Kumaraswamy Generalized Inverse Pareto Distribution

Abstract: This study introduces a new distribution referred to as Kumaraswamy generalized inverse Pareto distribution, which contains a number of distributions as special cases.The proposed distribution is more flexible and is applicable in the study of variety of fields. Several properties of the proposed distribution, including explicit expressions for the quantile, moments, moment generating function and are studied. Parameters of the proposed distribution have been estimated using maximum likelihood method and observed information matrix is derived. A real data set is used to compare the proposed distribution with widely known distributions.

Key words: Kumaraswamy generalized distribution, inverse Pareto distribution, quantile, moment generating function, maximum likelihood.

Introduction

The Pareto distribution is the most popular model for analyzing heavy tailed phenomena. The Pareto distribution was first proposed by Pareto (1986) as a model for the distribution of incomes and other financial variables, and other phenomena. The Pareto distribution has a wide range of applications in several fields such as life testing, economics, finance, engineering, and etc…(Evans et al. (2000)). Many generalized distributions of the Pareto distribution can be found in the literature. A few examples of these distributions are the beta-Pareto distribution by Akinsete et al. (2008), the Kumaraswamy- Pareto distribution by Bourguignon et al. (2013), the Kumaraswamy-generalized exponentiated Pareto distribution by Shams (2013), the Kumaraswamy transmuted Pareto distribution by Chhetri et al. (2017).

Klugman et al. (2012) introduced the cumulative distribution function (cdf) of the two-parameter inverse Pareto (IP) distribution as follows:

$$G(x)=\left(\frac{x}{\beta+x}\right)^{\alpha}\quad,\quad \alpha,\beta,x>0, \tag{1}$$

where α is shape parameter and β is scale parameter. Some authors dealing with many families of some well-known distributions which are more flexibility for modeling several types of data. Classical distributions do not provide adequate fits to the real data which are highly skewed. To overcome this drawback several methods of introducing additional shape parameters, and generating new families of distributions are available in the statistical literature. Some well-known generators are: the generalized-exponential by Gupta and Kundo (1999), Kumaraswamy generalized distribution by Cordeiro and de Castro (2011), generalized beta-generated by Alexander et al. (2012), weibull-generated by Bourguignon et al. (2014), Kumaraswamy weibull-generated by Hassan and Elgarhy (2016), generalized additive weibull-generated by Hassan et al. (2017), inverse weibull-generated Hassan and Nassr (2018), odd inverse Pareto-generated by Aldahlan et al. (2019), modified weibull-generated by Abdelall (2019) and more.

The aim of this paper is to propose a new and more flexible distribution, called Kumaraswamy generalized inverse Pareto (KGIP) distribution and provides expansions expressions for its cumulative and density functions.Numerous statistical properties of the KGIP distribution including quantile function, order statistics, moments and moment generating function are studied in the following section. The next section provides estimation of the parameters by maximum likelihood method in addition to simulation studies. Real data application is used to check the effectivenessof the proposed distribution. Finally, conclusions are provided in the last section.

Kumaraswamy Generalized Inverse Pareto Distribution

Cordeiro and de Castro (2011) defined the cdf of the Kumaraswamy generalized distribution as follows:

$$F(x) = 1 - \left[1 - (G(x))^a\right]^b, \ a, b > 0, \tag{2}$$

where a and b are additional shape parameters.Inserting (1) in (2), the cdf of the Kumaraswamy generalized inverse Pareto (KGIP) distribution with parameter $\underline{\xi} = (\mathrm{a}, \mathrm{b}, \alpha, \beta)$ is given by:

$$F_{KGIP}(x, \underline{\xi}) = 1 - \left[1 - \left(\frac{x}{\beta + x}\right)^{\alpha\, a}\right]^b, \ x > 0. \tag{3}$$

The probability density function (pdf) corresponding to (3), will be

$$f_{KGIP}(x,\underline{\xi}) = a\,b\,\alpha\,\beta \frac{x^{\alpha a-1}}{(\beta+x)^{\alpha a+1}} \left[1-\left(\frac{x}{\beta+x}\right)^{\alpha a}\right]^{b-1}, \ \alpha,\beta,a,b>0,\ x>0, \tag{4}$$

where α, a and b are shape parameters and β is scale parameter.

