

**THE EXTENSION OF DIAGRAM GROUP OVER SEMIGROUP
PRESENTATION**

ABSTRACT

In this paper, we will discuss the diagram groups from union of two semigroup presentations namely ${}^2S = \langle x, y : x = y \rangle$, ${}^3S = \langle a, b, c : a = b, b = c, c = a \rangle$ and their two complex graphs will be presented. The covering space will be determined by selecting normal subgroup from diagram group that previously obtained from ${}^2S \cup {}^3S$. finally, the number of generator and relations of the diagram group can be computed.

KEYWORDS

Generators, Relations, Diagram groups, Semigroup presentation.

INTRODUCTION

Graph theoretical and geometrical methods have played an important role in the development of semigroup presentation and diagram groups. This study addresses a new method for studying diagram groups.

For any given semigroup presentation, $S = \langle X : R \rangle$, the diagram group $D(S, U)$, where U is a positive word on X (Guba and Sapir 1997), can be obtained. The associated group with semigroup presentation is called $K(S)$. For a 2-complex graph, there is a fundamental group $\pi_1(K(S), U)$ with basepoint U . Kilibarda (1994, 1997) showed that

the fundamental group is isomorphic to diagram group $D(S, U)$. Therefore, it is sufficient to consider $\pi_1(K(S), U)$ instead of $D(S, U)$. This allows for constructing the fundamental group $\pi_1(K(S), U)$ from the union of two semigroup presentations.

In fact, Guba and Sapir (1997) have shown that if $S_1 = \langle X_1 : R_1 \rangle$, $S_2 = \langle X_2 : R_2 \rangle$ and $S = \langle X_1 \cup X_2 : R_1 \cup R_2 \rangle$ are semigroup presentations, then for $U_1, U_2 \in X^+$, $D(S, U_1 U_2)$ is isomorphic to the direct product of $D(S, U_1)$ and $D(S, U_2)$. Also they proved if one consider $S = \langle X_1 \cup X_2 : R_1 \cup R_2 \cup \{U_1 = U_2\} \rangle$ where X_1, X_2 are disjoint sets, and the congruence class of U_i modulo S_i does not contain words of the form YU_iZ , where Y, Z are words over X_1, X_2 and YZ are not empty, then $D(S, U_i)$ is isomorphic to the free product of $D(S_1, U_i)$ and $D(S_2, U_i)$. Upon that, it is recommended for future research to consider the semigroup presentation $S = \langle X_1 \cup X_2 : R_1 \cup R_2 \cup \{U_1 = U_2\} \rangle$ for the current method developed in this paper.

In [12] and [13] we obtained the connected 2-complex graphs 2K_i and ${}^3K_i, i \in \mathbb{N}$ that were obtained from ${}^2S = \langle x, y : x = y \rangle$, and ${}^3S = \langle a, b, c : a = b, b = c, c = a \rangle$ respectively.

In this paper we want determine the semigroup presentation of union of two semigroup presentation by adding a relation.

Let ${}^2S = \langle x, y : x = y \rangle$, ${}^3S = \langle a, b, c : a = b, b = c, c = a \rangle$ be semigroup presentations. Now we consider the semigroup presentation obtained from union of 2S and 3S by adding a relation $x = a$.

1. DETERMINING THE TWO COMPLEX GRAPH

In this section all connected two complex graph that are obtained from

$${}^5S = {}^2S \cup {}^3S = \langle x, y, a, b, c : x = y, a = b, b = c, c = a, x = a \rangle$$

will be constructed.

1. Let $L(U) = 1$, where U is positive words on 5S . so, the connected two complex graph 5K_1 is given by Figure 1.

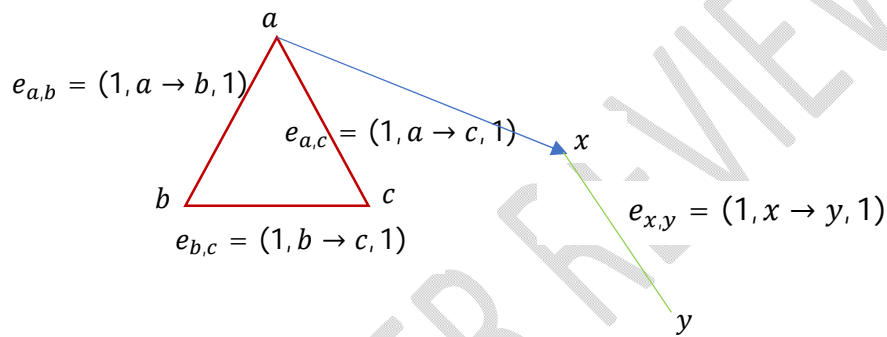


Figure 1 The connected two complex graph 5K_1

Note that when $L(U) = 1$, there will be five vertices and five edges in 5K_1 .

