__Original Research Article__

THE EXTENSIONOFDIAGRAMGROUPOVER SEMIGROUP PRESENTATION

ABSTACT

In this paper, we will discuss the diagram groups from union of two semigroup presentationsnamely $^{2}\mathrm{S} =< x , y : x = y >$, $^{3}\mathrm{S} =< a , b, c : a = b, b = c, c = a >$ and their two complex graphs will be presented. The covering space will be determined by selecting normal subgroup from diagram group that previously obtained from $^{2}\mathrm{S} \cup {}^{3}\mathrm{S}$. Finally, the number of generator and relations of the diagram group can be computed.

KEYWORDS

Generators, Relations, Diagram groups, Semigroup presentation.

INRODUCTION

Graph theoretical and geometrical methods have played an important role in the development of semigroup presentation and diagram groups. This study addresses a new method for studying diagram groups.

For any given semigroup presentation, $\mathrm{S} =< X: R >$, the diagram group $D(\mathrm{S}, U)$, where U is a positive word on X (Guba and Sapir 1997), can be obtained. The associated group with semigroup presentation is called $K(\mathrm{S})$. For a 2-complex graph, there is a fundamental group $\pi_1(K(\mathrm{S}), U)$ with basepoint U. Kilibarda (1994, 1997) showed that the fundamental group is isomorphic to diagram group $D(\mathsf{S}, U)$. Therefore, it is sufficient to consider $\pi_1(K(\mathsf{S}), U)$ instead of $D(\mathsf{S}, U)$. This allows for constructing the fundamental group $\pi_1(K(\mathsf{S}), U)$ from the union of two semigroup presentations.

In fact, Guba and Sapir (1997) have shown that if $\mathsf{S}_1 =< X_1 : R_1 >$, $\mathsf{S}_2 =< X_2 : R_2 >$ and $\mathsf{S} =< X_1 \cup X_2 : R_1 \cup R_2 >$ are semigroup presentations, then for $U_1, U_2 \in X^+, D(\mathsf{S}, U_1 U_2)$ is isomorphic to the direct product of $D(\mathsf{S}, U_1)$ and $D(\mathsf{S}, U_2)$. Also they proved if one consider $\mathsf{S} =< X_1 \cup X_2 : R_1 \cup R_2 \cup \{U_1 = U_2\} >$ where X_1, X_2 are disjoint sets, and the congruence class of U_i modulo S_i does not contain words of the form YU_iZ, where Y, Z are words over X_1, X_2 and YZ are not empty, then $D(\mathsf{S}, U_i)$ is isomorphic to the free product of $D(\mathsf{S}_1, U_i)$ and $D(\mathsf{S}_2, U_i)$. Upon that, it is recommended for future research to consider the semigroup presentation $\mathsf{S} =< X_1 \cup X_2 : R_1 \cup R_2 \cup \{U_1 = U_2\} >$ for the current method developed in this paper.

In [12] and [13] we obtained the connected 2-complex graphs ${}^2K_i$ and ${}^3K_i, i \in N$that were obtained from${}^2\mathsf{S} =< x, y : x = y >$, and ${}^3\mathsf{S} =< a, b, c : a = b, b = c, c = a >$ respectively.

In this paper we want determine the semigroup presentation of union of two semigroup presentation by adding a relation.

Let${}^2\mathsf{S} =< x, y : x = y >$, ${}^3\mathsf{S} =< a, b, c : a = b, b = c, c = a >$ be semigroup presentations. Now we consider the semigroup presentation obtained from union of ${}^2\mathsf{S}$ and ${}^3\mathsf{S}$ by adding a relation $x = a$.

1. DETERMINING THE TWO COMPLEX GRAPHS

In this section all connected two complex graph that are obtained from

$^5S = {}^2S \cup {}^3S = < x, y, a, b, c : x = y, a = b, b = c, c = a, , x = a >$

will be constructed.

