TIME STEPS DISTRIBUTION IN NUMERICAL TECHNIQUE: A COMPARATIVE ANALYSIS OF THIRD AND FOURTH ORDER RUNGE-KUTTA ALGORITHMS

ABSTRACT: To analyze a harmonically Duffing oscillator, this work used a combination of graphs, time steps distribution, adaptive time steps Runge-Kutta, and fourth order algorithms. The goal is to examine the performance of third and fourth order Runge-Kutta algorithms in finding chaotic solutions for a harmonically excited Duffing oscillator. Fourth-order algorithms favor larger time steps and are thus faster to execute than third-order algorithms in all circumstances studied. The accuracy of the data acquired with third order is worth the longer overall computation time steps period reported

Keywords- *Runge-Kutta, chaos, duffing, algorithms, time steps, differential equation, accuracy*

I. INTRODUCTION

Numerical techniques, such as the finite element method, are used to discretise these mathematical equations that are usually represented by partial differential equations representing the governing physics taking place, and the behaviour of the materials that make up the electronic or photonic device [1]. There is a specific Numerical Method for each type of problem that can be solved. Taylor Series Method, Euler Method, Runge Kutta Methods, Shooting Method, Finite Difference Methods, and so on are some of the numerical methods. Runge–Kutta method is an effective and widely used method for solving the initial-value problems of differential equations. Runge–Kutta method can be used to construct high order accurate numerical method by functions' self without needing the high order derivatives of functions [2, 3]. C.Runge and M.W. Kutta, two German mathematicians, invented the Runge-Kutta procedures in the past. The Runge-Kutta method's nomenclature was developed from the combination of the two names [4]. The importance of Runge-Kutta algorithms in solving issues involving nonlinear dynamics cannot be overstated. Numerical solutions to nonlinear dynamic problems have been the subject of a lot of research. When looking into the dynamics of a continuous-time system defined by an ordinary differential equation, it's common to start by looking for trajectories. The dynamics of the most often used family of numerical integration algorithms were elucidated by [5]. The authors' research revealed that Runge-Kutta integration should be used with caution when dealing with nonlinear systems. The connection between rigidity and chaos was explained in great detail. Because of their inefficiency, explicit Runge-Kutta algorithms should not be utilized for stiff situations, according to the conclusions of the study. Backward differentiation formulae methods or possibly implicit Runge Kutta procedures, according to the authors, are the best alternatives. The paper's conclusions revealed that dynamics is interested in periodic and chaotic behavior as well as issues with fixed point solutions. Bifurcation diagrams have been shown to be useful in the chaotic investigation of nonlinear electrical circuits [6]. The Runge-Kutta method was used to solve the required second order differential equations for ranges of suitable parameters. Bifurcation diagrams were created using the solutions found using this procedure. This work demonstrated how a bifurcation diagram may be used to investigate the dynamics of a nonlinear resonant circuit with a variety of control parameters. [7] research study used the Runge-Kutta sixth order method to provide numerical solutions for a system of second order robot arms. At various intervals, the precise solution of the system of equations defining the arm model of a robot was compared to the corresponding approximate solutions. The results and comparisons revealed that the numerical integration algorithm's efficiency is determined by the absolute error between exact and approximate answers. As a result of this discovery, the STWS algorithm is an A-stable approach that is not depending on Taylor's series. [8] explored the dynamics of a torsional system with harmonically variable drying friction torque. The example study involves nonlinear dynamics of a single degree of freedom torsional system with dry friction. The first model was a nonlinear system with a regularly varying normal load. After that, a multi-term harmonic balance method is reformulated (MHBM). The goal is to solve the nonlinear time-varying issue in the frequency domain directly. With a regularly fluctuating friction, the feasibility of MHBM is proved, and its accuracy is validated by numerical integration using the third order Runge-Kutta method. There has been developed a set of explicit third order new improved Runge-Kutta (NIRK) methods that only used two function evaluations per step [9]. The methodology suggested here has a lower computational cost than the standard third order Runge-Kutta method while keeping the same order of local accuracy due to a smaller number of function evaluations. [10] conducted a critical study of Ruge-Kutta discontinuous Galerkin (RKDG) approaches for nonlinear convection-dominated situations. The authors combined a Runge-Kutta time discretization that allows the method to be nonlinearly stable regardless of accuracy with a finite element space discretization by discontinuous approximations that incorporates the idea of numerical fluxes and slope limiters coined during the remarkable development of high resolution finite difference and finite volume schemes. RKDG methods are stable, high-order accurate, and highly parallelizable schemes that can easily handle intricate geometries and boundary conditions, according to this review. Its enormous applicability in Navier-Stokes equations and Hamilton-Jacobian equations were demonstrated in the review. This research has undoubtedly aided computational fluid dynamics. This method has mostly been used to investigate the dynamics of Duffing oscillators. The Duffing oscillator is made up of two simple coupled ordinary differential equations that must be solved. For numerical solutions of Duffing oscillator dynamics, the Runge-Kutta method has been widely employed. [6] Used bifurcation diagrams to explore the dynamical behavior of a Duffing oscillator. In order to solve pertinent second order differential equations, the authors used the third order Runge-Kutta method. While the bifurcation diagrams revealed the dynamics of the Duffing oscillator, they also indicated that the dynamics are highly dependent on the beginning conditions. [11] Demonstrated how the Duffing equation may be used to estimate sawdust particle emission properties. The study explains how to model sawdust particle motion as a two-dimensional continuous time series transformation system. In order to solve Duffing's model equation for sawdust particles, the authors used the Runge-Kutta algorithm. The solution was based on the perspective of displacement and velocity. The authors' findings demonstrated the high-profile viability of simulating sawdust dynamics as band saw emissions. The conclusion drawn from this research is that the findings will undoubtedly increase our understanding of sawdust emission studies. Despite the fact that the Runge-Kutta technique is widely used as a numerical tool in nonlinear dynamics, there is no denying that a research gap exists. According to the available literature, studies comparing the performance of different Runge-Kutta orders (second, third, fourth, sixth, and so on) are very limited. This work compares the performance of third and fourth order Runge-Kutta algorithms as instruments for finding chaotic solutions in a harmonically excited Duffing oscillator.

