

# A New Inverse Pareto Distribution

**Abstract:** New distribution of inverse Pareto using kumaraswamy generalized distribution called Kumaraswamy generalized inverse Pareto distribution is introduced in this paper. The proposed model is applicable in the study of variety of fields. Several properties of the proposed distribution, including explicit expressions for the quantile, moments, moment generating function and are studied. The maximum likelihood estimation is used to estimate parameters model and to derive the information matrix of it. Simulation study is carried out to evaluate the estimated parameters. The performance of the new model is examined using real data set comparing with widely known distributions.

**Key words:** Kumaraswamy generalized, inverse Pareto distribution, quantile, moment generating function, maximum likelihood.

## 1. Introduction

The Pareto distribution is the most popular model for analyzing heavy tailed phenomena. The Pareto distribution was first proposed by Pareto (1986) as a model for the distribution of incomes and other financial variables, and other phenomena. The Pareto distribution has a wide range of applications in several fields such as life testing, economics, finance, engineering, and etc...(Evans et al. (2000)). Many generalized distributions of the Pareto distribution can be found in the literature. A few examples of these distributions are the beta-Pareto distribution by Akinsete et al. (2008), the Kumaraswamy- Pareto distribution by Bourguignon et al. (2013), the Kumaraswamy-generalized exponentiated Pareto distribution by Shams (2013), the Kumaraswamy transmuted Pareto distribution by Chhetri et al. (2017).

Klugman et al. (2012) introduced the cumulative distribution function (cdf) of the two-parameter inverse Pareto (IP) distribution as follows:

$$G(x) = \left( \frac{x}{\beta + x} \right)^\alpha, \quad \alpha, \beta, x > 0, \quad (1)$$

where  $\alpha$  is shape parameter and  $\beta$  is scale parameter. Some authors deal with many families of some well-known distributions which are more flexible for modeling several types of data. Traditional models do not provide adequate fits to the real data

with highly skewed. To solve this problem several, introducing several methods including additional one shape parameters, two shape parameters to generating new families of distributions are available in the statistical literature. Some well-known generators are: the generalized-exponential by Gupta and Kundo (1999), Kumaraswamy generalized distribution by Cordeiro and de Castro (2011), generalized beta-generated by Alexander et al. (2012), weibull-generated by Bourguignon et al. (2014), Kumaraswamy weibull-generated by Hassan and Elgarhy (2016), generalized additive weibull-generated by Hassan et al. (2017), inverse weibull-generated Hassan and Nassr (2018), odd inverse Pareto-generated by Aldahlan et al. (2019), modified weibull-generated by Abdelall (2019) and more.

The goal of this article is to provide a new and more flexible distribution, called Kumaraswamy generalized inverse Pareto (KGIP) distribution and introduces expansions expressions for its cumulative and density functions. Numerous statistical properties of the KGIP distribution including quantile function, order statistics, moments and moment generating function are studied in second section. The third section provides parameter estimation using maximum likelihood method. The simulation study is introduced in fourth section. Fifth section displays the effectiveness of the proposed distribution by practical application on real data sets. Finally, the conclusion is provided.

## 2. Kumaraswamy Generalized Inverse Pareto Distribution

Cordeiro and de Castro (2011) defined the cdf of the Kumaraswamy generalized distribution as follows:

$$F(x) = 1 - \left[ 1 - (G(x))^a \right]^b, \quad a, b > 0, \quad (2)$$

where  $a$  and  $b$  are additional shape parameters. Using (1), (2), the cdf of the Kumaraswamy generalized inverse Pareto (KGIP) distribution with parameter  $\underline{\xi} = (a, b, \alpha, \beta)$  is given by:

$$F_{KGIP}(x, \underline{\xi}) = 1 - \left[ 1 - \left( \frac{x}{\beta + x} \right)^{\alpha a} \right]^b, \quad x > 0. \quad (3)$$

The corresponding probability density function (pdf), will be

$$f_{KGIP}(x, \underline{\xi}) = ab\alpha\beta \frac{x^{\alpha a-1}}{(\beta+x)^{\alpha a+1}} \left[ 1 - \left( \frac{x}{\beta+x} \right)^{\alpha a} \right]^{b-1}, \quad \alpha, \beta, a, b > 0, x > 0, \quad (4)$$

where  $\alpha$ ,  $a$  and  $b$  are shape parameters and  $\beta$  is scale parameter.

### Special Distributions:

Sub-models can be deduced from KGIP distribution as follows:

If  $b = 1$  in (4), we get the exponentiated inverse Pareto distribution with parameters  $a, \alpha$ , and  $\beta$ .

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If  $b = a = 1$  in (4), we get the inverse Pareto distribution with parameters  $\alpha$  and  $\beta$ .

