fOriginal Research Article

THE EXTENSIONOFDIAGRAMGROUPOVER SEMIGROUP PRESENTATION

ABSTACT

In this paper, we will discuss the diagram groups from union of two semigroup presentationsnamely $^2\mathcal{S} =< x\,, y\ : x = y\ >$, $^3\mathcal{S} =< a\,, b, c\ : a = b, b = c, c = a\ >$ and their two complex graphs will be presented. The covering space will be determined by selecting normal subgroup from diagram group that previously obtained from $^2\mathcal{S} \cup {}^3\mathcal{S}$. Finally, the number of generator and relations of the diagram group can be computed.

KEYWORDS

Generators, Relations, Diagram groups, Semigroup presentation.

1 INRODUCTION

Graph theoretical and geometrical methods have played an important role in the development of semigroup presentation and diagram groups. This study addresses a new method for studying diagram groups.

For any given semigroup presentation, $\mathcal{S} =< X: R >$, the diagram group $D(\mathcal{S}, U)$, where U is a positive word on X (Guba and Sapir 1997), can be obtained. The associated group with semigroup presentation is called $K(\mathcal{S})$. For a 2-complex graph, there is a fundamental group $\pi_1(K(\mathcal{S}), U)$ with basepoint U. Kilibarda (1994, 1997) showed that the fundamental group is isomorphic to diagram group $D(\mathsf{S}, U)$. Therefore, it is sufficient to consider $\pi_1(K(\mathsf{S}), U)$ instead of $D(\mathsf{S}, U)$. This allows for constructing the fundamental group $\pi_1(K(\mathsf{S}), U)$ from the union of two semigroup presentations.

In fact, Guba and Sapir (1997) have shown that if $\mathsf{S}_1 =< X_1 : R_1 >$, $\mathsf{S}_2 =< X_2 : R_2 >$ and $\mathsf{S} =< X_1 \cup X_2 : R_1 \cup R_2 >$ are semigroup presentations, then for $U_1, U_2 \in X^+, D(\mathsf{S}, U_1 U_2)$ is isomorphic to the direct product of $D(\mathsf{S}, U_1)$ and $D(\mathsf{S}, U_2)$. Also they proved if one consider $\mathsf{S} =< X_1 \cup X_2 : R_1 \cup R_2 \cup \{U_1 = U_2\} >$ where X_1, X_2 are disjoint sets, and the congruence class of U_i modulo S_i does not contain words of the form YU_iZ, where Y, Z are words over X_1, X_2 and YZ are not empty, then $D(\mathsf{S}, U_i)$ is isomorphic to the free product of $D(\mathsf{S}_1, U_i)$ and $D(\mathsf{S}_2, U_i)$. Upon that, it is recommended for future research to consider the semigroup presentation $\mathsf{S} =< X_1 \cup X_2 : R_1 \cup R_2 \cup \{U_1 = U_2\} >$ for the current method developed in this paper.

In [12] and [13] we obtained the connected 2-complex graphs ${}^2K_i$ and ${}^3K_i, i \in N$ that were obtained from ${}^2\mathsf{S} =< x , y : x = y >$, and ${}^3\mathsf{S} =< a , b , c : a = b , b = c , c = a >$ respectively.

In this paper we want determine the semigroup presentation of union of two semigroup presentation by adding a relation.

Let ${}^2\mathsf{S} =< x , y : x = y >$, ${}^3\mathsf{S} =< a , b , c : a = b , b = c , c = a >$ be semigroup presentations. Now we consider the semigroup presentation obtained from union of ${}^2\mathsf{S}$ and ${}^3\mathsf{S}$ by adding a relation $x = a$.

2 DETERMINING THE TWO COMPLEX GRAPHS

In this section all connected two complex graph that are obtained from

$^{5}S = {}^{2}S \cup {}^{3}S = < x, y, a, b, c \; : \; x = y, a = b, b = c, c = a, \; , x = a >$

will be constructed.

1. Let $L(U) = 1$, where U is positive words on ^{5}S. so, the connected two complex graph $^{5}K_1$ is given by Figure 1.