II. DUFFING OSCILLATOR

The investigated normalized governing equation for the dynamic behavior of a harmonically excited Duffing system is given as

$$\ddot{x} + y\dot{x}(1 - x^2) = P_0(Sin\ \omega t) \qquad 1$$

Where: x is displacement

$\dot{x}$ is velocity

$\ddot{x}$ is acceleration

y is the damping coefficient

P_0 is the amplitude strength of harmonic excitation

ω is the excitation frequency

t is time

[12, 13, 14 and 15] in their separate papers, postulated that a harmonically excited duffing oscillator with a damping coefficient of 0.0168, amplitude strength of 0.09, and excitation frequency of 1.0 exhibits chaotic behavior. Equation (1) was investigated using adaptive time steps Runge-Kutta algorithms across 150 excitations starting with a time step of Δt ($\Delta t = Excitation\ Period/1000$) in this paper. The stable answers from the latest fifty (50) excitation period computations were used to create the phase plot.

III. SELECTION OF TIME STEP

Using a constant step size to find solutions to ordinary differential equations in some dynamical systems that exhibit a sudden transition, according to [16], could be a major constraint. The choice of adaptive time step size becomes unavoidable in such engineering situations (chaotic dynamics). Equations 2 and 3 are the formulas for raising and lowering the time step (Δt) in this investigation, respectively.

$$\Delta t = \Delta t(0.95)(\varepsilon_t/\varepsilon)^{1/4} \quad (2)$$

$$\Delta t = \Delta t(0.95)(\varepsilon_t/\varepsilon)^{1/5} \quad (3)$$

Where ε_t is the tolerance

ε is the error

Equation 2 is deployed when the error is less than the tolerance ($\varepsilon < \varepsilon_t$) while equation 3 is deployed when the error is greater than the tolerance ($\varepsilon > \varepsilon_t$).