The survival and hazard rate functions of the KGIP distribution are obtained, respectively as follows:

$$S_{KGIP}(x, \underline{\xi}) = \left[ 1 - \left( \frac{x}{\beta+x} \right)^{\alpha a} \right]^b,$$

and

$$h_{KGIP}(x, \underline{\xi}) = ab\alpha\beta \frac{x^{\alpha a-1}}{(\beta+x)^{\alpha a+1}} \left[ 1 - \left( \frac{x}{\beta+x} \right)^{\alpha a} \right]^{-1}. \quad \text{Fig 1 and 2 display the graphs of}$$

pdf and hazard function of the KGIP distribution.

## 2.1 Model Expansions

Here, we give explicit expansions for the cdf and pdf of the KGIP model. By using the generalized binomial theorem

$$(1+z)^v = \sum_{i=0}^{\infty} \binom{v}{i} z^i, \quad 0 < z < 1. \quad (5)$$

Equation (3) can be rewritten as follows:

$$F_{KGIP}(x) = 1 - \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} \left( \frac{x}{\beta+x} \right)^{\alpha ai} = 1 - \sum_{i=0}^{\infty} \eta_i \tau(x, \underline{\theta}), \quad (6)$$

where

$\eta_i = (-1)^i \binom{b}{i}$ , and  $\tau(x, \underline{\theta})$  denotes the Inverse Pareto cdf with parameters

$\underline{\theta} = (\alpha a i, \beta)$ . Now, using the power series (5) in the last term of (4), we get

$$f_{KGIP}(x) = b \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)} \binom{b-1}{i} \left( \frac{\alpha a (i+1) \beta x^{\alpha a (i+1)-1}}{(\beta+x)^{\alpha a (i+1)+1}} \right) \quad (7)$$

$$= \sum_{i=0}^{\infty} k_i \psi(x, \underline{g}),$$

where

$k_i = (-1)^i \binom{b}{i+1}$ , and  $\psi(x, \underline{g})$  is the Inverse Pareto density function with parameters

$\underline{g} = (\alpha a (i+1), \beta)$ . Thus, the KGIP density function can be expressed as an infinite linear combination of the Inverse Pareto density. Thus, some of its statistical properties can be obtained directly from those properties of the Inverse Pareto distribution.

## 2.2 Quantile function and simulation

The quantile function of the KGIP distribution can be defined as follows:

$$Q(q) = \beta \left[ 1 - (1-q)^{\frac{1}{b}} \right]^{\frac{1}{\alpha a}} \left[ 1 - \left[ 1 - (1-q)^{\frac{1}{b}} \right]^{\frac{1}{\alpha a}} \right]^{-1}, \quad 0 < q < 1. \quad (8)$$

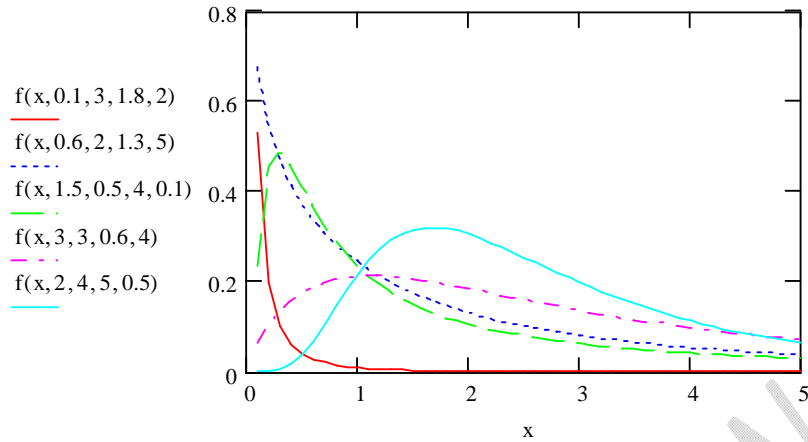
Median of KGIP distribution can be obtained by putting  $q=0.5$ , that is

$$Q(0.5) = \beta \left[ 1 - 0.5^{\frac{1}{b}} \right]^{\frac{1}{\alpha a}} \left[ 1 - \left[ 1 - 0.5^{\frac{1}{b}} \right]^{\frac{1}{\alpha a}} \right]^{-1}.$$