2. Let $L(U) = 2$. In this case there are $5^2 = 25$ possibilities vertices in the connected two complex graph 5K_2 (see Figure 2).

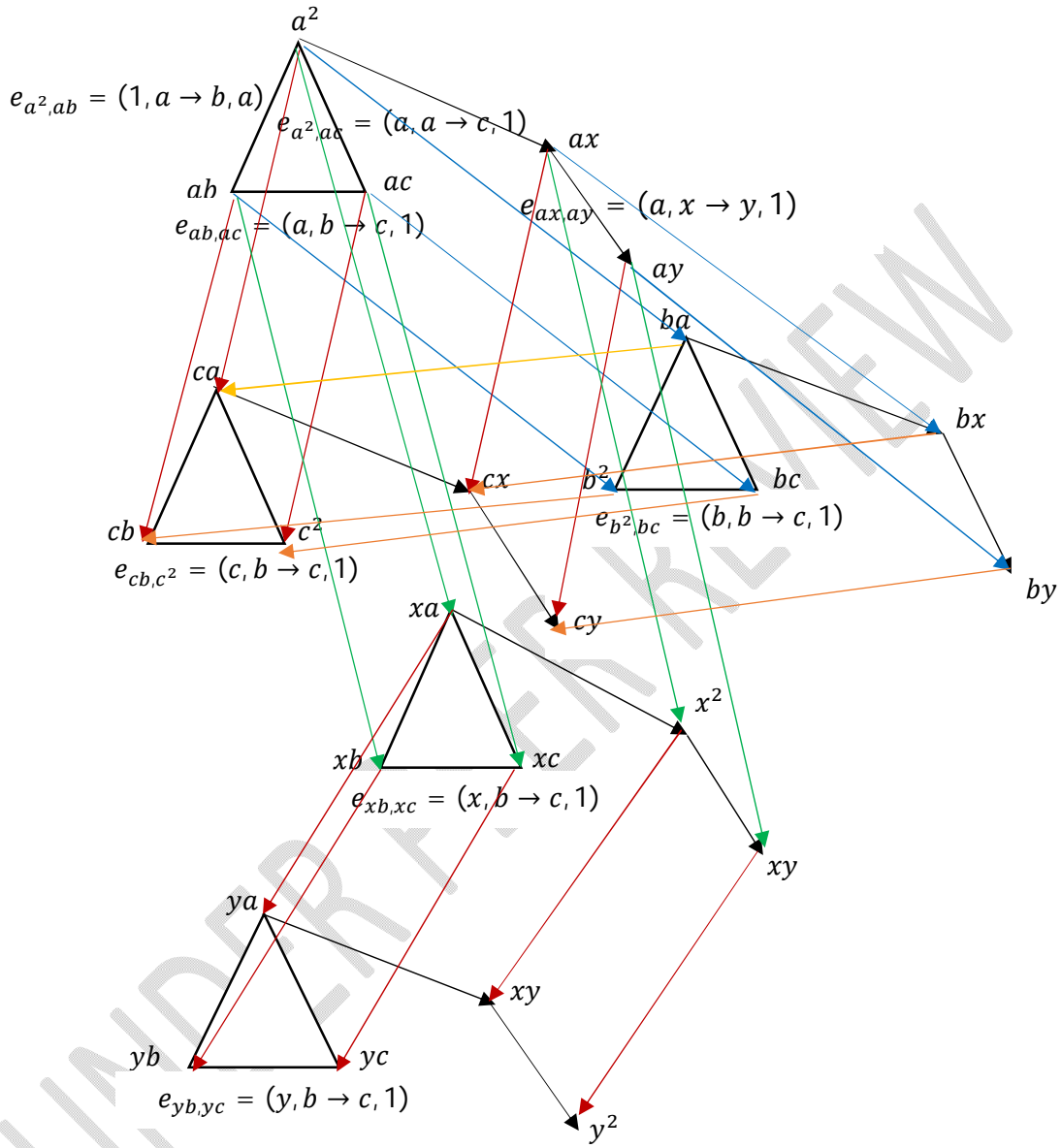


Figure 2 Connected 2-complex graph 5K_2

COROLLARY 1 A connected 2 —complex graph ${}^5K_n({}^5S)$ contains 5^n vertices.