1. Let $L(U) = 1$, where U is positive words on ^5S. so, the connected two complex graph $^5K_1$ is given by Figure 1.

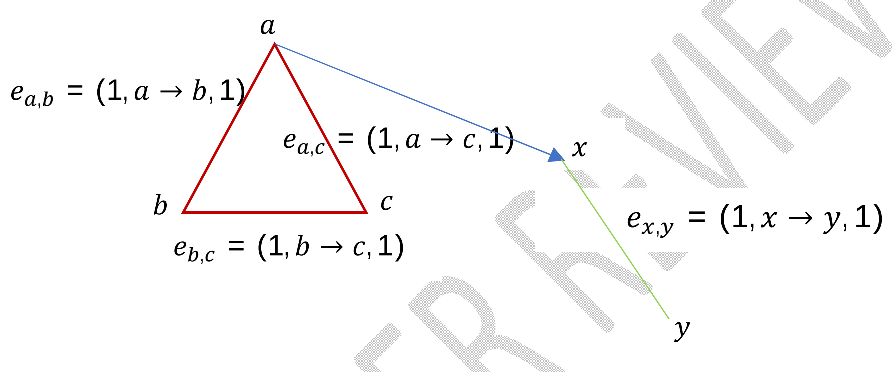

Figure 1The connectedtwo complex graph $^5K_1$

Note that when $L(U) = 1$, there will be five vertices and five edges in $^5K_1$.

2. Let $L(U) = 2$. In this case there are $5^2 = 25$ possibilities vertices inthe connected two complex graph $^5K_2$(see Figure 2).

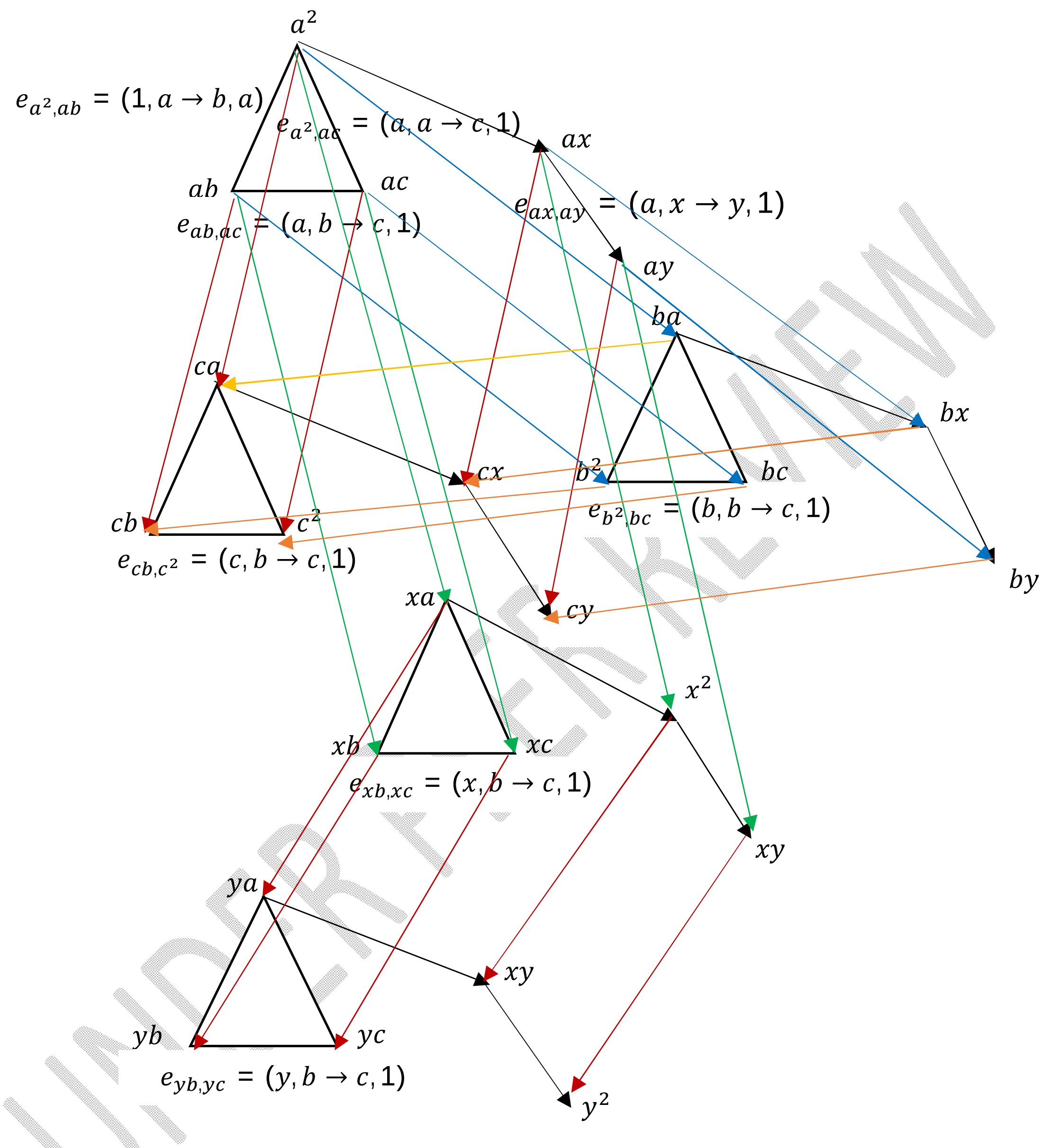

Figure 2 Connected2-complex graph $^5K_2$

COROLLARY 1 Aconnected2—complexgraph$^5K_n(^5S)$contains5^nvertices.