IV. DETAILS of THE STUDIED SAMPLES' PARAMETERS

Three separate samples were investigated using the information in table 1 below and the governing equation (1). Displacement ($x = 0.1$), initial velocity ($\dot{x}$), and excitation frequency (ω) are all common factors in all circumstances.

Table 1: Parameters for Investigated Samples

Samples	Damping Coefficient (y)	Excitation Amplitude (P_0)
1	0.1680	0.21
2	0.0168	0.09
3	0.0168	0.21

V. RESULTS

The investigation of sample 1 parameters using the Runge-Kutta third and fourth order algorithms generates figure 1 as seen below.

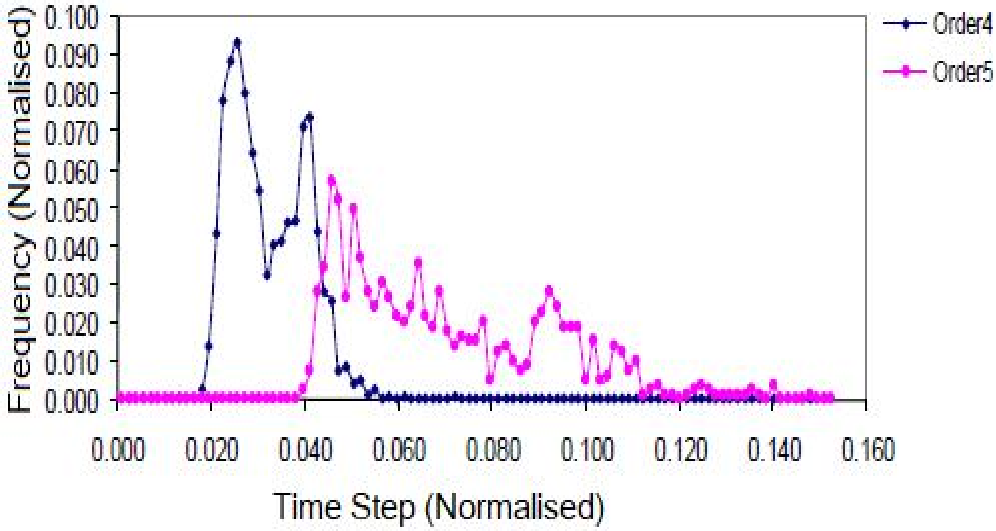

Figure 1: Sample 1 Time-Step Distribution

From figure 1, the third order algorithms' time steps distribution range is shorter, whereas the fourth order algorithms' time steps distribution range is longer. Higher computational time steps are not tolerated as well by third order algorithms as they are by fourth order algorithms. The third and fourth order algorithms' distributions peaked at 0.026 and 0.026 excitation periods, respectively.

