Clique Centrality and Global Clique Centrality of Graphs

Original Research Article

Abstract

We formally introduce in this paper two parameters in graph theory, namely, clique centrality and global clique centrality. Let G be a finite, simple and undirected graph of order n. A clique in G is a nonempty subset $W \subseteq V(G)$ such that the subgraph $\langle W \rangle_G$ induced by W is complete. The maximum size of any clique containing vertex $u \in V(G)$ is called the clique centrality of u in G. Normalizing the sum of the clique centralities of all the vertices of G will lead us to the global clique centrality of G, whose value ranges from $\frac{1}{m}$ to 1. In this paper, we study some general properties of the global clique centrality and then evaluate it for some parameterized families of graphs.

Keywords: clique, centrality, global clique centrality, social network

2020 Mathematics Subject Classification: 05C69, 05C35

1 Introduction

A clique in a graph G is a subset of the vertex set of G where any two vertices in the subset are connected by an edge in G. A clique's size is determined by how many vertices that make up the clique. Every node in a graph may be a member of one or more cliques of various sizes [1]. The largest size of a clique in G containing node $u \in G$ is referred to in this paper as the clique centrality of u. Note that the maximum of the clique centrality values for all the nodes of G is widely known as the clique number $\omega(G)$ of G.

Clique centrality may be added to the list of basic centrality indices at the node or vertex level of a graph, together with the more classical degree centrality, closeness centrality, betweeness centrality, harmonic centrality, and eigenvector centrality. This vertex level clique centrality can be upgraded to the corresponding graph level global clique centrality, which will be done in the next section, or to the clique centralization parameter following Freeman in 1979 [2], an exploration of which is being done separately..

In a social network G where the nodes or vertices represent people and the links or edges represent mutual acquaintances of the concerned people, a clique represents a particular subset of the people who all know each other. It is known that for an arbitrary network or graph G, the problem of finding a clique of maximum size (the maximum clique problem), and subsequently that of $\omega(G)$, is NP-hard (see for instance [3]). As a consequence, finding the global clique centrality of an arbitrary graph G could be very hard; nevertheless, its equivalent expression or simplified formula would be explored in some specifically structured graphs like those carried out in [5], [8], [9], [10], [11], [12], [13], and [14].

In this paper, we aim to show some general properties of the global clique centrality and then investigate it for some parameterized families of graphs such as path, cycle, complete graph, star, complete bipartite graph, crown, and complete split graph with the hope that the generated results in this study would be of use and help when one considers to study more complex graphs.

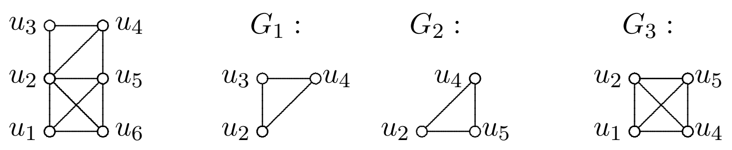
For basic graph theoretic terminologies not given here, please refer to [4]. Throughout this paper, all graphs are considered nonempty, finite, undirected, and simple with vertex set $V(G)$ and edge set $E(G)$.

2 Basic Concepts and Elementary Properties

For empahsis, two of the main concepts in this paper are formally defined below.

Definition 2.1. Let G be a nontrivial, connected, and simple graph. A clique in a graph G is a subset of $V(G)$ such that every pair of vertices in the subset are adjacent in G, that is, its induced subgraph is complete. A maximal clique is a clique that cannot be extended to a bigger clique by including one more adjacent vertex. The maximum cardinality of a clique containing vertex $u \in V(G)$ is called the clique centrality of u and is denoted by $\omega_G(u)$.

Example 2.2 In Figure 1 below, graph G has 3 maximal cliques, namely, $W_1 = \{u_2, u_3, u_4\}$, $W_2 = \{u_2, u_4, u_5\}$, and $W_3 = \{u_1, u_2, u_5, u_4\}$. Observe that the clique centralities of the 6 vertices are given as follows: $\omega_G(u_3) = \omega_G(u_4) = 3$, $\omega_G(u_1) = \omega_G(u_2) = \omega_G(u_5) = \omega_G(u_6) = 4$. From these values, we have $\omega(G) = 4$.



a. Graph G *b.* Subgraphs induced by the maximal cliques in G

Figure 1: Graph G and its subgraphs induced by its maximal cliques

Definition 2.3 The global clique centrality $\widehat{\omega}(G)$ of a graph G of order m is the ratio of the sum of the clique centralities of the vertices of G to the square of the order of G. In symbol,

$$\widehat{\omega}(G) = \frac{\sum_{u \in V(G)} \omega_G(u)}{m^2}.$$

The following are some general properties of the global clique centrality of a graph.