Special Distributions:

The following distributions are special sub-models of the KGIP distribution. If $b=1$ in (4), the KGIP reduces to the exponentiated inverse Pareto distribution with parameters a, α, and β. If $a=1$ in (4), the KGIP reduces to the exponentiated inverse Pareto distribution with parameters b,α, and β. If $b=a=1$ in (4), the KGIP reduces to the inverse Pareto distribution with parameters α and β.

The survival and hazard rate functions of the KGIP distribution are obtained, respectively as follows:

$$S_{KGIP}(x,\underline{\xi}) = \left[1-\left(\frac{x}{\beta+x}\right)^{\alpha a}\right]^{b},$$

and

$$h_{KGIP}(x,\underline{\xi}) = a\,b\,\alpha\,\beta \frac{x^{\alpha a-1}}{(\beta+x)^{\alpha a+1}} \left[1-\left(\frac{x}{\beta+x}\right)^{\alpha a}\right]^{-1}.$$

Fig 1 and 2 demonstrate the graphs of pdf and hazard function of the KGIP distribution for different values of parameters.

Expansions for cdf and pdf

Here, we give simple expansions for the KGIP cdf and pdf respectively. By using the generalized binomial theorem

$$(1+z)^{\nu} = \sum_{i=0}^{\infty} \binom{\nu}{i} z^{i}, \ 0<z<1. \tag{5}$$

Equation (3), can be rewritten as follows:

$$F_{KGIP}(x) = 1-\sum_{i=0}^{\infty}(-1)^{i}\binom{b}{i}\left(\frac{x}{\beta+x}\right)^{\alpha a i} = 1-\sum_{i=0}^{\infty}\eta_i\,\tau(x,\underline{\theta}), \tag{6}$$

where

$\eta_i = (-1)^i \binom{b}{i}$, and $\tau(x,\underline{\theta})$ denotes the Inverse Pareto cdf with parameters $\underline{\theta} = (\alpha a i, \beta)$. Now, using the power series (5) in the last term of (4), we get

$$f_{KGIP}(x) = \frac{b}{(i+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \left(\frac{\alpha\, a\,(i+1)\beta\, x^{\alpha\, a\,(i+1)-1}}{(\beta + x)^{\alpha\, a\,(i+1)+1}} \right) \tag{7}$$
$$= k_i\, \psi(x, \underline{\vartheta}),$$

where

$k_i = \dfrac{b}{(i+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i}$, and $\psi(x,\underline{\vartheta})$ denotes the Inverse Pareto distribution with parameters $\underline{\vartheta} = (\alpha a(i+1), \beta)$. Thus, the KGIP density function can be expressed as an infinite linear combination of the Inverse Pareto density. Thus, some of its mathematical properties can be obtained directly from those properties of the Inverse Pareto distribution.

Quantile function and simulation

Quantile function is defined as an inverse of the distribution function. The quantile function of the KGIP distribution is the solution of the following equation:

$$Q(q) = \beta \left[1 - (1-q)^{\frac{1}{b}} \right]^{\frac{1}{\alpha\, a}} \left[1 - \left[1 - (1-q)^{\frac{1}{b}} \right]^{\frac{1}{\alpha\, a}} \right]^{-1}, \quad 0 < q < 1. \tag{8}$$

Median of KGIP distribution can be obtained by putting q=0.5, that is

$$Q(0.5) = \beta \left[1 - 0.5^{\frac{1}{b}} \right]^{\frac{1}{\alpha\, a}} \left[1 - \left[1 - 0.5^{\frac{1}{b}} \right]^{\frac{1}{\alpha\, a}} \right]^{-1}.$$

For Simulating the KGIP random variable, let q be a uniform variate on the unit interval (0,1). Thus, by means of the inverse transformation method, we consider the random variable X given by:

$$X = \beta \left[1 - (1-U)^{\frac{1}{b}} \right]^{\frac{1}{\alpha\, a}} \left[1 - \left[1 - (1-U)^{\frac{1}{b}} \right]^{\frac{1}{\alpha\, a}} \right]^{-1},$$

Which follows (4), i.e, X distributed $\mathrm{KGIP}(\mathrm{a}, \mathrm{b}, \alpha, \beta)$.