COROLLARY 2 Vertices v_1 and v_2 are connected if and only if $L(v_1) = L(v_2)$.

LEMMA 3 If $L((W_1)) = L(W_2)$ then $\pi_1({}^5K_n({}^5S), W_1) = \pi_1({}^5K_n({}^5S), W_2)$.

LEMMA 4 Vertices of ${}^5K_n({}^5S)$ are all words of length n .

LEMMA 5 (Rotman 1995, 2002): The map $f_N : {}^5K_N \rightarrow {}^5K$, $f_N(N[\alpha]) = v, f_N(N[\alpha], x) = x$ is a mapping of connected 2-complex graphs.

LEMMA 7 (Rotman 1995, 2002): The map $f_N : {}^5K_N \rightarrow {}^5K$, $f_N(N[\alpha]) = v, f_N(N[\alpha], x) = x$ is locally bijective.

THEOREM 1: Consider the following connected two complex graph 5K_1 as shown in Figure 1, such that $G = \pi_1({}^5K_1, a)$ contains μ , where $\mu = \langle e_{a,b}e_{b,c}e_{a,c} \rangle$. If N is the smallest normal subgroup of G containing $\langle \mu^2 \rangle$, then the covering complex ${}^5(K_N)_1$ for 5K_1 is a hexagonal shape plus one triangle.

PROOF:

From 5K_1 , $\pi_1({}^5K_1)$ can be obtained. Fix a vertex a in 5K_1 . Now, for any normal subgroup of $\pi_1({}^5K_1, a)$, there exists a unique covering space. Start by choosing basic $N[\mu]$ where μ is a path such that $i(\mu) = a, \tau(\mu) = v$ for every vertex v in 5K_1 . As a result, these basic $N[1], N[e_{a,b}]$, and $N[e_{a,b}e_{b,c}]$ will be designated, and then all possible edges can be determined, as shown in Table 1.

Table 1 Edges from $N[1]$ in 2K_N

Edges	Initial	Terminal
$(N[1], e_{a,b})$	$N[1]$	$N[e_{a,b}]$
$(N[1], e_{a,b}e_{b,c}e_{a,c}e_{a,b}e_{b,c})$	$N[1]$	$N[e_{a,b}e_{b,c}e_{a,c}e_{a,b}e_{b,c}]$

Since $f_N[N[1]] = a$ and $star(a) = 3$, then $star(N[1]) = 3$. Consider a vertex a ; the vertex in 5K_N is $N[1]$, and $N[1]$ in 5K_N maps to a . From $a \rightarrow b$ in 5K_1 , the vertex in 5K_N is $N[e_{a,b}]$, and the edge is $(N[1], e_{a,b})$. $N[e_{a,b}]$ in 5K_N maps to b in 5K_1 , as shown in Figure 3.

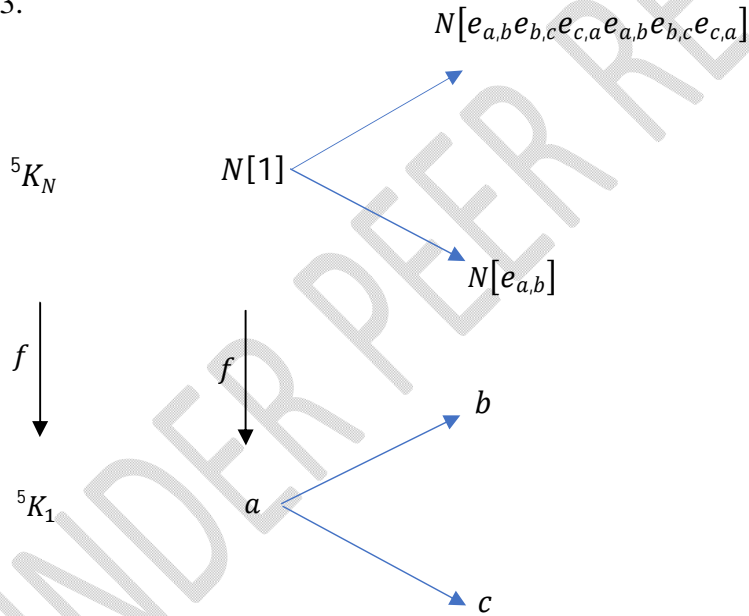


Figure 3 Mapping from ${}^5(K_N)_1$ to 5K_1

Similarly, the same applied procedure is used to determine the vertices and the edges.