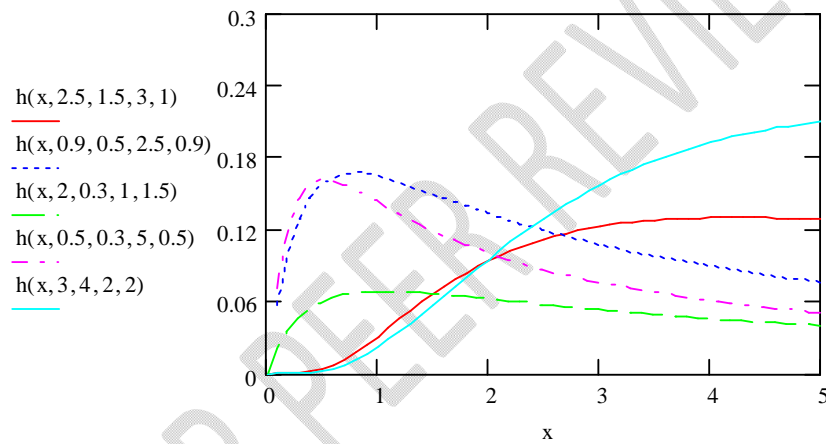
For **simulating** the KGIP random variable, let  $q$  be a uniform variate on the unit interval  $(0,1)$ . Thus, by means of the inverse transformation method, we consider the random variable  $X$  given by:

$$X = \beta \left[ 1 - (1-U)^{\frac{1}{b}} \right]^{\frac{1}{\alpha a}} \left[ 1 - \left[ 1 - (1-U)^{\frac{1}{b}} \right]^{\frac{1}{\alpha a}} \right]^{-1},$$

which follows (4), i.e,  $X$  distributed  $KGIP(a, b, \alpha, \beta)$ .



**Figure 1.** The KGIP  $(a, b, \alpha, \beta)$  density function.



**Figure 2.** The KGIP  $(a, b, \alpha, \beta)$  hazard function.

### 2.3 Skewness and Kurtosis

The Bowley Skewness measure introduced by Kenny and Keeping (1962) based on quantile function as follows:

$$BS = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}.$$

Also, the Moors kurtosis measure introduced by Moors (1988) as follows:

$$MK = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)},$$

where  $Q(\cdot)$  is the quantile function. Figure 3 illustrates plots of the skewness and kurtosis measures of the KGIP distribution for different values of the parameter  $b$  as

a function of  $\alpha$  and fixed values of  $a$  and  $\beta$ . These plots indicate that these measures decreases as  $b = 0.5, 0.8, 1, 2$  (increases) for fixed  $a$  and  $\beta$ .

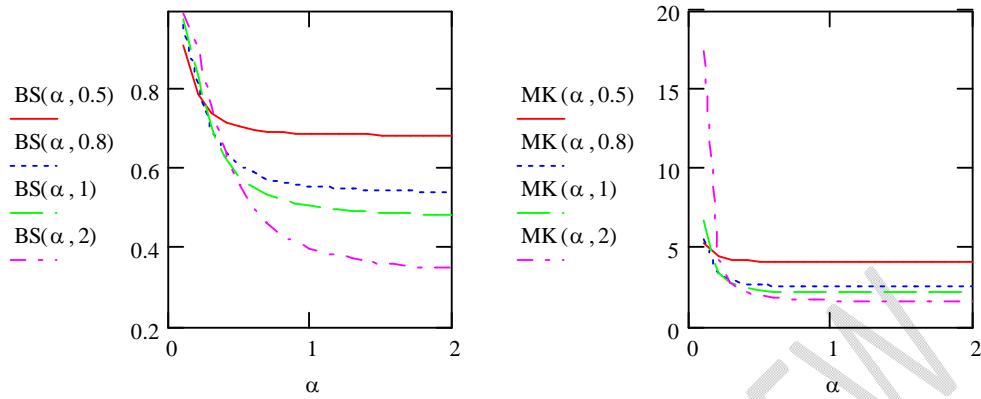


Figure3. KGIP skewness and kurtosis measures.

## 2.4 Moments

The  $r^{\text{th}}$  moment for KGIP random variable  $X$  is given by:

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f_{KGIP}(x) dx$$

Using (7), we have

$$\mu'_r = \sum_{i=0}^{\infty} k_i \int_0^{\infty} x^r \frac{\alpha a (i+1) \beta x^{\alpha a (i+1) - 1}}{(\beta + x)^{\alpha a (i+1) + 1}} dx,$$

Let  $z = \frac{x}{\beta + x}$ , after simplification, we obtain

$$\mu'_r = \sum_{i=0}^{\infty} \alpha a \beta^r (i+1) k_i \int_0^1 z^{\alpha a (i+1) + r - 1} (1-z)^{-r} dz,$$

Using beta function, we get

$$\mu'_r = \sum_{i=0}^{\infty} \alpha a \beta^r (i+1) k_i B(\alpha a (i+1) + r, 1-r)$$

$$= \sum_{i=0}^{\infty} \beta^r k_i \frac{\Gamma(\alpha a (i+1) + r) \Gamma(1-r)}{\Gamma(\alpha a (i+1))}, \quad -\alpha a (i+1) < r < 1,$$

where  $B(c, d) = \int_0^1 z^{c-1} (1-z)^{d-1} dz = \frac{\Gamma(c) \Gamma(d)}{\Gamma(c+d)}$  is the beta function, and  $\Gamma(\cdot)$  is the

gamma function. If  $r$  is a negative integer, the  $r^{\text{th}}$  moment is obtained as follows:

$$\mu'_r = \sum_{i=0}^{\infty} \beta^r k_i \frac{(-r)!}{(\alpha a(i+1)-1)\dots(\alpha a(i+1)+r)}.$$