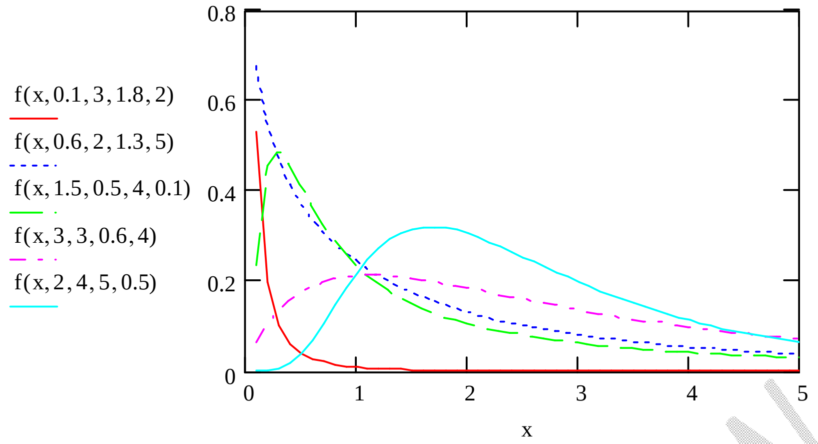

Figure 1. The pdf of KGIP (a, b, α, β) distribution for various values of parameters.

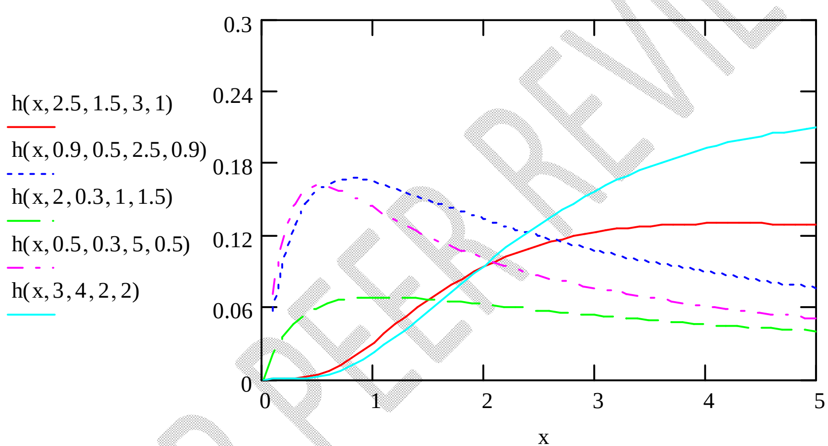

Figure 2. Increasing, decreasing shapes of hazard function of KGIP (a, b, α, β) distribution.

Skewness and Kurtosis

Kenny and Keeping (1962) proposed skewness based on quartiles called the Bowleyskewness which is defined as follows:

$$\mathrm{BS} = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}.$$

Further, the Moors kurtosis (Moors (1988)) based on octiles is defined as follows:

$$\mathrm{MK} = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)},$$

where $Q(.)$ is the quantile function. Figure 3 illustrates plots of the skewness and kurtosis of the KGIP distribution for some choices of the parameter b as a function of

α and fixed values of a and β. These plots indicate that the skewness and kurtosis decrease when $b = 0.5, 0.8, 1, 2$ (increases) for fixed a and β.

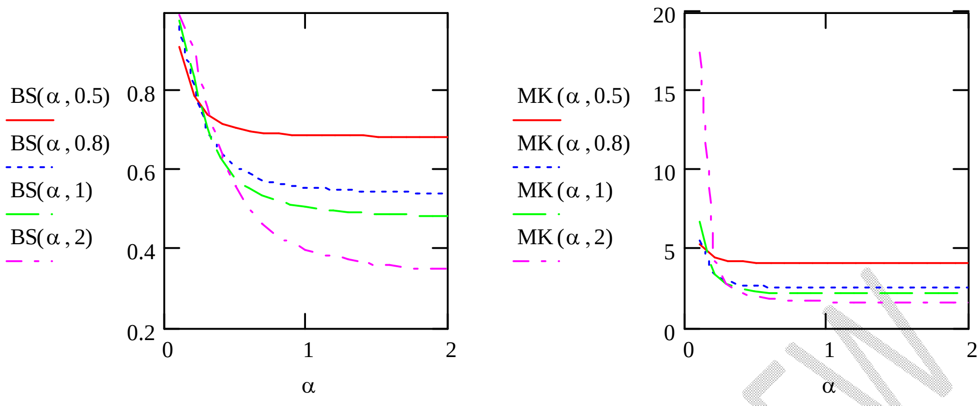

Figure3. plots of skewness and kurtosis for KGIP distribution based on quantile.

