G \cong K_m$ . □

**Theorem 2.5** For any graph  $G$  of order  $m$ , we have

$$\frac{1}{m} \leq \widehat{\omega}(G).$$

Moreover,  $\widehat{\omega}(G) = \frac{1}{m}$  iff  $G = \overline{K_m}$ .

*Proof.* We know that  $1 \leq \omega_G(u)$  for each  $u \in V(G)$ . Thus, we have

$$\frac{1}{m} = \frac{m}{m^2} = \frac{\sum_{u \in V(G)} [1]}{m^2} \leq \frac{\sum_{u \in V(G)} \omega_G(u)}{m^2} = \widehat{\omega}(G),$$

where  $\widehat{\omega}(G) = \frac{1}{m}$  if and only if  $\omega_G(u) = 1$  for each  $u \in V(G)$ ; that is  $\widehat{\omega}(G) = \frac{1}{m}$  if and only if  $G$  is the null graph  $\overline{K_m}$ . □

**Theorem 2.6** Let  $G$  be any graph and  $H$  a spanning subgraph of  $G$ . Then

$$\widehat{\omega}(H) \leq \widehat{\omega}(G).$$

*Proof.* Since  $H$  is a spanning subgraph of  $G$ , it follows that  $V(H) = V(G)$  and  $E(H) \subseteq E(G)$ . Consequently, we have  $\omega_H(u) \leq \omega_G(u)$ , so that

$$\widehat{\omega}(H) = \frac{\sum_{u \in V(H)} \omega_H(u)}{m^2} \leq \frac{\sum_{u \in V(G)} \omega_G(u)}{m^2} = \widehat{\omega}(G).$$

□

**Theorem 2.7** For any real number  $\epsilon > 0$ , there exists a graph  $G$  so that  $\widehat{\omega}(G) < \epsilon$ .

*Proof.* For  $\epsilon > 0$ , choose a particular integer  $m_0$  such that  $\frac{1}{m_0} < \epsilon$ . Then choose the null graph  $G = \overline{K_{m_0}}$ . By Theorem 2.5 above, we have  $\widehat{\omega}(G) = \frac{1}{m_0} < \epsilon$ . □

### 3 Global Clique Centrality of Some Families of Graphs

Recall that the path  $P_m$  of order  $m$  is a sequence of distinct vertices  $v_1, v_2, \dots, v_m$  with  $m - 1$  distinct edges  $v_1v_2, v_2v_3, \dots, v_{m-1}v_m$ . The cycle  $C_m$  of order  $m$  is the graph consisting of  $m$  distinct vertices  $v_1, v_2, \dots, v_m$  and distinct edges  $v_1v_2, v_2v_3, \dots, v_{m-1}v_m, v_mv_1$ . The skeletal diagrams of these parameterized graphs are shown in Figure 2 below.

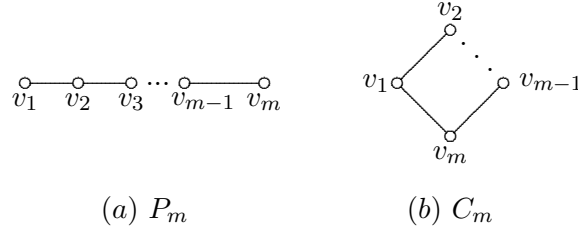


Figure 2: The path  $P_m$  and cycle  $C_m$

**Proposition 3.1.** Let  $G$  be the path  $P_m = [v_1, v_2, \dots, v_m]$  of order  $m \geq 3$  or the cycle  $C_m = [v_1, v_2, \dots, v_m, v_1]$  of order  $m \geq 4$ . Then

$$\widehat{\omega}(P_m) = \widehat{\omega}(C_m) = \frac{2}{m}$$

*Proof.* Note that in  $P_m$  and  $C_m$ ,  $\omega_{P_m}(v_i) = \omega_{C_m}(v_i) = 2$ , for every  $i = 1, 2, \dots, m$ , so that by Definition 2.3, we have

$$\begin{aligned} \widehat{\omega}(P_m) = \widehat{\omega}(C_m) &= \sum_{i=1}^m \frac{\omega(x_i)}{m^2}, \text{ for every } x_i \in V(P_m) \text{ (or } x_i \in V(C_m)) \\ &= \frac{2m}{m^2} \\ &= \frac{2}{m}. \end{aligned}$$

□

For the other parameterized families of graphs, we consider the complete split graph, complete bipartite graph, crown graph, fan graph, wheel graph, star graph, helm graph, and the sunlet graph.