The  $r^{\text{th}}$  incomplete moment for KGIP random variable  $X$  is then equal to:

$$w_s(t) = \int_0^t x^s f_{KGIP}(x) dx$$

$$= \sum_{i=0}^{\infty} \alpha a \beta^s (i+1) k_i B\left(\frac{t}{\beta+t}, \alpha a(i+1)+s, 1-s\right), \quad -\alpha a(i+1) < r < 1, \quad (9)$$

where  $B(x, c, d) = \int_0^x z^{c-1} (1-z)^{d-1} dz$  is the incomplete beta function. Also, the moment generating function of the proposed distribution is defined as:

$$M_X(t) = E[e^{tX}] = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r = \sum_{r,i=0}^{\infty} k_i \frac{(t\beta)^r \Gamma(\alpha a(i+1)+r) \Gamma(1-r)}{\Gamma(\alpha a(i+1)) \Gamma(r+1)}.$$

## 2.5 Order Statistic

Taking a simple random sample of size  $n$  from  $KGIP(\underline{\xi})$  with cdf and pdf  $F_{KGIP}(x; \underline{\xi})$  and  $f_{KGIP}(x; \underline{\xi})$  given by (3) and (4) respectively. The ordered statistics sample obtained from random sample is  $X_{1:n} \leq X_{2:n} \leq \dots, X_{n:n}$ . The pdf of the  $r$ th order statistic  $X_{r:n}$  is:

$$f_{r:n}(x; \underline{\xi}) = \frac{1}{B(r, n-r+1)} f(x; \underline{\xi}) [F(x; \underline{\xi})]^{r-1} [1-F(x; \underline{\xi})]^{n-r}, \quad (10)$$

where  $B(.,.)$  is the beta function. Since  $0 < F(x; \underline{\xi}) < 1$  for  $x > 0$ , we can use the binomial expansion of  $[1-F(x; \underline{\xi})]^{n-r}$  given as follows

$$[1-F(x; \underline{\xi})]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x; \underline{\xi})]^i. \quad (11)$$

Substituting from (11) into (10), we obtain

$$f_{r:n}(x; \underline{\xi}) = \frac{1}{B(r, n-r+1)} f(x; \underline{\xi}) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x; \underline{\xi})]^{i+r-1}. \quad (12)$$

Based on (6) and (7), we can write

$$f(x; \underline{\xi}) F(x; \underline{\xi})^{i+r-1} = ab \sum_{k,j,l=0}^{\infty} \frac{(-1)^{k+l} \alpha \beta}{(\beta+x)^{\alpha a(l+1)+1}} \binom{i+r-1}{k} \binom{b(l+1)-1}{l} x^{\alpha a(l+1)-1}. \quad (13)$$

Inserting (13) in (12), the pdf of  $X_{r:n}$  written as

$$f_{r,n}(x, \underline{\xi}) = \sum_{k,l,i=0}^{\infty} \omega_{k,l,i} \psi(x, \underline{g}),$$

where  $\omega_{k,j,t,l} = \sum_{i=0}^{n-r} \frac{(-1)^{k+l+i} a b}{B(r, n-r+1)} \binom{n-r}{i} \binom{i+r-1}{k} \binom{b(l+1)-1}{l}$ ,  $\psi(x, \underline{g})$  denotes the

Inverse Pareto distribution with parameters  $\underline{g} = (\alpha a(l+1), \beta)$ . Thus, the pdf of the KGIP order statistics is a linear combination of the Inverse Pareto density. Also we can define first order statistics  $X_{1:n} = \min(X_1; X_2; \dots; X_n)$ , and the last order statistics as  $X_{n:n} = \max(X_1; X_2; \dots; X_n)$ .

## 2.6 Mean deviations

The useful measures of variation for population are mean deviations about mean and median. If the KGIP has mean ( $\mu$ ) and median ( $m$ ), the mean deviations about mean and median are respectively, can be defined as:

$$D(\mu) = 2\mu F_{KGIP}(\mu) - 2w_{1,KGIP}(\mu), \text{ and } D(m) = \mu - 2w_{1,KGIP}(m),$$

where  $w_1(\cdot)$  is the first incomplete moment of KGIP obtained from (9) for  $s = 1$ .