Similarly, the parameters of sample 2 gives rise to Figure 2 as seen below

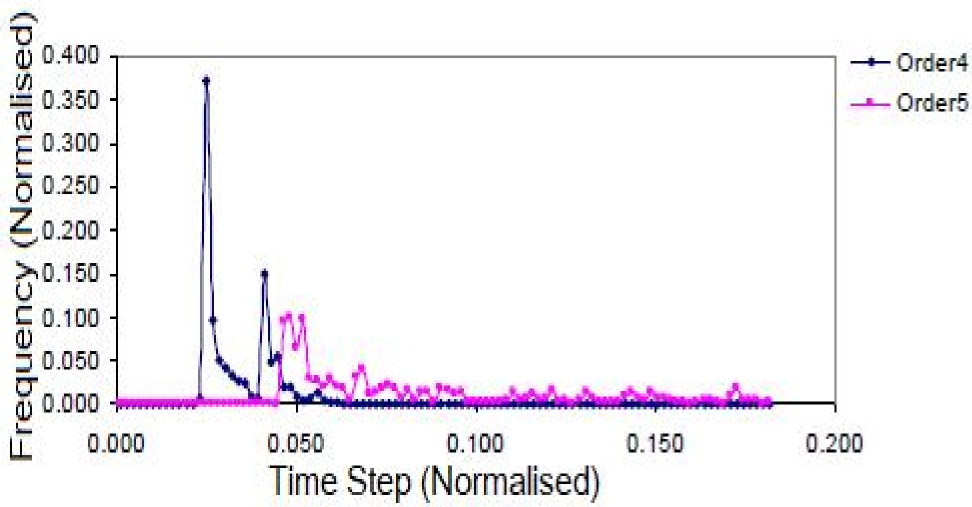

Figure 2: Sample 2 Time-Step Distribution

Figure 1 is similar to Figure 2 in shape, however the frequency intensities, on the other hand, are vastly different. The third and fourth order algorithms' distributions in Figure 2 peaked at 0.025 and 0.048 excitation periods, respectively as against what was obtained in Figure 1.

Like in the cases of Samples 1 and 2, , the parameters of sample 3 gives rise to Figure 3 as seen below

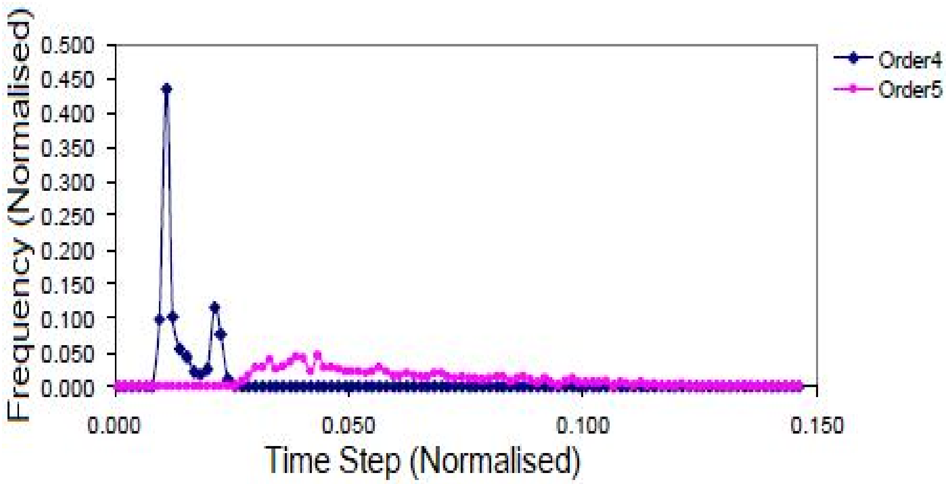

Figure 3: Sample 3 Time-Step Distribution

Figure 3 can be compared to figures 1 and 2 in terms of quality. However, the frequency intensities differ, and the third and fourth order algorithms' distributions peaked at 0.011 and 0.043 periods, respectively, for the third and fourth order algorithms.

Figures 4, 5, 6, 7, 8 and 9 shows the phase graphs obtained using the Runge-Kutta third and fourth orders; they are only similar but not exact for sample 1 and sample 2.