The complete split graph  $CS(m, k)$  of order  $m + k$  is the graph whose vertex set  $V(CS(m, k))$  comprises of two sets  $A = \{u_1, u_2, \dots, u_m\}$  and  $B = \{v_1, v_2, \dots, v_k\}$  that induce a clique and an empty graph, respectively, where  $u_i v_j$  is an edge,  $1 \leq i \leq m, 1 \leq j \leq k$  [5]. A graph  $G$  is called bipartite if its vertex set  $V(G)$  can be partitioned into two nonempty subsets  $V_1$  and  $V_2$ , called the partite sets of  $G$ , such that every edge of  $G$  has one end vertex in  $V_1$  and another end vertex in  $V_2$ . A bipartite graph  $G$  is called a complete bipartite graph if every vertex in one particular partite set is adjacent to all the vertices in the other partite set [6]. The crown graph  $G_{m,m}$  of order  $2m$  is the graph with vertex set  $V(G_{m,m}) = \{u_1, u_2, \dots, u_m\} \cup \{v_1, v_2, \dots, v_m\}$  and whose edges are produced by connecting  $u_i$  to  $v_j$  whenever  $i \neq j$  [7]. Figure 3 provides the skeletal diagrams of these graphs.

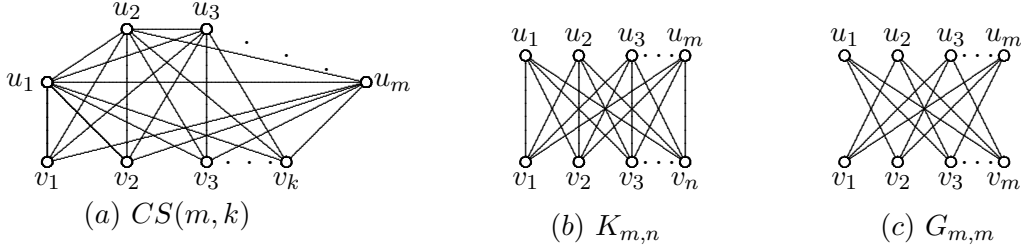


Figure 3: The complete split graph  $CS(m, k)$ , complete bipartite  $K_{m,n}$ , and crown graph  $G_{m,m}$

**Proposition 3.2.** The global clique centrality of each of graphs  $CS(m, k)$ ,  $K_{m,n}$ , and  $G_{m,m}$  are given as follows:

- i.  $\widehat{\omega}(CS(m, k)) = \frac{m+1}{m+k}$ ;
- ii.  $\widehat{\omega}(K_{m,n}) = \frac{2}{m+n}$ ;
- iii.  $\widehat{\omega}(G_{m,m}) = \frac{1}{m}$ .

*Proof.* The proofs are done by applying Definition 2.3 and in similar fashion as the path graph  $P_m$  and cycle  $C_m$  in Proposition 3.1 and thus they are omitted in this paper.  $\square$

The next group of parameterized families of graphs are graphs with order  $m+1$ , namely, the fan graph  $F_{m+1}$ , wheel graph  $W_{m+1}$ , and the star graph  $K_{1,m}$ . The fan graph  $F_{m+1}$  of order  $m+1$  is the graph formed by connecting one extra vertex  $v_0$  to every vertex  $u_i$  of the path  $P_m = [u_1, u_2, \dots, u_m]$  [6]. The wheel graph  $W_{m+1}$  of order  $m+1$  is the graph obtained by adjoining a vertex  $v_0$  to every vertex  $u_i$  of the cycle  $C_m = [u_1, u_2, \dots, u_m, u_1]$  [7]. The star  $K_{1,m}$  of order  $m+1$  is the graph formed by linking each of the  $m$  pairwise non-adjacent vertices  $u_i, 1 \leq i \leq m$ , to a single vertex  $v_0$  [7]. Figure 4 provides the skeletal diagrams of these graphs.