## 3. Parameter Estimation of KGIP distribution

Using the maximum likelihood estimation, we derived the estimated parameters of the KGIP distribution. Let  $X_1, X_2, \dots, X_n$  be a random sample from KGIP( $\underline{\xi}$ ) where  $\underline{\xi} = (a, b, \alpha, \beta)$  be the vector of the parameters, the log-likelihood function,  $\ell$ , is given by:

$$\begin{aligned} \ell(\underline{\xi}) = & n \log a + n \log b + n \log \alpha + n \log \beta + (\alpha a - 1) \sum_{i=1}^n \log(x_i) - (\alpha a + 1) \sum_{i=1}^n \log(\beta + x_i) \\ & + (b - 1) \sum_{i=1}^n \log \left[ 1 - \left( \frac{x_i}{\beta + x_i} \right)^{\alpha a} \right]. \end{aligned} \quad (14)$$

The score vector  $U(\underline{\xi}) = (\partial \ell / \partial a, \partial \ell / \partial b, \partial \ell / \partial \alpha, \partial \ell / \partial \beta)^T$ , where the components to the parameters in  $\underline{\xi}$  are given by differentiating (14). By setting  $z_i = x_i / (\beta + x_i)^{-1}$ ,

$$U_a = \frac{n}{a} + \alpha \sum_{i=1}^n \log(x_i) - \alpha \sum_{i=1}^n \log(\beta + x_i) - (b - 1) \sum_{i=1}^n \left[ \frac{\alpha z_i^{\alpha a} \log(z_i)}{(1 - z_i^{\alpha a})} \right], \quad (15)$$

$$U_b = \frac{n}{b} + \sum_{i=1}^n \log(1 - z_i^{\alpha a}), \quad (16)$$

$$U_\alpha = \frac{n}{\alpha} + a \sum_{i=1}^n \log(x_i) - a \sum_{i=1}^n \log(\beta + x_i) - (b-1) \sum_{i=1}^n \left[ \frac{a z_i^{\alpha a} \log(z_i)}{(1 - z_i^{\alpha a})} \right], \quad (17)$$

and

$$U_\beta = \frac{n}{\beta} - (\alpha a + 1) \sum_{i=1}^n (\beta + x_i)^{-1} + (b-1) \sum_{i=1}^n \left[ \frac{\alpha a z_i^{\alpha a}}{(\beta + x_i)(1 - z_i^{\alpha a})} \right]. \quad (18)$$

The maximum likelihood estimates (MLEs) of the unknown parameters are obtained by setting system of non-linear (15) - (18) equations to zero and solve them simultaneously. From equation (16), The MLE of b can be rewritten in the closed form as follows

$$\hat{b} = \frac{-n}{\sum_{i=1}^n \log[1 - z_i^{\hat{\alpha} \hat{a}}]}, \quad (19)$$

When estimates  $(\hat{a}, \hat{\alpha})$  known. Substituting from (19) into (15), (17), and (18), we get the MLEs  $(\hat{a}, \hat{\alpha}, \hat{\beta})$ . These equations cannot be solved analytically and statistical software can be used to solve them simultaneously. For interval estimation of the parameters, we require the 4×4 observed information matrix  $J(\underline{\xi}) = \{U_{s,t}\}$ , where  $s, t = (a, b, \alpha, \beta)$  given in Appendix A. To construct the approximate confidence intervals for the parameters  $a, b, \alpha$ , and  $\beta$  of the KGIP distribution, the multivariate normal  $N_4(0, J(\underline{\xi})^{-1})$  distribution can be used. Here,  $J(\underline{\xi})$  is the total observed information matrix evaluated at  $\underline{\xi}$ . The asymptotic  $100(1-\eta)\%$  confidence intervals for parameters  $a, b, \alpha$ , and  $\beta$  are respectively given by:

$$\hat{a} \pm Z_{(\eta/2)} \sqrt{\text{var}(\hat{a})}, \hat{b} \pm Z_{(\eta/2)} \sqrt{\text{var}(\hat{b})}, \hat{\alpha} \pm Z_{(\eta/2)} \sqrt{\text{var}(\hat{\alpha})}, \text{ and } \hat{\beta} \pm Z_{(\eta/2)} \sqrt{\text{var}(\hat{\beta})},$$

where  $Z_{(\eta/2)}$  is the upper  $(\eta/2)^{th}$  percentile of the standard normal distribution and the

$\text{Var}(\cdot)$ 's denote the diagonal elements of  $J(\underline{\xi})^{-1}$  corresponding to a, b,  $\alpha$ , and  $\beta$ .