Moments

The rthmoment for KGIP random variable X is given by:

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r \, f_{KGIP}(x)\, dx$$

Using (7), we have

$$\mu_r' = \int_0^{\infty} x^r \, k_i \, \frac{\alpha\, a\,(i+1)\,\beta\, x^{\alpha\, a\,(i+1)-1}}{(\beta + x)^{\alpha\, a\,(i+1)+1}}\, dx,$$

Let $z = \dfrac{x}{(\beta + x)}$, after simplification, we obtain

$$\mu_r' = \alpha\, a\, \beta^r \,(i+1)\, k_i \int_0^1 z^{\alpha\, a\,(i+1)+r-1}\,(1-z)^{-r}\, dz,$$

Using beta function, we get

$$\mu_r' = \alpha\, a\, \beta^r\,(i+1)\, k_i\, B(\alpha\, a\,(i+1) + r, 1 - r)$$

$$= \beta^r k_i \frac{\Gamma(\alpha\, a\,(i+1) + r)\,\Gamma(1-r)}{\Gamma(\alpha\, a\,(i+1))}, \qquad -\alpha\, a\,(i+1) < r < 1,$$

where $B(c,d) = \int_0^1 z^{c-1}(1-z)^{d-1} dz = \dfrac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}$ is the beta function, and $\Gamma(.)$ is the gamma function. If r is a negative integer, the rth moment is obtained as follows:

$$\mu_r' = \beta^r k_i \frac{(-r)!}{(\alpha\, a\,(i+1) - 1)\ldots(\alpha\, a\,(i+1) + r)}.$$

The r^th incomplete moment for KGIP random variable X is then equal to:

$$w_s(t) = \int_0^t x^s \; f_{KGIP}(x)\, dx$$

$$= \alpha\, a\, \beta^s\, (i+1)\, k_i\; B\left(\frac{t}{\beta + t},\; \alpha\, a\,(i+1) + s,\; 1 - s\right), \quad -\alpha\, a\,(i+1) < r < 1, \tag{9}$$

where $B(x, c, d) = \int_0^x z^{c-1} (1-z)^{d-1} dz$ is the incomplete beta function.

The moment generating function of the KGIP distribution is given by:

$$M_X(t) = E\left[e^{tx}\right] = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{r=0}^{\infty} \frac{t^r}{r!} \beta^r k_i \frac{\Gamma(\alpha\, a\,(i+1) + r)\Gamma(1-r)}{\Gamma(\alpha\, a\,(i+1))}.$$

Order Statistic

Let $X_1, X_2, ..., X_n$ be a simple random sample of size n from KGIP($\underline{\xi}$) with cdf $F_{KGIP}(x; \underline{\xi})$ and pdf $f_{KGIP}(x; \underline{\xi})$ given by (3) and (4) respectively. Let $X_{1:n} \le X_{2:n} \le ..., X_{n:n}$ denote the order statistics obtained from this sample. The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x; \underline{\xi}) = \frac{1}{B(r, n-r+1)} f(x; \underline{\xi}) \left[F(x; \underline{\xi})\right]^{r-1} \left[1 - F(x; \underline{\xi})\right]^{n-r}, \tag{10}$$

where $f(x; \underline{\xi})$ and $F(x; \underline{\xi})$ are the pdf and cdf of KGIP($\underline{\xi}$) distribution given by (4) and (3) respectively and B(., .) is the beta function, also we define first order statistics $X_{1:n} = \min(X_1; X_2; ...; X_n)$,and the last order statistics as $X_{n:n} = \max(X_1; X_2; ...; X_n)$. Since $0 < F(x; \underline{\xi}) < 1$ for $x > 0$, we can use the binomial expansion of $\left[1 - F(x; \underline{\xi})\right]^{n-r}$ given as follows

$$\left[1 - F(x; \underline{\xi})\right]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \left[F(x; \underline{\xi})\right]^i. \tag{11}$$

Substituting from (11) into (10), we obtain

$$f_{r:n}(x; \underline{\xi}) = \frac{1}{B(r, n-r+1)} f(x; \underline{\xi}) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \left[F(x; \underline{\xi})\right]^{i+r-1}. \tag{12}$$