COROLLARY2 Vertices$v_1$and$v_2$areconnectedifandonlyif$L(v_1) = L(v_2)$.

LEMMA3 If$L((W_1)) = L(W_2))$then$\pi_1({}^5K_n({}^5S),W_1) = \pi_1({}^5K_n({}^5S),W_2)$.

LEMMA4 Verticesof${}^5K_n({}^5S)$areallwordsoflengthn.

LEMMA5 (Rotman 1995, 2002)**:** The map$f_N :{}^5K_N \rightarrow {}^5K, f_N(N[\alpha]) = v , f_N(N[\alpha], x) = x$is a mapping of connected 2-complex graphs.

LEMMA 7 (Rotman 1995, 2002)**:** The map$f_N :{}^5K_N \rightarrow {}^5K,\ f_N(N[\alpha]) = v , f_N(N[\alpha], x) = x$is locally bijective.

THEOREM 1: Consider the following connected two complex graph ${}^5K_1$ as shown in Figure 1, such that $G = \pi_1({}^5K_1, a)$ contains μ, where $\mu = < e_{a,b}e_{b,c}e_{a,c} >$. If N is the smallest normal subgroup of G containing $< \mu^2 >$, then the covering complex${}^5(K_N)_1$for${}^5K_1$ is a hexagonal shapeplus one triangle.

PROOF:

From ${}^5K_1$, $\pi_1({}^5K_1)$ can be obtained. Fix a vertex a in ${}^5K_1$. Now, for any normal subgroup of $\pi_1({}^5K_1, a)$, there exists a unique covering space. Start by choosing basic$N[\mu]$ where μ is a path such that $i(\mu) = a , \tau(\mu) = v$for every vertex v in ${}^5K_1$. As a result, these basic $N[1]$, $N[e_{a,b}]$, and $N[e_{a,b}e_{b,c}]$ will be designated, and then all possible edges can be determined, as shown in Table 1.

Table 1 Edges from $N[1]$ in ${}^{2}K_N$

Edges	**Initial**	**Terminal**
$(N[1], e_{a,b})$	$N[1]$	$N[e_{a,b}]$
$(N[1], e_{a,b}e_{b,c}e_{a,c}e_{a,b}e_{b,c})$	$N[1]$	$N[e_{a,b}e_{b,c}e_{a,c}e_{a,b}e_{b,c}]$

Since $f_N[N[1]] = a$ and $star(a) = 3$, then $star(N[1]) = 3$. Consider a vertex a; the vertex in ${}^{5}K_N$ is $N[1]$, and $N[1]$ in ${}^{5}K_N$ maps to a. From $a \to b$ in ${}^{5}K_1$, the vertex in ${}^{5}K_N$ is $N[e_{a,b}]$, and the edge is $(N[1], e_{a,b})$. $N[e_{a,b}]$ in ${}^{5}K_N$ maps to b in ${}^{5}K_1$, as shown in Figure 3.

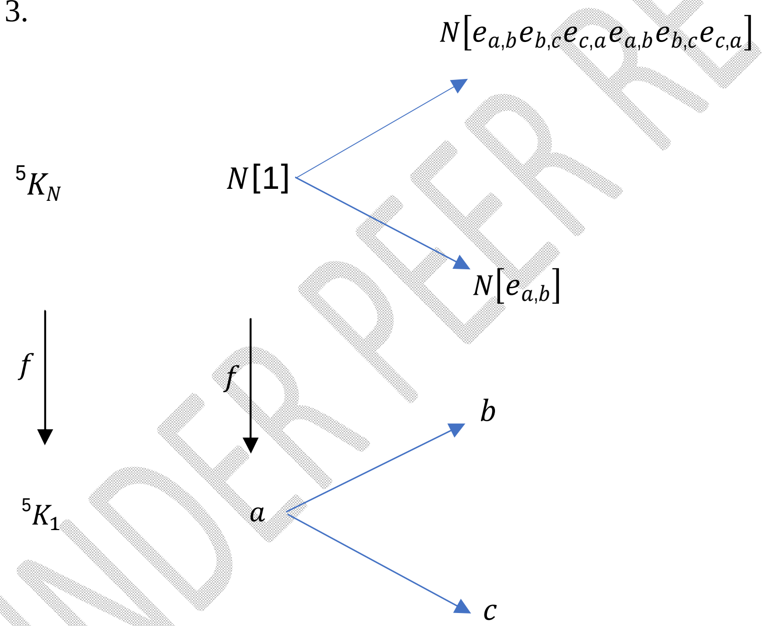

Figure 3 Mapping from ${}^{5}(K_N)_1$ to ${}^{5}K_1$

Similarly, the same applied procedure is used to determine the vertices and the edges.