#### 4. Simulation study

Simulation study has been performed for average MLEs, Mean Square Error (MSE), and Bias. The KGIP random number generation was performed of size  $n = 50(50) 200$

for  $N = 1000$  replications. The parameter values used in the data generation processes are followed in Table 1. Bias and MSE are calculated by:  $Bias = \frac{1}{N} \sum_{i=1}^N (\hat{\xi}_i - \xi)$ ,

$$MSE = \frac{1}{N} \sum_{i=1}^N (\hat{\xi}_i - \xi)^2,$$

where  $\xi = (a, b, \alpha, \beta)$ . Simulation results were obtained for several combinations of parameters. Table 1 lists the means of the MLEs of the four parameters KGIP with respective MSEs, and Biases. It can be illustrated from table that, the MSEs and Biases decreases for all parameter combinations when sample size increase.

**Table 1:** Mean estimates, corresponding MSE and Bias of the MLE of  $(a, b, \alpha, \beta)$ .

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| Actual Values  | n   | Mean          |               |                    |                   | Bias          |               |                    |                   | MSE           |               |                    |                   |
|----------------|-----|---------------|---------------|--------------------|-------------------|---------------|---------------|--------------------|-------------------|---------------|---------------|--------------------|-------------------|
|                |     | ( $\hat{a}$ ) | ( $\hat{b}$ ) | ( $\hat{\alpha}$ ) | ( $\hat{\beta}$ ) | ( $\hat{a}$ ) | ( $\hat{b}$ ) | ( $\hat{\alpha}$ ) | ( $\hat{\beta}$ ) | ( $\hat{a}$ ) | ( $\hat{b}$ ) | ( $\hat{\alpha}$ ) | ( $\hat{\beta}$ ) |
| a = 0.5        | 50  | 0.734         | 1.955         | 2.158              | 0.509             | 0.234         | 1.455         | 0.668              | 0.093             | 0.055         | 2.122         | 0.437              | 0.00014           |
| b = 0.5        | 100 | 0.716         | 1.887         | 2.160              | 0.511             | 0.216         | 1.387         | 0.650              | 0.017             | 0.047         | 1.924         | 0.436              | 0.00013           |
| $\alpha = 1.5$ | 150 | 0.714         | 1.861         | 2.140              | 0.510             | 0.213         | 1.361         | 0.640              | 0.011             | 0.046         | 1.853         | 0.410              | 0.00012           |
| $\beta = 0.5$  | 200 | 0.707         | 1.848         | 2.130              | 0.509             | 0.207         | 1.348         | 0.631              | 0.010             | 0.043         | 1.818         | 0.395              | 0.00012           |
| a = 0.5        | 50  | 1.249         | 1.955         | 1.250              | 0.509             | 0.749         | 0.455         | 0.750              | 0.093             | 0.561         | 0.212         | 0.562              | 0.00015           |
| b = 1.5        | 100 | 1.236         | 1.885         | 1.237              | 0.511             | 0.736         | 0.387         | 0.737              | 0.018             | 0.542         | 0.150         | 0.543              | 0.00014           |
| $\alpha = 0.5$ | 150 | 1.232         | 1.860         | 1.233              | 0.511             | 0.732         | 0.361         | 0.733              | 0.015             | 0.536         | 0.131         | 0.537              | 0.00013           |
| $\beta = 0.5$  | 200 | 1.229         | 1.848         | 1.229              | 0.510             | 0.728         | 0.348         | 0.729              | 0.012             | 0.531         | 0.121         | 0.532              | 0.00012           |
| a = 0.5        | 50  | 1.249         | 1.955         | 1.249              | 0.509             | 0.749         | 1.455         | 0.749              | 0.009             | 0.562         | 2.122         | 0.561              | 0.00015           |
| b = 0.5        | 100 | 1.236         | 1.887         | 1.237              | 0.510             | 0.736         | 1.387         | 0.737              | 0.016             | 0.542         | 1.924         | 0.543              | 0.00014           |
| $\alpha = 0.5$ | 150 | 1.232         | 1.861         | 1.233              | 0.510             | 0.732         | 1.361         | 0.733              | 0.014             | 0.537         | 1.853         | 0.536              | 0.00012           |
| $\beta = 0.5$  | 200 | 1.229         | 1.848         | 1.229              | 0.509             | 0.729         | 1.348         | 0.729              | 0.011             | 0.531         | 1.818         | 0.532              | 0.00011           |
| a = 0.5        | 50  | 0.721         | 1.955         | 2.163              | 0.509             | 0.221         | 0.455         | 0.663              | 0.093             | 0.049         | 0.212         | 0.441              | 0.00015           |
| b = 1.5        | 100 | 0.714         | 1.887         | 2.142              | 0.511             | 0.214         | 0.387         | 0.642              | 0.018             | 0.046         | 0.150         | 0.412              | 0.00014           |
| $\alpha = 1.5$ | 150 | 0.712         | 1.861         | 2.135              | 0.510             | 0.212         | 0.361         | 0.635              | 0.011             | 0.045         | 0.131         | 0.403              | 0.00013           |
| $\beta = 0.5$  | 200 | 0.710         | 1.848         | 2.129              | 0.510             | 0.210         | 0.348         | 0.629              | 0.014             | 0.044         | 0.121         | 0.395              | 0.00012           |
| a = 0.5        | 50  | 0.703         | 2.507         | 2.011              | 0.777             | 0.203         | 1.007         | 0.511              | -0.023            | 0.041         | 1.023         | 0.262              | 0.00063           |
| b = 1.5        | 100 | 0.693         | 2.424         | 2.002              | 0.776             | 0.193         | 0.924         | 0.508              | -0.024            | 0.037         | 0.856         | 0.259              | 0.00059           |
| $\alpha = 1.5$ | 150 | 0.687         | 2.395         | 2.008              | 0.776             | 0.187         | 0.895         | 0.505              | -0.024            | 0.035         | 0.801         | 0.257              | 0.00052           |
| $\beta = 0.8$  | 200 | 0.685         | 2.382         | 2.005              | 0.776             | 0.185         | 0.882         | 0.502              | -0.025            | 0.034         | 0.778         | 0.255              | 0.00051           |
| a = 0.5        | 50  | 0.608         | 3.991         | 1.874              | 1.873             | 0.114         | 2.684         | 0.386              | -0.124            | 0.013         | 6.211         | 0.149              | 0.018             |
| b = 1.5        | 100 | 0.614         | 4.184         | 1.882              | 1.876             | 0.108         | 2.491         | 0.384              | -0.127            | 0.012         | 6.010         | 0.147              | 0.017             |
| $\alpha = 1.5$ | 150 | 0.603         | 3.931         | 1.883              | 1.870             | 0.103         | 2.431         | 0.383              | -0.131            | 0.011         | 5.912         | 0.146              | 0.016             |
| $\beta = 2$    | 200 | 0.601         | 3.908         | 1.886              | 1.870             | 0.101         | 2.408         | 0.374              | -0.133            | 0.010         | 5.800         | 0.144              | 0.014             |