Based on (6) and (7), we can write

$$f(x,\underline{\xi})\,F(x,\underline{\xi})^{i+r-1} = a\,b \sum_{k,j,t,l=0}^{\infty} \frac{(-1)^{k+l}\,\alpha\,\beta}{(\beta+x)^{\alpha\,a(l+1)+1}} \binom{(i+r-1)}{k} \binom{b(l+1)-1}{l} x^{\alpha\,a(l+1)-1}. \qquad (13)$$

Inserting (13) in (12), the pdf of $X_{r:n}$ written as

$$f_{r;n}(x,\underline{\xi}) = \sum_{k,l,i=0}^{\infty} \omega_{k,l,i}\, \psi(x,\underline{\vartheta}),$$

where $\omega_{k,j,t,l} = \sum_{i=0}^{n-r} \frac{(-1)^{k+l+i}\,a\,b}{B(r,n-r+1)} \binom{n-r}{i} \binom{(i+r-1)}{k} \binom{b(l+1)-1}{l}$, $\psi(x,\underline{\vartheta})$ denotes the Inverse Pareto distribution with parameters $\underline{\vartheta} = (\alpha a(l+1), \beta)$. Thus, the pdf of the KGIP order statistics is a linear combination of the Inverse Pareto density.

Mean deviations

The mean deviations about the mean and the median are useful measures of variation for population. Let μ and m are the mean and the median of the KGIP distribution, respectively. The mean deviations about mean and about the median are defined by:

$$D(\mu) = \int_0^{\infty} |\,x-\mu\,|\, f_{KGIP}(x)\,dx = 2\mu\,F(\mu) - 2\,w_1(\mu),$$

and

$$D(m) = \int_0^{\infty} |\,x-m\,|\, f_{KGIP}(x)\,dx = \mu - 2\,w_1(m),$$

where $w_1(.)$ is the first incomplete moment can be calculated from (9) for $s=1$.

Estimation of the parameters of KGIP distribution

In this section, we discuss the estimation of the parameters of the KGIP distribution by using the method of maximum likelihood. Let $X_1, X_2, ..., X_n$ be a random sample from KGIP$(\underline{\xi})$ where $\underline{\xi} = (\mathrm{a}, \mathrm{b}, \alpha, \beta)$ be the vector of the parameters, the log-likelihood function, ℓ , is given by:

$$\ell(\underline{\xi}) = n\log a + n\log b + n\log\alpha + n\log\beta + (\alpha\,a-1)\sum_{i=1}^{n}\log(x_i) - (\alpha\,a+1)\sum_{i=1}^{n}\log(\beta+x_i)$$

$$+ (b-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{x_i}{\beta+x_i}\right)^{\alpha\,a}\right]. \qquad (14)$$

The score vector $U(\underline{\xi}) = (\partial\ell/\partial a, \partial\ell/\partial \mathrm{b}, \partial\ell/\partial\alpha, \partial\ell/\partial\beta)^T$, where the components to the parameters in $\underline{\xi}$ are given by differentiating (14). By setting $z_i = x_i\,(\beta + x_i)^{-1}$,

$$U_a = \frac{n}{a} + \alpha \sum_{i=1}^{n} \log(x_i) - \alpha \sum_{i=1}^{n} \log(\beta + x_i) - (b-1) \sum_{i=1}^{n} \left[\frac{\alpha\, z_i^{\alpha a} \log(z_i)}{(1 - z_i^{\alpha a})} \right], \tag{15}$$

$$U_b = \frac{n}{b} + \sum_{i=1}^{n} \log(1 - z_i^{\alpha a}), \tag{16}$$

$$U_\alpha = \frac{n}{\alpha} + a \sum_{i=1}^{n} \log(x_i) - a \sum_{i=1}^{n} \log(\beta + x_i) - (b-1) \sum_{i=1}^{n} \left[\frac{a\, z_i^{\alpha a} \log(z_i)}{(1 - z_i^{\alpha a})} \right], \tag{17}$$

and

$$U_\beta = \frac{n}{\beta} - (\alpha\, a + 1) \sum_{i=1}^{n} (\beta + x_i)^{-1} + (b-1) \sum_{i=1}^{n} \left[\frac{\alpha\, a\, z_i^{\alpha a}}{(\beta + x_i)(1 - z_i^{\alpha a})} \right]. \tag{18}$$