Theorem 2.4 Let G be any graph of order m. Then

$$\widehat{\omega}(G) \leq \frac{\omega(G)}{m} \leq 1.$$

Moreover, $\widehat{\omega}(G) = 1$ iff $G = K_m$.

Proof. Note that

$$\widehat{\omega}(G) = \frac{\sum_{u \in V(G)} \omega_G(u)}{m^2} \leq \frac{\sum_{u \in V(G)} \omega(G)}{m^2} = \frac{m \cdot \omega(G)}{m^2} = \frac{\omega(G)}{m}$$

Clearly,

$$\widehat{\omega}(G) \leq \frac{\omega(G)}{m}$$

Since $\omega(G) \leq m$, it follows that

$$\widehat{\omega}(G) \leq \frac{\omega(G)}{m} \leq \frac{m}{m} = 1.$$

Moreover, $\widehat{\omega}(G) = 1$ iff $\omega_G(u) = m \ \forall \ u \in V(G)$, that is, $\widehat{\omega}(G) = 1$ iff $G \cong K_m$. □

Theorem 2.5 For any graph G of order m, we have

$$\frac{1}{m} \leq \widehat{\omega}(G).$$

Moreover, $\widehat{\omega}(G) = \frac{1}{m}$ iff $G = \overline{K_m}$.

Proof. We know that $1 \leq \omega_G(u)$ for each $u \in V(G)$. Thus, we have

$$\frac{1}{m} = \frac{m}{m^2} = \frac{\sum_{u \in V(G)} [1]}{m^2} \leq \frac{\sum_{u \in V(G)} \omega_G(u)}{m^2} = \widehat{\omega}(G),$$

where $\widehat{\omega}(G) = \frac{1}{m}$ if and only if $\omega_G(u) = 1$ for each $u \in V(G)$; that is $\widehat{\omega}(G) = \frac{1}{m}$ if and only if G is the null graph $\overline{K_m}$. □

Theorem 2.6 Let G be any graph and H a spanning subgraph of G. Then

$$\widehat{\omega}(H) \leq \widehat{\omega}(G).$$

Proof. Since H is a spanning subgraph of G, it follows that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. Consequently, we have $\omega_H(u) \leq \omega_G(u)$, so that

$$\widehat{\omega}(H) = \frac{\sum_{u \in V(H)} \omega_H(u)}{m^2} \leq \frac{\sum_{u \in V(G)} \omega_G(u)}{m^2} = \widehat{\omega}(G).$$

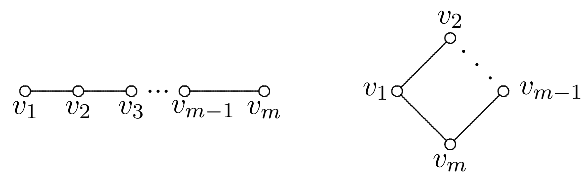
□

Theorem 2.7 For any real number $\epsilon > 0$, there exists a graph G so that $\widehat{\omega}(G) < \epsilon$.

Proof. For $\epsilon > 0$, choose a particular integer m_0 such that $\frac{1}{m_0} < \epsilon$. Then choose the null graph $G = \overline{K_{m_0}}$. By Theorem 2.5 above, we have $\widehat{\omega}(G) = \frac{1}{m_0} < \epsilon$. □

3 Global Clique Centrality of Some Familiies of Graphs

Recall that the path P_m of order m is a sequence of distinct vertices $v_1, v_2, ..., v_m$ with $m-1$ distinct edges $v_1v_2, v_2v_3, ..., v_{m-1}v_m$. The cycle C_m of order m is the graph consisting of m distinct vertices $v_1, v_2, ..., v_m$ and distinct edges $v_1v_2, v_2v_3, ..., v_{m-1}v_m, v_mv_1$. The skeletal diagrams of these parameterized graphs are shown in Figure 2 below.