## 5. Data Analysis

The importance and flexibility of the KGIP distribution can be examined by application to real data set. We compare the fit of KGIP with some competitive distributions namely: the Kumaraswamy-Pareto (KP), Exponentiated Pareto (EP), Beta Pareto (BP), and Pareto distributions (Bourguignon et al. (2013)). The data set correspond to the exceedances of flood peaks of the Wheaton River near Careross in Yukon Territory, Canada. The data consist of 72 excessdances for the years 1958-1984, rounded to one decimal place. They were analyzed by Choulakian and Stephens (2001) and are listed in Table 2.

**Table2:** Exceedances of Wheaton River flood data.

|     |      |      |      |      |      |      |      |      |      |      |      |
|-----|------|------|------|------|------|------|------|------|------|------|------|
| 1.7 | 2.2  | 14.4 | 1.1  | 0.4  | 20.6 | 5.3  | 0.7  | 1.9  | 13.0 | 12.0 | 9.3  |
| 1.4 | 18.7 | 8.5  | 25.5 | 11.6 | 14.1 | 22.1 | 1.1  | 2.5  | 14.4 | 1.7  | 37.6 |
| 0.6 | 2.2  | 39.0 | 0.3  | 15.0 | 11.0 | 7.3  | 22.9 | 1.7  | 0.1  | 1.1  | 0.6  |
| 9.0 | 1.7  | 7.0  | 20.1 | 0.4  | 2.8  | 14.1 | 9.9  | 10.4 | 10.7 | 30.0 | 3.6  |
| 5.6 | 30.8 | 13.3 | 4.2  | 25.5 | 3.4  | 11.9 | 21.5 | 27.6 | 36.4 | 2.7  | 64.0 |
| 1.5 | 2.5  | 27.4 | 1.0  | 27.1 | 20.2 | 16.8 | 5.3  | 9.7  | 27.5 | 2.5  | 27.0 |

In order to compare the distributions, we calculated the MLEs of the parameters,  $-2$  log-likelihood ( $-2\ell$ ) and the goodness of fit statistics measures namely: Akaike Information Criteria (AIC), and Bayesian Information Criteria(BIC), Consistent Akaike Information Criteria (CAIC), and Kolmogorov-Smirnov statistic (K-S) for all distributions and listed in Table 3. The KGIP distribution has the lowest AIC, CAIC, BIC and K-S among all the fitted other distributions. **All required computations are performed using Mathcad package.** From the results, the proposed distribution is the best model under these data than other competitive distributions.