The maximum likelihood estimates (MLEs) of the unknown parameters are obtained by setting system of non-linear (15) - (18) equations to zero and solve them simultaneously. From equation (16), we obtain the maximum likelihood estimate of b in a closed form as follows

$$\hat{b} = \frac{-n}{\sum_{i=1}^{n} \log\left[1 - z_i^{\hat{\alpha}\,\hat{a}}\right]}, \tag{19}$$

When estimates $(\hat{a}, \hat{\alpha})$ known. Substituting from (19) into (15), (17), and (18) , we get the MLEs $(\hat{a}, \hat{\alpha}, \hat{\beta})$. These equations cannot be solved analytically and statistical software can be used to solve them simultaneously. For interval estimation of the parameters, we require the 4×4 observed information matrix $J(\underline{\xi}) = \{U_{s,t}\}$, where $s,t = (\mathrm{a}, \mathrm{b}, \alpha, \beta)$ given in Appendix A. To construct the approximate confidence intervals for the parameters a, b, α, and β of the KGIP distribution, the multivariate normal $N_4\left(0, J(\hat{\underline{\xi}})^{-1}\right)$ distribution can be used. Here, $J(\hat{\underline{\xi}})$ is the total observed information matrix evaluated at $\hat{\underline{\xi}}$. The asymptotic $100(1-\eta)\%$ confidence intervals for parameters a, b, α, and β are respectively given by:

$\hat{a} \pm Z_{\eta/2}\sqrt{\mathrm{var}(\hat{a})}$, $\hat{b} \pm Z_{\eta/2}\sqrt{\mathrm{var}(\hat{b})}$, $\hat{\alpha} \pm Z_{\eta/2}\sqrt{\mathrm{var}(\hat{\alpha})}$, and $\hat{\beta} \pm Z_{\eta/2}\sqrt{\mathrm{var}(\hat{\beta})}$, where $Z_{(\eta/2)}$ is the upper $(\eta/2)^{th}$ percentile of the standard normal distribution and the $Var(.)'s$ denote the diagonal elements of $J(\hat{\underset{\sim}{\xi}})^{-1}$ corresponding to a, b, α, and β.

Simulation study

Simulation study has been performed for average MLEs, Mean Square Error (MSE), and Bias. The KGIP random number generation was performed using the inversion method for $N = 1000$ replications of size $n = 50\,(50)\,200$. The true parameter values used in the data generation processes are followed in Table 1. Bias and MSE are calculated by: $Bias = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\xi}_i - \xi\right)$, $MSE = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\xi}_i - \xi\right)^2$,

where $\xi = (a, b, \alpha, \beta)$. Simulation results were obtained for several combinations of parameters. Table 1 lists the means of the MLEs of the four parameters KGIP with respective MSEs, and Biases. It can be illustrated from table that, the MSEs and Bias decreases for all parameter combinations when sample size increase.

Table 1: Mean estimates, corresponding MSE and Bias of the MLE of (a, b, α, β).