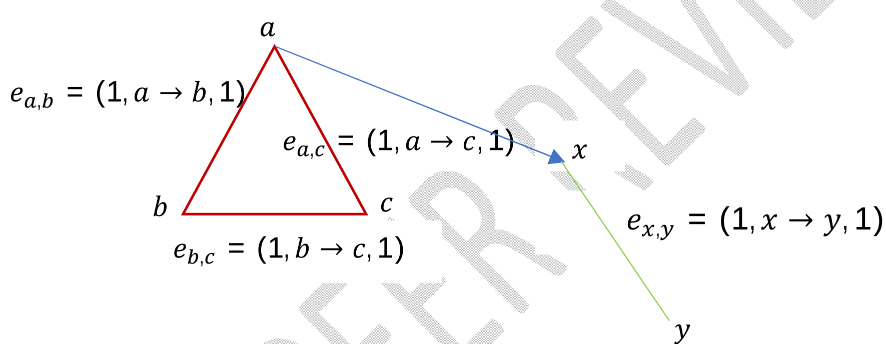

Figure 1The connectedtwo complex graph $^{5}K_1$

Note that when $L(U) = 1$, there will be five vertices and five edges in $^{5}K_1$.

2. Let $L(U) = 2$. In this case there are $5^2 = 25$ possibilities vertices inthe connected two complex graph $^{5}K_2$(see Figure 2).

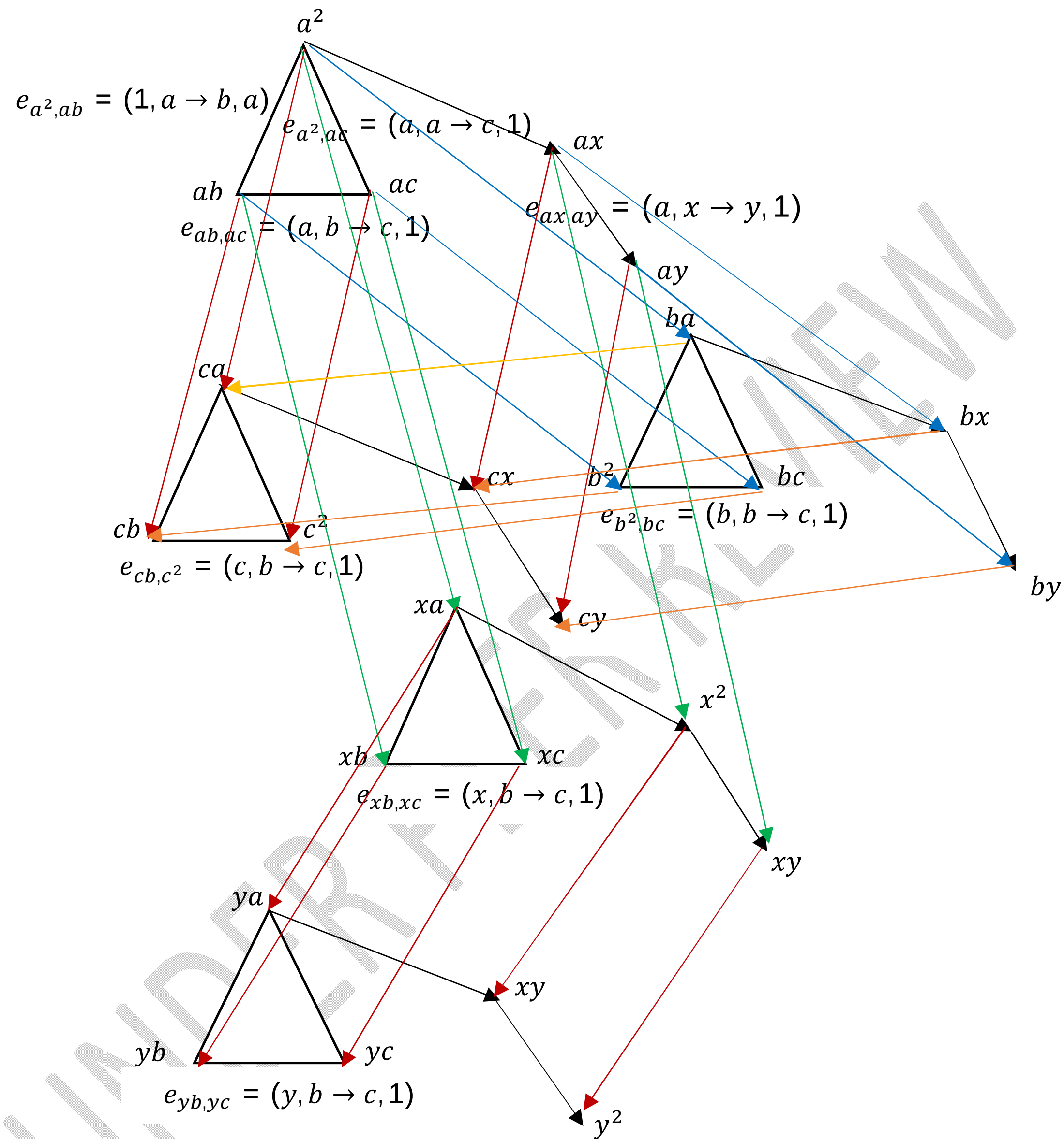

Figure 2 Connected2-complex graph $^5K_2$

COROLLARY 1 Aconnected2—complexgraph^5K_ncontains5^nvertices.