**Table 3.**MLEs,  $-2$ log-likelihood, AIC, CAIC, and K-S.

| Model       | a            | b            | $\alpha$     | $\beta$      | $-2\ell$     | AIC          | BIC          | CAIC         | K-S          |
|-------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| Pareto      | 1            | 1            | 0.2438       | 0.1          | 606.2        | 608.2        | 610.4        | 608.2        | 0.332        |
| EP          | 2.8797       | 1            | 0.4241       | 0.1          | 574.6        | 578.6        | 583.2        | 578.8        | 0.198        |
| BP          | 3.1473       | 85.7508      | 0.0088       | 0.1          | 567.4        | 573.4        | 580.3        | 573.8        | 0.174        |
| KP          | 2.8553       | 85.8468      | 0.0528       | 0.1          | 542.4        | 548.4        | 555.3        | 548.8        | 0.170        |
| <b>KGIP</b> | <b>1.112</b> | <b>2.712</b> | <b>0.865</b> | <b>2.191</b> | <b>507.7</b> | <b>515.8</b> | <b>524.9</b> | <b>516.4</b> | <b>0.168</b> |

## 6. Conclusion

In this article, we introduce a new generalization of the inverse Pareto distribution, termed as KGIP distribution, which can be quite flexible in analyzing real data in different fields. Study several statistical properties of the proposed distribution. Maximum likelihood estimation has been used to estimate the model parameters. The simulation study is used to assess the performance of the estimated parameters. The applicability of the proposed distribution is examined by applying to a real data set. The KGIP distribution provides better fit than other compared distributions.

## Appendix A

Elements of the information matrix  $J(\underline{\xi}) = \{U_{s,t}\}$ , where  $s, t = (a, b, \alpha, \beta)$  are given by:

$$U_{aa} = \frac{-n}{a^2} - (b-1) \sum_{i=1}^n \left[ \frac{\alpha^2 z_i^{\alpha a} \log(z_i)^2}{(1-z_i^{\alpha a})} \left[ 1 + \frac{z_i^{\alpha a}}{(1-z_i^{\alpha a})} \right] \right],$$

$$U_{ab} = -\alpha \sum_{i=1}^n \left[ \frac{z_i^{\alpha a} \log(z_i)}{(1-z_i^{\alpha a})} \right],$$

$$U_{a\alpha} = \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \log(\beta + x_i) - (b-1) \sum_{i=1}^n \left[ \frac{z_i^{\alpha a} \log(z_i)}{(1-z_i^{\alpha a})} \left[ 1 + \alpha a \log(z_i) + \frac{\alpha a z_i^{\alpha a} \log(z_i)}{(1-z_i^{\alpha a})} \right] \right],$$

$$U_{a\beta} = -\alpha \sum_{i=1}^n (\beta + x_i)^{-1} + (b-1) \sum_{i=1}^n \left[ \frac{\alpha z_i^{\alpha a}}{(\beta + x_i)(1-z_i^{\alpha a})} \left[ 1 + a \log(z_i) + \frac{\alpha a z_i^{\alpha a} \log(z_i)}{(1-z_i^{\alpha a})} \right] \right],$$

$$U_{bb} = \frac{-n}{b^2},$$

$$U_{b\alpha} = -a \sum_{i=1}^n \left[ \frac{z_i^{\alpha a} \log(z_i)}{(1-z_i^{\alpha a})} \right],$$

$$U_{b\beta} = \alpha a \sum_{i=1}^n \left[ \frac{z_i^{\alpha a}}{(\beta + x_i)(1-z_i^{\alpha a})} \right],$$

$$U_{\alpha\alpha} = \frac{-n}{\alpha^2} - (b-1) \sum_{i=1}^n \left[ \frac{a^2 z_i^{\alpha a} \log(z_i)^2}{(1-z_i^{\alpha a})} \left[ 1 + \frac{z_i^{\alpha a}}{(1-z_i^{\alpha a})} \right] \right],$$

$$U_{\alpha\beta} = -a \sum_{i=1}^n (\beta + x_i)^{-1} + (b-1) \sum_{i=1}^n \left[ \frac{a z_i^{\alpha a}}{(\beta + x_i)(1-z_i^{\alpha a})} \left[ 1 + \alpha a \log(z_i) + \frac{\alpha a z_i^{\alpha a} \log(z_i)}{(1-z_i^{\alpha a})} \right] \right],$$

and

$$U_{\beta\beta} = \frac{-n}{\beta^2} + (\alpha a + 1) \sum_{i=1}^n (\beta + x_i)^{-2} - (b-1) \sum_{i=1}^n \left[ \frac{\alpha a z_i^{\alpha a}}{(\beta + x_i)^2 (1 - z_i^{\alpha a})} \left[ (\alpha a + 1) + \frac{z_i^{\alpha a}}{(1 - z_i^{\alpha a})} \right] \right].$$

where  $z_i = x_i (\beta + x_i)^{-1}$ .

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