Actual Values	n	Mean				Bias				MSE			
		$(\hat{a})$	$(\hat{b})$	$(\hat{\alpha})$	$(\hat{\beta})$	$(\hat{a})$	$(\hat{b})$	$(\hat{\alpha})$	$(\hat{\beta})$	$(\hat{a})$	$(\hat{b})$	$(\hat{\alpha})$	$(\hat{\beta})$
$a = 0.5$	50	0.734	1.955	2.158	0.509	0.234	1.455	0.668	0.093	0.055	2.122	0.437	0.00014
$b = 0.5$	100	0.716	1.887	2.160	0.511	0.216	1.387	0.650	0.017	0.047	1.924	0.436	0.00013
$\alpha = 1.5$	150	0.714	1.861	2.140	0.510	0.213	1.361	0.640	0.011	0.046	1.853	0.410	0.00012
$\beta = 0.5$	200	0.707	1.848	2.130	0.509	0.207	1.348	0.631	0.010	0.043	1.818	0.395	0.00012
$a = 0.5$	50	1.249	1.955	1.250	0.509	0.749	0.455	0.750	0.093	0.561	0.212	0.562	0.00015
$b = 1.5$	100	1.236	1.885	1.237	0.511	0.736	0.387	0.737	0.018	0.542	0.150	0.543	0.00014
$\alpha = 0.5$	150	1.232	1.860	1.233	0.511	0.732	0.361	0.733	0.015	0.536	0.131	0.537	0.00013
$\beta = 0.5$	200	1.229	1.848	1.229	0.510	0.728	0.348	0.729	0.012	0.531	0.121	0.532	0.00012
$a = 0.5$	50	1.249	1.955	1.249	0.509	0.749	1.455	0.749	0.009	0.562	2.122	0.561	0.00015
$b = 0.5$	100	1.236	1.887	1.237	0.510	0.736	1.387	0.737	0.016	0.542	1.924	0.543	0.00014
$\alpha = 0.5$	150	1.232	1.861	1.233	0.510	0.732	1.361	0.733	0.014	0.537	1.853	0.536	0.00012
$\beta = 0.5$	200	1.229	1.848	1.229	0.509	0.729	1.348	0.729	0.011	0.531	1.818	0.532	0.00011
$a = 0.5$	50	0.721	1.955	2.163	0.509	0.221	0.455	0.663	0.093	0.049	0.212	0.441	0.00015
$b = 1.5$	100	0.714	1.887	2.142	0.511	0.214	0.387	0.642	0.018	0.046	0.150	0.412	0.00014
$\alpha = 1.5$	150	0.712	1.861	2.135	0.510	0.212	0.361	0.635	0.011	0.045	0.131	0.403	0.00013
$\beta = 0.5$	200	0.710	1.848	2.129	0.510	0.210	0.348	0.629	0.014	0.044	0.121	0.395	0.00012
$a = 0.5$	50	0.703	2.507	2.011	0.777	0.203	1.007	0.511	-0.023	0.041	1.023	0.262	0.00063
$b = 1.5$	100	0.693	2.424	2.002	0.776	0.193	0.924	0.508	-0.024	0.037	0.856	0.259	0.00059
$\alpha = 1.5$	150	0.687	2.395	2.008	0.776	0.187	0.895	0.505	-0.024	0.035	0.801	0.257	0.00052
$\beta = 0.8$	200	0.685	2.382	2.005	0.776	0.185	0.882	0.502	-0.025	0.034	0.778	0.255	0.00051
$a = 0.5$	50	0.608	3.991	1.874	1.873	0.114	2.684	0.386	-0.124	0.013	6.211	0.149	0.018
$b = 1.5$	100	0.614	4.184	1.882	1.876	0.108	2.491	0.384	-0.127	0.012	6.010	0.147	0.017
$\alpha = 1.5$	150	0.603	3.931	1.883	1.870	0.103	2.431	0.383	-0.131	0.011	5.912	0.146	0.016
$\beta = 2$	200	0.601	3.908	1.886	1.870	0.101	2.408	0.374	-0.133	0.010	5.800	0.144	0.014

Data Analysis

In this section, we use a real data set to compare the fits of the KGIP distributionwith some distributions such as the Kumaraswamy-Pareto (KP), Exponentiated Pareto (EP), Beta Pareto (BP), and Pareto distributions (Bourguignon et al. (2013)). The data set correspond to the exceedances of flood peaks of the Wheaton River near Careross in Yukon Territory, Canada. The data consist of 72 excessdances for the years 1958-1984, rounded to one decimal place. They were analyzed by Choulakian and Stephens (2001) and are listed in Table 2.

Table2: Exceedances of Wheaton River flood data.

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0	12.0	9.3
1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1	2.5	14.4	1.7	37.6
0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9	1.7	0.1	1.1	0.6
9.0	1.7	7.0	20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.0

Table 3 provide MLEs and some descriptive statistics as the values of the -2 log-likelihood (-2ℓ), Akaike Information Criteria (AIC), and Bayesian Information Criteria(BIC), Consistent Akaike Information Criteria (CAIC), and Kolmogorove-Smirnov statistic (K-S). Since the values of AIC, CAIC, BIC and K-S are smaller for the KGIP distribution compared with those values of the other distributions. The proposeddistribution provides a better fitto these data than the KP, EP, BP, and Pareto distributions.

Table 3.MLEs,-2log-likelihood, AIC, CAIC, and K-S.