COROLLARY2 Vertices$v_1$and$v_2$areconnectedifandonlyif$L(v_1) = L(v_2)$.

LEMMA3 If$L((W_1)) = L(W_2)$then$\pi_1(^5K_n, W_1) = \pi_1(^5K_n, W_2)$.

LEMMA4 Verticesof^5K_nareallwordsoflengthn.

LEMMA5 (Rotman 1995, 2002): Let $f : K' \to K$be a mapping of 2-complexes graphs. If $\tilde{v}$ is a vertex in K', then there is induced monomorphism

$$f^*: \pi_1(K', v') \to \pi_1\big(K, f(v')\big)$$

defined by $f^*[\alpha'] = [f(\alpha')]$.

LEMMA6 (Rotman 1995, 2002):The mapping$f^*: \pi_1(K', v') \to \pi_1\big(K, f(v')\big)$ is an injective if f is a locally bijective.

LEMMA7(Rotman 1995, 2002)**:** The mapf_N :$^5K_N \to {}^5K$,$f_N(N[\alpha]) = v$, $f_N(N[\alpha], x) = x$is a mapping of connected 2-complex graphs.

LEMMA 8 (Rotman 1995, 2002)**:** The mapf_N :$^5K_N \to {}^5K$, $f_N(N[\alpha]) = v$, $f_N(N[\alpha], x) = x$is locally bijective.

THEOREM 1: Consider the following connected two complex graph $^5K_1$ as shown in Figure 1, such that $G = \pi_1(^5K_1, a)$ contains μ, where $\mu =< e_{a,b}e_{b,c}e_{a,c} >$. If N is the smallest normal subgroup of G containing $< \mu^2 >$, then the covering complex$^5(K_N)_1$for$^5K_1$ is a hexagonal shapeplus one triangle.

PROOF:

From $^5K_1$, $\pi_1(^5K_1)$ can be obtained. Fix a vertex a in $^5K_1$. Now, for any normal subgroup of $\pi_1(^5K_1, a)$, there exists a unique covering space. Start by choosing basic$N[\mu]$ where μ is a path such that $i(\mu) = a$, $\tau(\mu) = v$for every vertex v in $^5K_1$. As a result, these basic $N[1]$, $N[e_{a,b}]$, and $N[e_{a,b}e_{b,c}]$ will be designated, and then all possible edges can be determined, as shown in Table 1.

Table 1 Edges from $N[1]$ in ${}^{2}K_N$

Edges	**Initial**	**Terminal**
$(N[1], e_{a,b})$	$N[1]$	$N[e_{a,b}]$
$(N[1], e_{a,b}e_{b,c}e_{a,c}e_{a,b}e_{b,c})$	$N[1]$	$N[e_{a,b}e_{b,c}e_{a,c}e_{a,b}e_{b,c}]$

Since $f_N[N[1]] = a$ and $star(a) = 3$, then $star(N[1]) = 3$. Consider a vertex a; the vertex in ${}^{5}K_N$ is $N[1]$, and $N[1]$ in ${}^{5}K_N$ maps to a. From $a \to b$ in ${}^{5}K_1$, the vertex in ${}^{5}K_N$ is $N[e_{a,b}]$, and the edge is $(N[1], e_{a,b})$. $N[e_{a,b}]$ in ${}^{5}K_N$ maps to b in ${}^{5}K_1$, as shown in Figure 3.