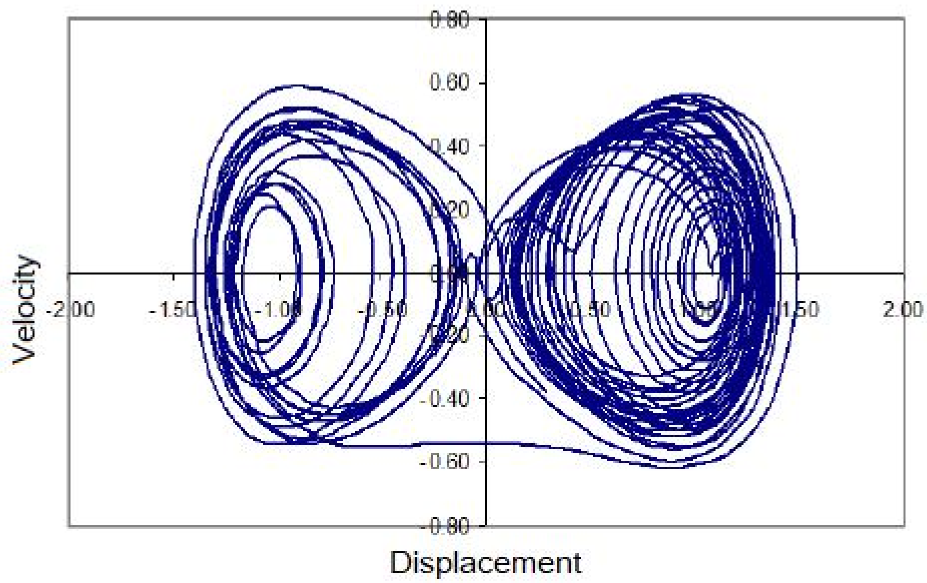

Figure 4: Velocity – Displacement Graph for Third Oder Sample 1

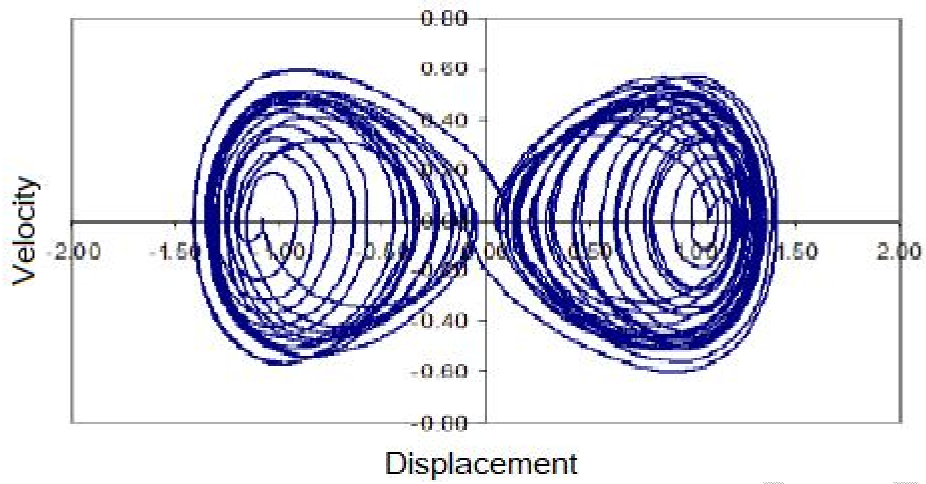

Figure 5: Velocity – Displacement Graph for Fourth Oder Sample 1

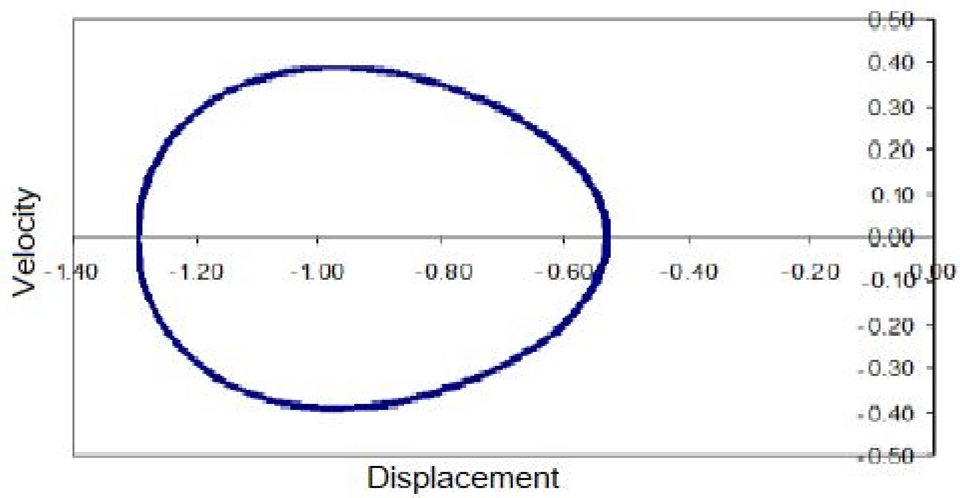

Figure 6: Velocity – Displacement Graph for Third Oder Sample 2

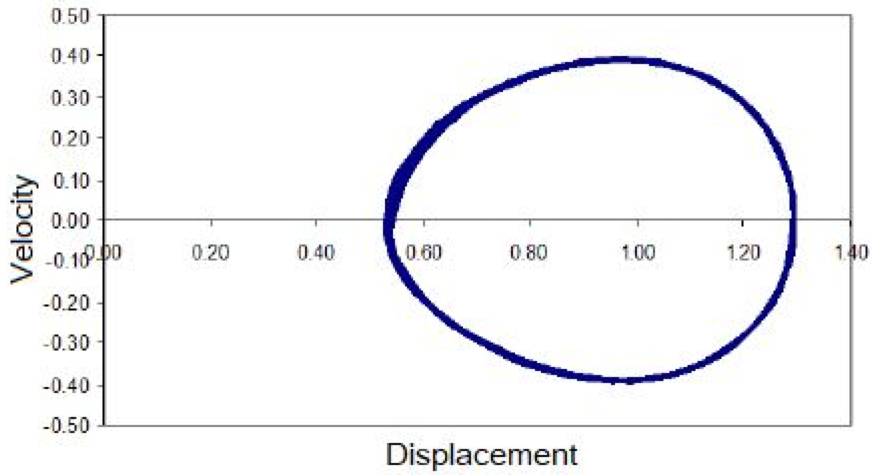

Figure 7: Velocity – Displacement Graph for Fourth Oder Sample 2

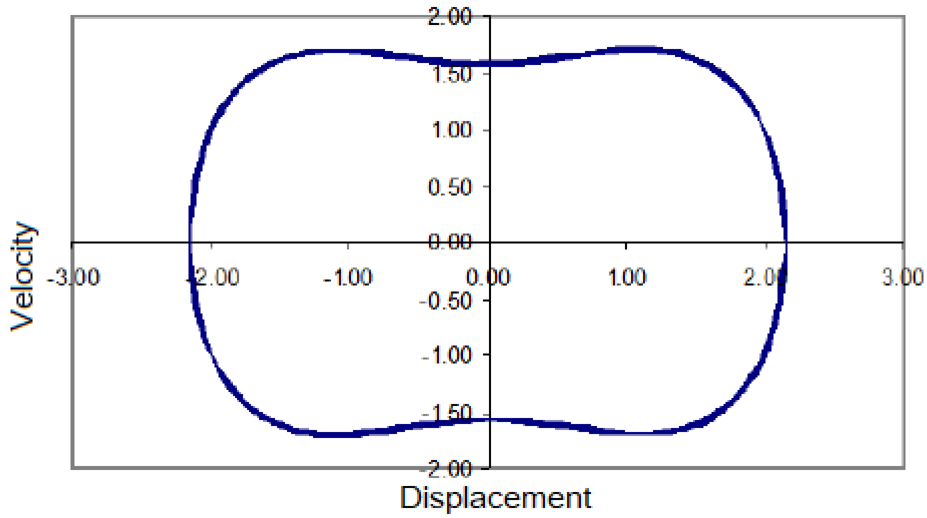

Figure 8: Velocity – Displacement Graph for Third Oder Sample 3

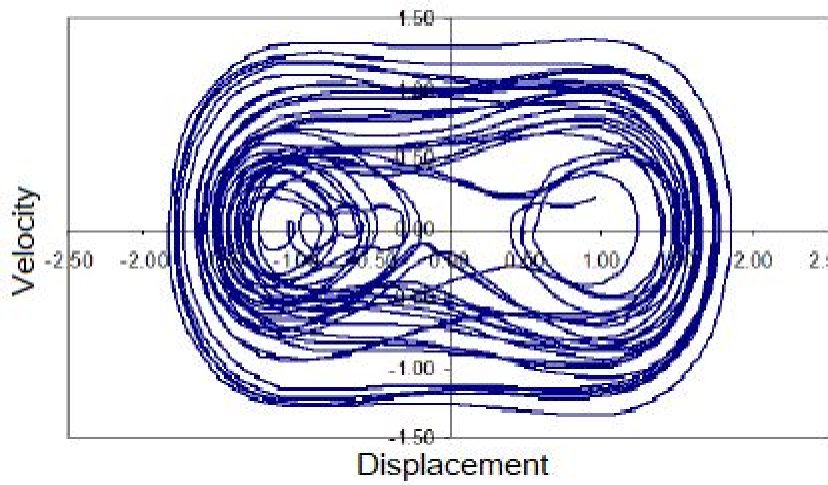

Figure 9: Velocity – Displacement Graph for Fourth Oder Sample 3

A deeper examination of the graph for Sample 2 reveals that the third order algorithm's solutions are bounded to the negative side of the displacement, whereas the fourth order algorithm's solutions are bounded to the positive side of the displacement. Figure 4 shows a graph that is remarkably similar to that of [14].