Table 1 and Table 2 summarize the results of all possible vertices and the edges respectively.

Vertex in 5K_1	Vertex v in ${}^5(K_N)_1$
a	$N[1]$
b	$N[e_{a,b}]$
c	$N[e_{a,b}e_{b,c}]$
a	$N[e_{a,b}e_{b,c}e_{c,a}]$
b	$N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}]$
c	$N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c}]$
x	$N[e_{a,x}]$
y	$N[e_{a,x}e_{x,y}]$

Table 2 Vertices in 5K_1 and ${}^5(K_N)_1$

Edges in 5K_1	Edges in ${}^5(K_N)_1$
$e_{a,b}$	$(N[1], e_{a,b})$
$e_{a,b}e_{b,c}$	$(N[e_{a,b}], e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(N[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a})$
$e_{a,b}$	$(N[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})$
$e_{a,b}e_{b,c}$	$(N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(N[1], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,x}$	$(N[1], e_{a,x})$
$e_{a,x}e_{x,y}$	$(N[e_{a,x}], e_{a,x}e_{x,y})$

Now suppose $f_N : {}^5(K_N)_1 \rightarrow {}^5K_1$ defined by $f_N(N[1]) = a$, $f_N(N[e_{a,x}]) = x$, $f_N(N[\alpha], e_{a,x}) = e_{a,x}$. This map can be viewed as locally bijective. For this reason, ${}^5(K_N)_1$ is the covering space for 5K_1 and it is of hexagonal shape plus one triangle. Therefore, the covering space ${}^5(K_N)_1$ for 5K_1 in this case is of hexagonal shape plus one triangle, as shown in Figure 4.

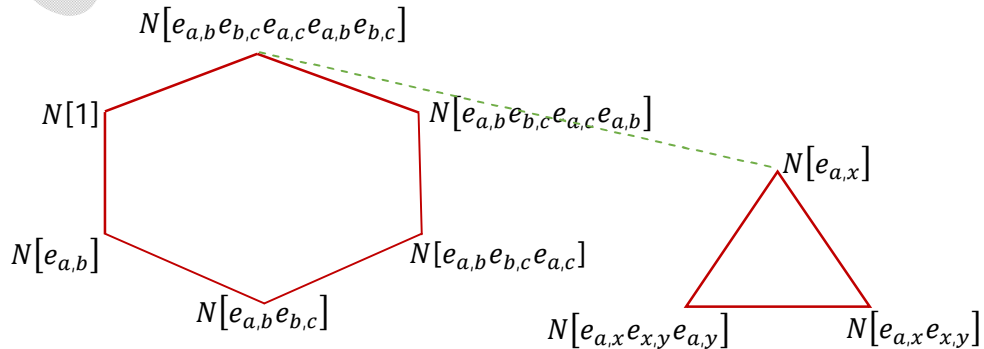


Figure 4 Covering complex ${}^5(K_N)_1$

Since a is a vertex of the connected two complex 5K_1 , and $N[1]$ lies over a , then by Theorem 2.8.12, $f_N^*: \pi_1({}^5(K_N)_1, N[1]) \rightarrow \pi_1({}^5K_1, a)$ is injective. Therefore, $f_N^*: \pi_1({}^5(K_N)_1, N[1]) \rightarrow \text{Im} f_N^* = N$. As a result, $N = \pi_1({}^5(K_N)_1, N[1])$ can be considered as a subgroup of $G = \pi_1({}^5K_1, a)$.

The generators for $\pi_1({}^5(K_N)_1, N[1])$ are computed here using maximal subtree methods. Select a maximal subtree $\Gamma({}^5K_N)$ for ${}^5(K_N)_1$ (see Figure 5).