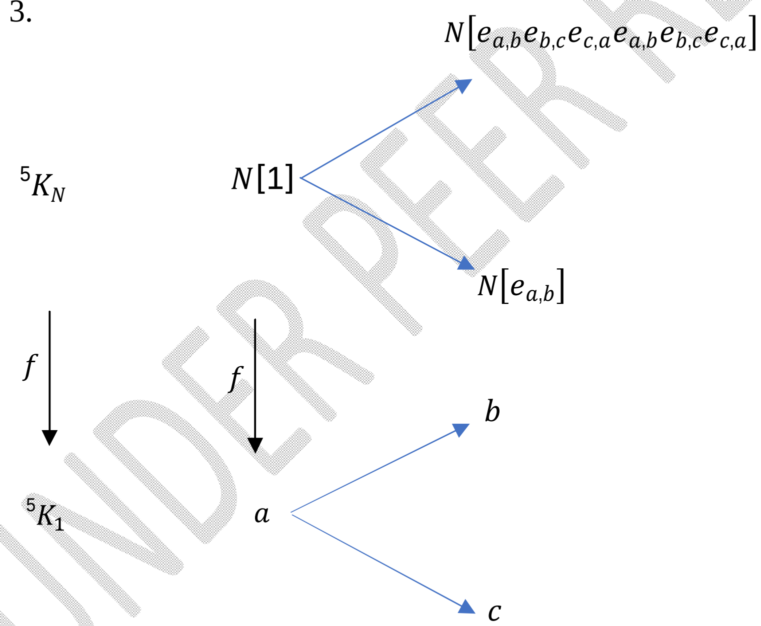

Figure 3 Mapping from ${}^{5}(K_N)_1$ to ${}^{5}K_1$

Similarly, the same applied procedure is used to determine the vertices and the edges.

Table 2 and Table 3 summarize the results of all possible vertices and the edges respectively.

Table 2 Vertices in $^5K_1$ and $^5(K_N)_1$

Vertex in $^5K_1$	**Vertex v in $^5(K_N)_1$**
a	$N[1]$
b	$N[e_{a,b}]$
c	$N[e_{a,b}e_{b,c}]$
a	$N[e_{a,b}e_{b,c}e_{c,a}]$
b	$N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}]$
c	$N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c}]$
x	$N[e_{a,x}]$
y	$N[e_{a,x}e_{x,y}]$

Table 3Edges in $^5K_1$and $^5(K_N)_1$

Edges in $^5K_1$	**Edges in $^5(K_N)_1$**
$e_{a,b}$	$(N[1]\,, e_{a,b})$
$e_{a,b}e_{b,c}$	$(N[e_{a,b}]\,, e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(N[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a})$
$e_{a,b}$	$(N[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})$
$e_{a,b}e_{b,c}$	$(N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(N[1], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,x}$	$(N[1]\,, e_{a,x})$

$e_{a,x}e_{x,y}$	$(N[e_{a,x}], e_{a,x}e_{x,y})$

Now suppose $f_N : {}^5(K_N)_1 \to {}^5K_1$ defined by $f_N(N[1]) = a$, $f_N(N[e_{a,x}]) = x$, $f_N(N[\alpha], e_{a,x}) = e_{a,x}$. This map can be viewed as locally bijective. For this reason, ${}^5(K_N)_1$ is the covering spacefor ${}^5K_1$ and it is of hexagonal shape plus one triangle.Therefore, the covering space ${}^5(K_N)_1$ for ${}^5K_1$ in this case is of hexagonal shapeplus one triangle, as shown in Figure 4.

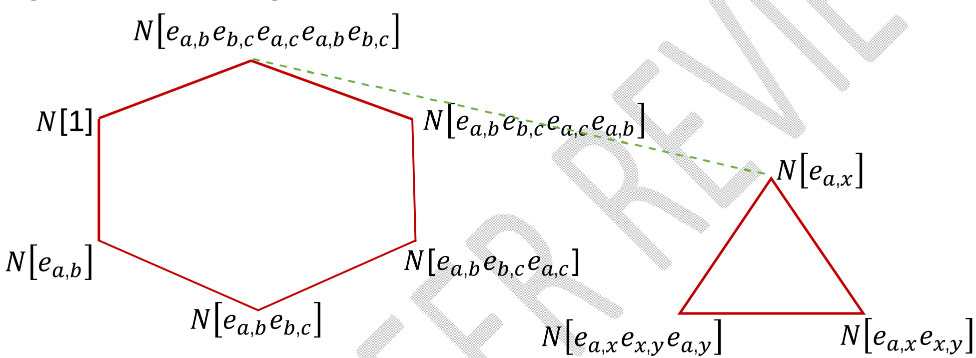

Figure 4 Covering complex ${}^5(K_N)_1$

Since a is a vertex of the connected two complex ${}^5K_1$, and $N[1]$ lies over a, then by LEMMA 6, $f_N^*: \pi_1({}^5(K_N)_1, N[1]) \to \pi_1({}^5K_1, a)$ is injective. Therefore, $f_N^*: \pi_1({}^5(K_N)_1, N[1]) \to Imf_N^* = N$. As a result, $N = \pi_1({}^5(K_N)_1, N[1])$ can be considered as a subgroup of $G = \pi_1({}^5K_1, a)$.