(a) P_m (b) C_m

Figure 2: The path P_m and cycle C_m

Proposition 3.1. Let G be the path $P_m = [v_1, v_2, ..., v_m]$ of order $m \geq 2$ or the cycle $C_m = [v_1, v_2, ..., v_m, v_1]$ of order $m \geq 3$. Then

$$\widehat{\omega}(P_m) = \widehat{\omega}(C_m) = \frac{2}{m}$$

Proof. Note that in P_m and C_m, $\omega_{P_m}(v_i) = \omega_{C_m}(v_i) = 2$, for every $i = 1, 2, ..., m$, so that by Definition 2.3, we have

$$\begin{aligned}\widehat{\omega}(P_m) = \widehat{\omega}(C_m) &= \sum_{i=1}^{m} \frac{\omega(x_i)}{m^2}, \text{ for every } x_i \in V(P_m) \text{ (or } x_i \in V(C_m))\\ &= \frac{2m}{m^2}\\ &= \frac{2}{m}.\end{aligned}$$

□

For the other parameterized families of graphs, we consider the complete split graph, complete bipartite graph, crown graph, fan graph, wheel graph, star graph, helm graph, and the sunlet graph.

The complete split graph $CS(m,k)$ of order $m+k$ is the graph whose vertex set $V(CS(m,k))$ comprises of two sets $A = \{u_1, u_2, ..., u_m\}$ and $B = \{v_1, v_2, ..., v_k\}$ that induce a clique and an empty graph, respectively, where u_iv_j is an edge, $1 \leq i \leq m, 1 \leq j \leq k$ [5]. A graph G is called bipartite if its vertex set $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2, called the partite sets of G, such that every edge of G has one end vertex in V_1 and another end vertex in V_2. A bipartite graph G is called a complete bipartite graph if every vertex in one particular partite set is adjacent to all the vertices in the other partite set [6]. The crown graph $G_{m,m}$ of order $2m$ is the graph with vertex set $V(G_{m,m}) = \{u_1, u_2, ..., u_m\} \bigcup \{v_1, v_2, ..., v_m\}$ and whose edges are produced by connecting u_i to v_j whenever $i \neq j$ [7]. Figure 3 provides the skeletal diagrams of these graphs.

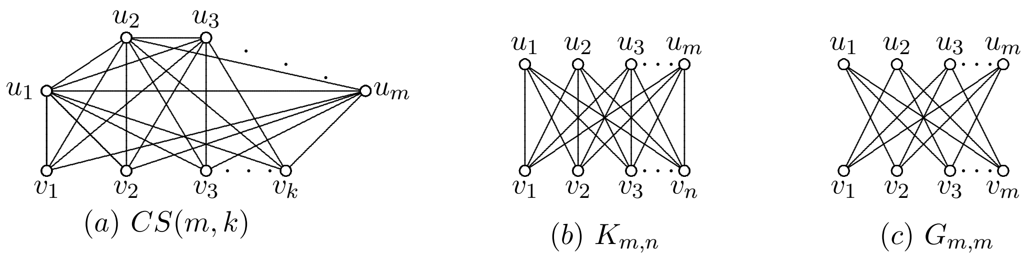


Figure 3: The complete split graph $CS(m,k)$, complete bipartite $K_{m,n}$, and crown graph $G_{m,m}$

Proposition 3.2. The global clique centrality of each of graphs $CS(m,k)$, $K_{m,n}$, and $G_{m,m}$ are given as follows:

i. $\widehat{\omega}(CS(m,k)) = \frac{m+1}{m+k}$;

ii. $\widehat{\omega}(K_{m,n}) = \frac{2}{m+n}$;

iii. $\widehat{\omega}(G_{m,m}) = \frac{1}{m}$.

Proof. The proofs are done by applying Definition 2.3 and in similar fashion as the path graph P_m and cycle C_m in Proposition 3.1 and thus they are omitted in this paper. □

The next group of parmeterized families of graphs are graphs with oder $m+1$, namely, the fan graph F_{m+1}, wheel graph W_{m+1}, and the star graph $K_{1,m}$. The fan graph F_{m+1} of order $m+1$ is the graph formed by connecting one extra vertex v_0 to every vertex u_i of the path $P_m = [u_1, u_2, ..., u_m]$ [6]. The wheel graph W_{m+1} of order $m+1$ is the graph obtained by adjoining a veretx v_0 to every vertex u_i of the cycle $C_m = [u_1, u_2, ..., u_m, u_1]$ [7]. The star $K_{1,m}$ of order $m+1$ is the graph formed by linking each of the m pairwise non-adjacent vertices $u_i, 1 \leq i \leq m$, to a single vertex v_0 [7]. Figure 4 provides the skeletal diagrams of these graphs.