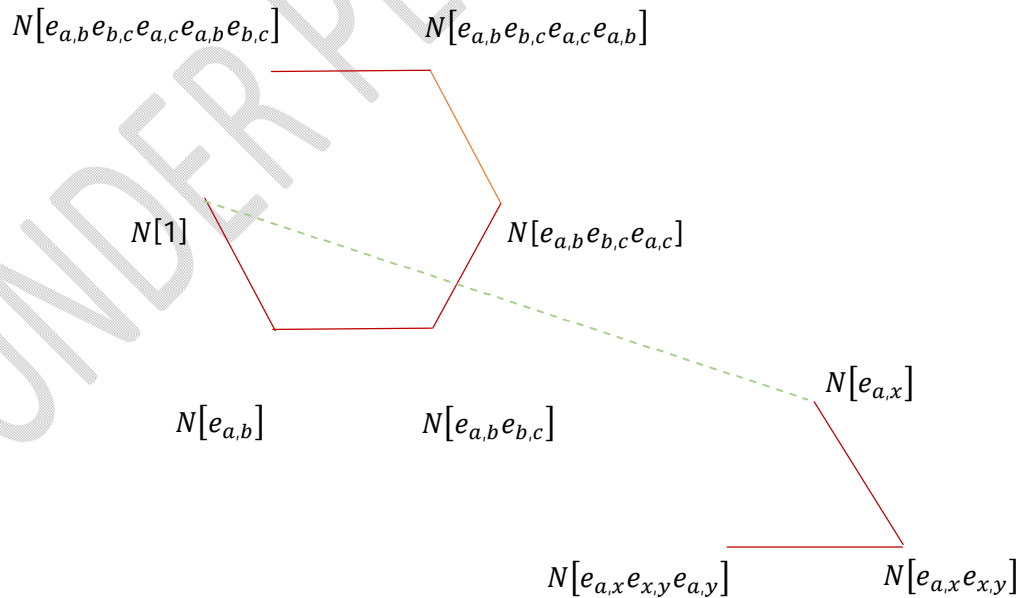


Figure 5 Maximal subtree $\Gamma({}^5(K_N)_1)$

The generators for the fundamental group $\pi_1({}^5(K_N)_1, N[1])$ will be:

$$\begin{aligned}
g_1({}^5K_N) &= (N[1], e_{a,b})(N[e_{a,b}], e_{a,b}e_{b,c})(N[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a}) \\
&(N[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})(N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c}) \\
&(N[1], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})^{-1}. \\
g_2(\pi_1({}^5K_N)) &= (N[1], e_{a,x})(N[e_{a,x}], e_{a,x}e_{x,y})(N[e_{a,x}e_{x,y}], e_{a,x}e_{x,y}e_{a,y}) \\
&(N[e_{a,x}], e_{a,x}e_{x,y})^{-1}(N[1], e_{a,x})^{-1}.
\end{aligned}$$

THEOREM 2: Let the following semigroup presentation

$${}^5S = \langle x, y, a, b, c : x = y, a = b, b = c, c = a, x = a \rangle$$

If the number of all vertices of two complex graph 5K_N is 5^n , then the number of all vertices of the covering space ${}^5(K_N)_n$ is $(5)^n + 3$.

PROOF: By induction, for $k = 1$ the number of all vertices in ${}^5(K_N)_1$ is 5. Thus for $k = 1$ is true (see Figure 1). Now assume $v_k = (5)^k + 3$ be the number of all vertices in ${}^5(K_N)_k$.

We will prove the number of all vertices of the covering space ${}^5(K_N)_{k+1}$ is $(5)^{k+1} + 3$.

By the definition of ${}^5K_{k+1}$ is five copies of 5K_k and assumption, then the number of all vertices of the covering space ${}^5(K_N)_{k+1}$ is $v_{k+1} = 5 \cdot (5)^k + 3 = (5)^{k+1} + 3$.

THEOREM 3: Consider the semigroup presentation ${}^5S = \langle x, y, a, b, c : x = y, a = b, b = c, c = a, x = a \rangle$. The number of all edges in the covering space ${}^5(K_N)_n$ is $e_n = n5^n + 3$.

PROOF: By induction, for $k = 1$ the number of all vertices in ${}^5(K_N)_1$ is $e_1 = 1(5) + 3 = 8$.

Now let $e_k = k5^k + 3$ be the number of all edges the covering space ${}^5(K_N)_k$. We will prove that the number of all edges in ${}^5(K_N)_{k+1}$ is $e_{k+1} = (k + 1)5^{k+1} + 3$. By using last theorem

$$\begin{aligned} e_{k+1} &= 5e_k + 5^{k+1} + 3 = 5k5^k + 5^{k+1} + 3 \\ &= k \cdot 5^{k+1} + 5^{k+1} + 3 \\ &= (k + 1)5^{k+1} + 3 . \end{aligned}$$

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