The generators for $\pi_1({}^5(K_N)_1, N[1])$ are computed here using maximal subtree methods. Select a maximal subtree $\mathrm{T}({}^5K_N)$ for ${}^5(K_N)_1$ (see Figure 5).

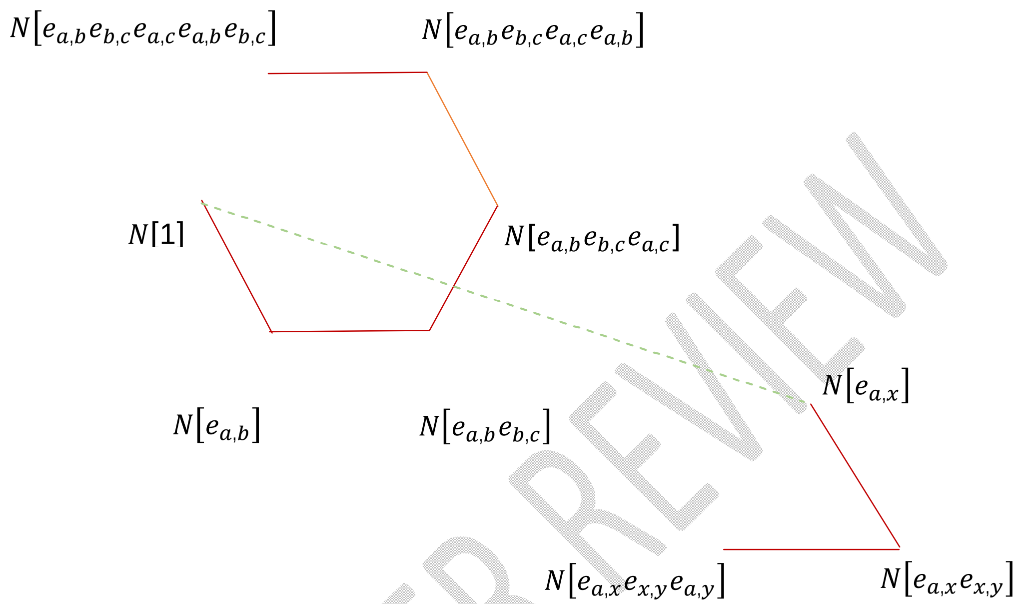

Figure 5 MaximalsubtreeT$(^5(K_N)_1)$

The generators for the fundamental group $\pi_1(^5(K_N)_1, N[1])$will be:

$g_1(^5K_N) =$

$(N[1]\,, e_{a,b})(N[e_{a,b}]\,, e_{a,b}e_{b,c})(N[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a})(N[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})(N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}$

$$g_2(\pi_1(^5K_N)) = (N[1], e_{a,x})(N[e_{a,x}], e_{a,x}e_{x,y})(N[e_{a,x}e_{x,y}], e_{a,x}e_{x,y}e_{a,y})$$

$$(N[e_{a,x}], e_{a,x}e_{x,y})^{-1}(N[1], e_{a,x})^{-1}.$$

THEOREM 2: Let the following semigroup presentation

$$^5S =< x\,,y\,,a\,,b\,,c\ :\ x = y, a = b\,, b = c\,, c = a,\ \ , x = a >$$

If the number of all vertices of two complex graph ^5K_N is 5^n, then the number of all vertices of the covering space $^5(K_N)_n$ is $(5)^n + 3$.

PROOF: By induction, for $k = 1$ the number of all vertices in $^5(K_N)_1$ is 5. Thus for $k = 1$ is true (see Figure 1). Now assume $v_k = (5)^k + 3$ be the number of all vertices in $^5(K_N)_k$.

We will prove the number of all vertices of the covering space $^5(K_N)_{k+1}$ is $(5)^{k+1} + 3$. By the definition of $^5K_{k+1}$ is five copies of ^5K_k and assumption, then the number of all vertices of the covering space $^5(K_N)_{k+1}$ is $v_{k+1} = 5 \cdot (5)^k + 3 = (5)^{k+1} + 3$.

THEOREM 3: Consider the semigroup presentation

$$^5\mathrm{S} = < x , y , a , b , c \ : \ x = y, a = b , b = c , c = a, \ , x = a >.$$

The number of all edges in the covering space $^5(K_N)_n$ is $e_n = n5^n + 3$.