Table 1 and Table 2 summarize the results of all possible vertices and the edges respectively.

Vertex in $^5K_1$	**Vertex v in $^5(K_N)_1$**
a	$N[1]$
b	$N[e_{a,b}]$
c	$N[e_{a,b}e_{b,c}]$
a	$N[e_{a,b}e_{b,c}e_{c,a}]$
b	$N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}]$
c	$N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c}]$
x	$N[e_{a,x}]$
y	$N[e_{a,x}e_{x,y}]$

Table 2 Vertices in ${}^5K_1$and ${}^5(K_N)_1$

Edges in ${}^5K_1$	**Edges in ${}^5(K_N)_1$**
$e_{a,b}$	$(N[1]\,, e_{a,b})$
$e_{a,b}e_{b,c}$	$(N[e_{a,b}]\,, e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(N[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a})$
$e_{a,b}$	$(N[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})$
$e_{a,b}e_{b,c}$	$(N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(N[1], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,x}$	$(N[1]\,, e_{a,x})$
$e_{a,x}e_{x,y}$	$(N[e_{a,x}]\,, e_{a,x}e_{x,y})$

Now suppose $f_N : {}^5(K_N)_1 \to {}^5K_1$defined by $f_N(N[1]) = a\,, f_N(N[e_{a,x}]) = x,$ $f_N(N[\alpha], e_{a,x}) = e_{a,x}$. This map can be viewed as locally bijective. For this reason, ${}^5(K_N)_1$is the covering spacefor${}^5K_1$ and it is of hexagonal shape plus one triangle.Therefore, the covering space${}^5(K_N)_1$for${}^5K_1$in this case is of hexagonal shapeplus one triangle, as shown in Figure 4.

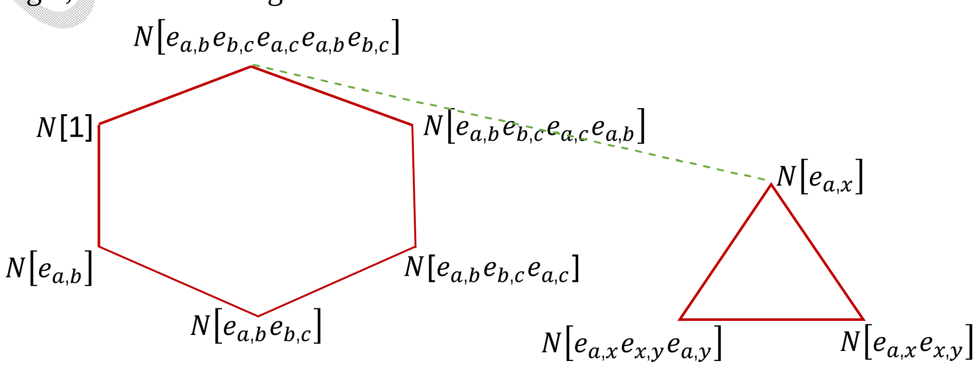

Figure 4 Covering complex${}^5(K_N)_1$

Since a is a vertex of the connected two complex ${}^5K_1$, and $N[1]$ lies over a, then by Theorem 2.8.12, $f_N^*: \pi_1({}^5(K_N)_1, N[1]) \rightarrow \pi_1({}^5K_1, a)$ is injective. Therefore, $f_N^*: \pi_1({}^5(K_N)_1, N[1]) \rightarrow Imf_N^* = N$. As a result, $N = \pi_1({}^5(K_N)_1, N[1])$ can be considered as a subgroup of $G = \pi_1({}^5K_1, a)$.

The generators for $\pi_1({}^5(K_N)_1, N[1])$ are computed here using maximal subtree methods. Select a maximal subtree$\mathsf{T}({}^5K_N)$for ${}^5(K_N)_1$(see Figure 5).