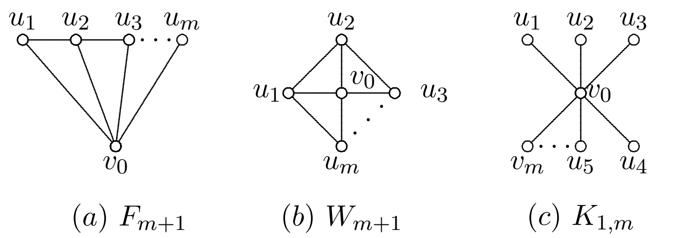


Figure 4: The fan F_{m+1}, wheel W_{m+1}, and star $K_{1,m}$

Proposition 3.3. The global clique centralities of F_{m+1}, W_{m+1} and $K_{1,m}$ are given as follows:

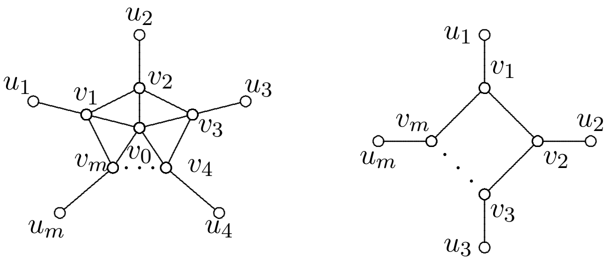
i. $\widehat{\omega}(F_{m+1}) = \frac{3}{m+1}, \quad m \geq 2$;

ii. $\widehat{\omega}(W_{m+1}) = \begin{cases} \frac{4}{m+1} & if \quad m = 3, \\ \frac{3}{m+1} & if \quad m \geq 4; \end{cases}$

iii. $\widehat{\omega}(K_{1,m}) = \frac{2}{m+1}$.

Proof. Again, the proofs are done in a similar manner as Proposition 3.1 and are also omitted in this paper. □

Finally, the helm graph H_m is a graph formed from a wheel W_{m+1} by attaching a pendant vertex at each of the vertices of the m-cycle. Observe that H_m contains three types of vertices: an apex vertex v_0 of degree m, m vertices $v_1, v_2, ..., v_m$ of degree 4, and m pendant vertices $u_1, u_2, ..., u_m$ of degree 1 [8]. The sunlet graph S_m is the graph of order $2m$ formed from a cycle by attaching a pendant vertex u_i to every vertex v_i of the m-cycle [8]. The graphs are illustrated in Figure 5 below.



(a) H_m (b) S_m

Figure 5: The helm graph H_m and the sunlet graph S_m

Proposition 3.4. The global clique centralities of H_m and S_m are given as follows:

i. $\widehat{\omega}(H_m) = \begin{cases} \frac{6m+4}{(2m+1)^2} & if \quad m = 3, \\ \frac{5m+3}{(2m+1)^2} & if \quad m \geq 4; \end{cases}$

ii $\widehat{\omega}(S_m) = \frac{1}{m}$.

Proof. The global clique centralities above are obtained by applying Definition 2.3 and the proofs are similar to that of Proposition 3.1. □

4 Conclusion

In this paper, we successfully introduced the clique centrality at the vertex level and the global clique centrality at the graph level, where the second parameter can be used to compare two graphs of the same order. We were able to generate some general properties for the global clique centrality of graphs. As planned, we were also able to show the corresponding formulas for the global clique centrality of some known parameterized families of graphs such as path, cycle, complete split graph, wheel graph, fan graph, crown graph, complete graph, complete bipartite graph, helm graph, and sunlet graph. Our final goal in this paper was to obtain the corresponding clique centrality and global clique centrality formulas of the aforementioned graphs, similar to those done in [5], [8] [9], [10], [11], [12], [13], and [14], with the hope that the generated results in this study would be of help when one considers to study more complex graphs.

Some open problems: Given the value $\widehat{\omega}(G)$ for a graph G of order m, determine the following:
a. $\widehat{\omega}(\bar{G})$, where $\bar{G}$ is the complement of G;
b. $\widehat{\omega}(L(G))$, where $L(G)$ is the line graph of G;
c. $\widehat{\omega}(G^2)$, where G^2 is the second power of G.

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