Furthermore, tables 2 and 3 interpretations significantly support the third order algorithm's outcomes being more consistent and reliable than its fourth order version. Overall, a comparison of the graphs in conjunction with the time steps distribution suggests that the third order algorithm is more trustworthy than the fourth order at the cost of additional computation steps each period.

Table 2: Corresponding Graph Parameters for Third Order Algorithm

Samples	**Constant Time Steps**	**Variable Time Steps (Section 1)**	**Variable Time Steps (Section 2)**
1	A	A	Nil
2	D	C	Irregular C
3	E	E	Irregular E

Table 3: Corresponding Graph Parameters for Fourth Order Algorithm

Samples	**Constant Time Steps**	**Variable Time Steps (Section 1)**	**Variable Time Steps (Section 2)**
1	Nil	B	Roughly A
2	Near B	D	C
3	E	F	F

Where: A, B, C, D, E, F is equivalent to Figures 4, 5, 6, 7, 8 and 9.

Tables 4 and 5 illustrates that when compared to constant time steps, adaptive third order can be twenty five (25) times faster to execute (as seen in Samples 1 and 2). Similarly, as compared to its corresponding time steps, adaptive fourth order can be fifty (50) times faster to perform (as seen in the three samples).

Table 4: Total Number of Variable Steps for Third Order Algorithm

Samples	Constant Time Steps	Adaptive Time Steps (Section 1)	Adaptive Time Steps (Section 2)
1	50,000	1,598	1,588
2	50,000	1,598	1,594
3	50,000	3,667	3,704

Table 5: Total Number of Variable Steps for Fourth Order Algorithm

Samples	Constant Time Steps	Adaptive Time Steps (Section 1)	Adaptive Time Steps (Section 2)
1	50,000	785	764
2	50,000	788	792
3	50,000	962	959

Tables 4 and 5 also show that adaptive fourth order may be computed four times faster than its counterpart third order, as seen in Sample 3. However, the accuracy of computed findings may be questioned. The percentage of total steps taken by third and fourth order algorithms to find steady solutions is section independent.

VI. CONCLUSION

The performance of two Runge-Kutta algorithms to find the chaotic stable solutions of a harmonically excited Duffing oscillator was visually demonstrated in this paper. The study found that the Runge-Kutta fourth order can be four times faster to execute than the comparable third order, but at the expense of the computed findings' dependability.

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