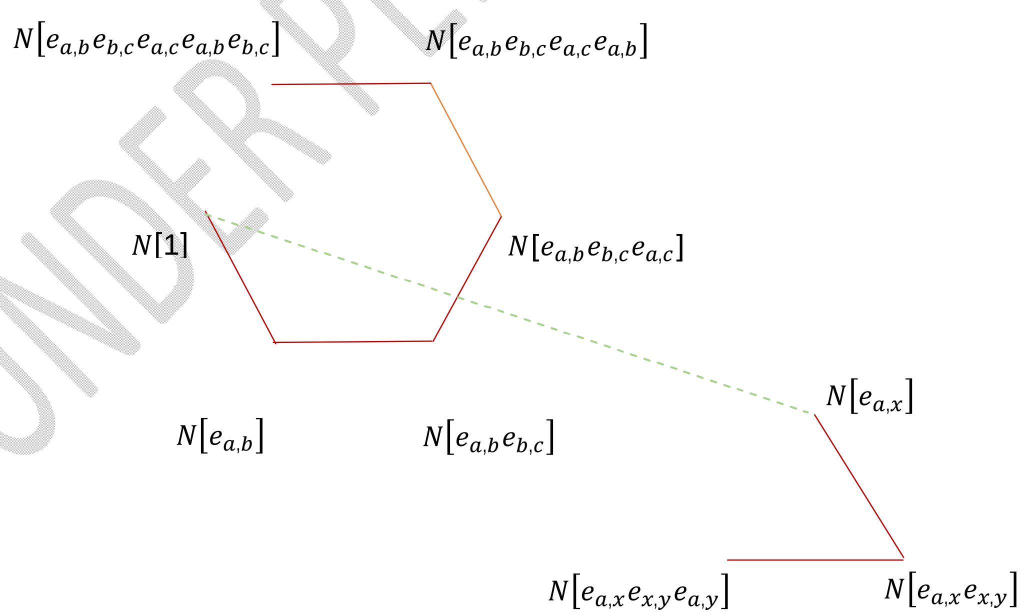

Figure 5 Maximalsubtree$\mathsf{T}({}^5(K_N)_1)$

The generators for the fundamental group $\pi_1({}^5(K_N)_1, N[1])$ will be:

$g_1({}^5K_N) =$

$(N[1]\,, e_{a,b})(N[e_{a,b}]\,, e_{a,b}e_{b,c})(N[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a})(N[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})(N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}$

$g_2(\pi_1({}^5K_N)) = (N[1], e_{a,x})(N[e_{a,x}], e_{a,x}e_{x,y})(N[e_{a,x}e_{x,y}], e_{a,x}e_{x,y}e_{a,y})$

$$(N[e_{a,x}], e_{a,x}e_{x,y})^{-1}(N[1], e_{a,x})^{-1}.$$

THEOREM 2: Let the following semigroup presentation

${}^5S =< x\,, y\,, a\,, b\,, c\ :\ x = y, a = b\,, b = c\,, c = a,\ \ , x = a >$

If the number of all vertices of two complex graph ${}^5K_N$ is 5^n, then the number of all vertices of the covering space ${}^5(K_N)_n$ is $(5)^n + 3$.

PROOF: By induction, for $k = 1$ the number of all vertices in ${}^5(K_N)_1$ is 5. Thus for $k = 1$ is true (see Figure 1). Now assume $v_k = (5)^k + 3$ be the number of all vertices in ${}^5(K_N)_k$.

We will prove the number of all vertices of the covering space ${}^5(K_N)_{k+1}$ is $(5)^{k+1} + 3$. By the definition of ${}^5K_{k+1}$ is five copies of ${}^5K_k$ and assumption, then the number of all vertices of the covering space ${}^5(K_N)_{k+1}$ is $v_{k+1} = 5 \cdot (5)^k + 3 = (5)^{k+1} + 3$.

THEOREM 3: Consider the semigroup presentation ${}^5S =< x\,, y\,, a\,, b\,, c\ :\ x = y, a = b\,, b = c\,, c = a,\ \ , x = a >$. The number of all edges in the covering space ${}^5(K_N)_n$ is $e_n = n5^n + 3$.

PROOF: By induction, for $k = 1$ the number of all vertices in ${}^5(K_N)_1$ is $e_1 = 1(5) + 3$=8. Now let $e_k = k5^k + 3$ be the number of all edges the covering space ${}^5(K_N)_k$. We will prove that the number of all edges in ${}^5(K_N)_{k+1}$ is $e_{k+1} = (k + 1)(5)^{k+1} + 3$. By using last theorem

$$e_{k+1} = 5e_k + 5^{k+1} + 3 = 5k5^k + 5^{k+1} + 3$$

$= k \cdot 5^{k+1} + 5^{k+1} + 3$

$= (k + 1)5^{k+1} + 3.$

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