PROOF: By induction, for $k = 1$ the number of all vertices in $^5(K_N)_1$ is $e_1 = 1(5) + 3 = 8$. Now let $e_k = k5^k + 3$ be the number of all edges the covering space $^5(K_N)_k$. We will prove that the number of all edges in $^5(K_N)_{k+1}$ is $e_{k+1} = (k + 1)(5)^{k+1} + 3$. By using last theorem

$$e_{k+1} = 5e_k + 5^{k+1} + 3 = 5k5^k + 5^{k+1} + 3$$
$$= k \cdot 5^{k+1} + 5^{k+1} + 3$$
$$= (k + 1)5^{k+1} + 3.$$

3 CONCLUSION

The paper provided, a new technique which has been explored to study diagram groups that was previously obtained from a union of two semigroup presentations

$$^5\mathrm{S} = {}^2\mathrm{S} \cup {}^3\mathrm{S} = < x , y , a , b , c \ : \ x = y, a = b , b = c , c = a, \ , x = a >$$

By adding a relation.

The paper discussed how to determine the covering complex $^5(K_N)_1$ for the connected two complex graph $^5K_1$ by selection normal subgroup from the diagram group.Also, this paper discussed how the generators and the relations for the fundamental group $\pi_1(^5(K_N)_1, N[1])$ were calculated by using maximal tree methods. Finally,the number of all vertices and edges in the covering space $^5(K_N)_1$ were computed.

REFERENCES

[1] Ahmad, A.G. 1995. The application of pictures of decision problems and relative presentations. PhD thesis. University of Glasgow.

[2] Ahmad, A.G. 2003. Triviality problems for diagram groups. *Jour. Inst. Math. Com. Sci.* **16**(2):105-107.

[3] Ahmad, A.G., & Al-Odhari, A.M., 2004. The graph of diagram groups constructed from natural numbers semigroup with a repeating generator. *Jour. Inst. Math. Com. Sci.* **17**(1):67-69.

[4] Al-Odhari, A.M. & Ahmad, A.G. 2004. The graph of diagram groups with length one constructed from natural semigroup. *Jour. Inst. Math. Com. Sci.* **16**(2):251-253.

[5] Cohen, D. 1989. *Combinatorial Group Theory with Topological Approach*. Cambridge University Press

[6] Guba, V. & Sapir, M. 1997. *Diagram Groups*. Memoirs of the American Mathematical Society.

[7] Guba, V. & Sapir, M. 1999. On subgroups of R. Thompson's group F and other diagram groups. Mathem. Sb. **190**(8):3-16(Russian). English transl. in: *Sbornik: Mathematics.***190**(8): 1077-1130.

[8] Guba, V. 2002. Some properties of periodic words. *Mathematical notes, Traslated from mathematicheskieZameki.***72**(3): 330-337.

[9] Guba, V. & Sapir, M. 2002. Rigidity properties of diagram groups. *Inter. Jour. Algeb. Comp.***2**(1) and (2): 9-17.

[10] Guba, V. & Sapir, M. 2006a. Diagram groups and directed 2-complexes homotopy and homology. *Journal of Pure and Applied Algebra* 205 (1):1-47.

[11] Guba, V. & Sapir, M. 2006b. Diagram groups and totally orderable. *Journal of Pure and Applied Algebra* 205 (1): 48-73.

[12] Kalthom, M.& Ahmad, A.G. 2011. Covering graph for diagram group from semigroup presentation $^{2}S = < a, b : a = b >$. *International Journal of Countempoary Mathematical Sciences,* 21(6): 1019-1028.

[13] Kalthom, M.& Ahmad, A.G. 2011. Covering Space for diagram group from semigroup presentation $^{3}S = < a, b, c : a = b, b = c, c = a >$,. *International Journal of Countempoary Mathematical Sciences*, 25(28): 1341-1350.

[14] Kilibarda, V. 1997. On the algebra of semigroup diagrams. *Intr. Jour. Algeb. Comp.* **7**:313-338.

[15] Kilibarda, V.1994. On the algebra of semigroup diagrams. PhD thesis, Univ. of Nebraska.

[16] Rotman, J. 1995. *An Introduction to Theory of Groups,* 2nd edition. New York: Springer-Verlag.

[17] Rotman, J. 2002. *Advanced Modern Algebra*. New Jersey: Pearson Education, In.