Model	a	b	α	β	-2ℓ	AIC	BIC	CAIC	K-S
Pareto	1	1	0.2438	0.1	606.2	608.2	610.4	608.2	0.332
EP	2.8797	1	0.4241	0.1	574.6	578.6	583.2	578.8	0.198
BP	3.1473	85.7508	0.0088	0.1	567.4	573.4	580.3	573.8	0.174
KP	2.8553	85.8468	0.0528	0.1	542.4	548.4	555.3	548.8	0.170
KGIP	**1.112**	**2.712**	**0.865**	**2.191**	**507.7**	**515.8**	**524.9**	**516.4**	**0.168**

Conclusion

The proposed distribution, termed as KGIP distribution, which extended of the inverse Pareto with two extra shape parameters. Different properties of the distribution have been derived including quantile, skewness, order statistics and moment generating function. Maximum likelihood estimation has been used to provide parameter estimates of the unknown parameters. The proposed distribution has been applied to real data set, which indicates its better fit as compared to other distributions.

Appendix A

The elements of the 4×4 observed information matrix $J(\underline{\xi})=\{U_{s,t}\}$, where $s,t=(a, b, \alpha, \beta)$ are given by:

$$U_{aa}=\frac{-n}{a^2}-(b-1)\sum_{i=1}^{n}\left[\frac{\alpha^2 z_i^{\alpha a}\log(z_i)^2}{(1-z_i^{\alpha a})}\left[1+\frac{z_i^{\alpha a}}{(1-z_i^{\alpha a})}\right]\right],$$

$$U_{ab}=-\alpha\sum_{i=1}^{n}\left[\frac{z_i^{\alpha a}\log(z_i)}{(1-z_i^{\alpha a})}\right],$$

$$U_{a\alpha}=\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\log(\beta+x_i)-(b-1)\sum_{i=1}^{n}\left[\frac{z_i^{\alpha a}\log(z_i)}{(1-z_i^{\alpha a})}\left[1+\alpha a\log(z_i)+\frac{\alpha a z_i^{\alpha a}\log(z_i)}{(1-z_i^{\alpha a})}\right]\right],$$

$$U_{a\beta}=-\alpha\sum_{i=1}^{n}(\beta+x_i)^{-1}+(b-1)\sum_{i=1}^{n}\left[\frac{\alpha z_i^{\alpha a}}{(\beta+x_i)(1-z_i^{\alpha a})}\left[1+a\log(z_i)+\frac{\alpha a z_i^{\alpha a}\log(z_i)}{(1-z_i^{\alpha a})}\right]\right],$$

$$U_{bb}=\frac{-n}{b^2},$$

$$U_{b\alpha}=-a\sum_{i=1}^{n}\left[\frac{z_i^{\alpha a}\log(z_i)}{(1-z_i^{\alpha a})}\right],$$

$$U_{b\beta}=\alpha a\sum_{i=1}^{n}\left[\frac{z_i^{\alpha a}}{(\beta+x_i)(1-z_i^{\alpha a})}\right],$$

$$U_{\alpha\alpha}=\frac{-n}{\alpha^2}-(b-1)\sum_{i=1}^{n}\left[\frac{a^2 z_i^{\alpha a}\log(z_i)^2}{(1-z_i^{\alpha a})}\left[1+\frac{z_i^{\alpha a}}{(1-z_i^{\alpha a})}\right]\right],$$

$$U_{\alpha\beta}=-a\sum_{i=1}^{n}(\beta+x_i)^{-1}+(b-1)\sum_{i=1}^{n}\left[\frac{a z_i^{\alpha a}}{(\beta+x_i)(1-z_i^{\alpha a})}\left[1+\alpha a\log(z_i)+\frac{\alpha a z_i^{\alpha a}\log(z_i)}{(1-z_i^{\alpha a})}\right]\right],$$

and

$$U_{\beta\beta}=\frac{-n}{\beta^2}+(\alpha a+1)\sum_{i=1}^{n}(\beta+x_i)^{-2}-(b-1)\sum_{i=1}^{n}\left[\frac{\alpha a z_i^{\alpha a}}{(\beta+x_i)^2(1-z_i^{\alpha a})}\left[(\alpha a+1)+\frac{z_i^{\alpha a}}{(1-z_i^{\alpha a})}\right]\right].$$

where $z_i=x_i(\beta+x_i)^{-1}$.

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