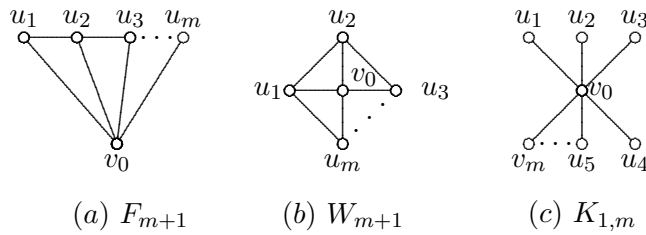


Figure 4: The fan  $F_{m+1}$ , wheel  $W_{m+1}$ , and star  $K_{1,m}$

**Proposition 3.3.** The global clique centralities of  $F_{m+1}$ ,  $W_{m+1}$  and  $K_{1,m}$  are given as follows:

- i.  $\widehat{\omega}(F_{m+1}) = \frac{3}{m+1}$ ,  $m \geq 2$ ;

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$$\text{ii. } \widehat{\omega}(W_{m+1}) = \begin{cases} \mathbf{1} & \text{if } m = 3, \\ \frac{3}{m+1} & \text{if } m \geq 4; \end{cases}$$

$$\text{iii. } \widehat{\omega}(K_{1,m}) = \frac{2}{m+1}.$$

*Proof.* Again, the proofs are done in a similar manner as Proposition 3.1 and are also omitted in this paper.  $\square$

Finally, the helm graph  $H_m$  is a graph formed from a wheel  $W_{m+1}$  by attaching a pendant vertex at each of the vertices of the  $m$ -cycle. Observe that  $H_m$  contains three types of vertices: an apex vertex  $v_0$  of degree  $m$ ,  $m$  vertices  $v_1, v_2, \dots, v_m$  of degree 4, and  $m$  pendant vertices  $u_1, u_2, \dots, u_m$  of degree 1 [8]. The sunlet graph  $S_m$  is the graph of order  $2m$  formed from a cycle by attaching a pendant vertex  $u_i$  to every vertex  $v_i$  of the  $m$ -cycle [8]. The graphs are illustrated in Figure 5 below.

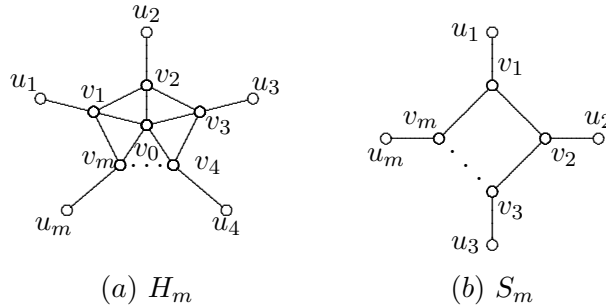


Figure 5: The helm graph  $H_m$  and the sunlet graph  $S_m$

**Proposition 3.4.** The global clique centralities of  $H_m$  and  $S_m$  are given as follows:

$$\text{i. } \widehat{\omega}(H_m) = \begin{cases} \frac{22}{49} & \text{if } m = 3, \\ \frac{5m+3}{(2m+1)^2} & \text{if } m \geq 4; \end{cases}$$

$$\text{ii } \widehat{\omega}(S_m) = \begin{cases} \frac{5}{12} & \text{if } m = 3, \\ \frac{1}{m} & \text{if } m \geq 4. \end{cases}$$

*Proof.* The global clique centralities above are obtained by applying Definition 2.3 and the proofs are similar to that of Proposition 3.1.  $\square$

## 4 Conclusion

In this paper, we successfully introduced the clique centrality at the vertex level and the global clique centrality at the graph level, where the second parameter can be used to compare two graphs of the same order. We were able to generate some general properties for the global clique centrality of graphs. As planned, we were also able to show the corresponding formulas for the global clique centrality of some known parameterized families of graphs such as path, cycle, complete split graph, wheel graph, fan graph, crown graph, complete graph, complete bipartite graph, helm graph, and sunlet graph. Our final goal in this paper was to obtain the corresponding clique centrality and global clique centrality formulas of the aforementioned graphs, similar to those done in [5], [8] [9], [10], [11], [12], [13], and [14], with the hope that the generated results in this study would be of

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help when one considers to study more complex graphs.

Some open problems: Given the value  $\widehat{\omega}(G)$  for a graph  $G$  of order  $m$ , determine the following:

- a.  $\widehat{\omega}(\overline{G})$ , where  $\overline{G}$  is the complement of  $G$ ;
- b.  $\widehat{\omega}(L(G))$ , where  $L(G)$  is the line graph of  $G$ ;
- c.  $\widehat{\omega}(G^2)$ , where  $G^2$  is the second power of  $G$ .

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## 6 Competing Interests

The authors declare that they have no competing interests.

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