Short Research Article

On propositions pertaining to the Riemann Hypothesis

Abstract

In this paper, we discuss methods that lead to identifying non-zeroes of the Riemann zeta function.

1 Introduction

In this paper, we study the Riemann zeta function (Riemann [1859], Stein and Shakarchi [2010]) from the point of view of establishing a method to determine whether a point would qualify as a non-zero point. This is based on prior perspectives as presented in Basu [2022]. The results perhaps might be viewed as complementary to theorems in Mangoldt [1905], Hardy [1914] and Hardy and Littlewood [1921]. As noted in Basu [2022], the previous papers and theorems related to the Riemann Hypothesis may be found in Conrey [2003], Lagarias [2002], Bump et al. [2000], Platt and Trudgian [2021] and Borwein et al. [2008].

2 Riemann Hypothesis

We build on some results from Basu [2022]. The Riemann zeta function is defined as follows in this paper. Let us first define the region of interest explicitly as a subset of $\mathbb{R}^2$ as

$$S = \{(\sigma, t) \in \mathbb{R}^2 : \sigma \in (0,1); t \neq 0\}. \tag{1}$$

The riemann zeta function is defined for each $s \in S$ as

$$\zeta(s) := \sum_{n=1}^{\infty} (1 - \frac{1}{2^{1-s}}) \times \frac{(-1)^{n+1}}{n^s}, \tag{2}$$

where in the infinite sum, we have the vector

$$\frac{1}{n^s} = e^{-\sigma \ln(n)}(\cos(-t\ln(n)), \sin(-t\ln(n))), \tag{3}$$

obtained using Euler's formula (Stein and Shakarchi [2010]), multiplied by the vector $1 - \frac{1}{2^{1-s}}$ as defined by the binary operation of multiplication of complex numbers i.e. $\mathbb{R}^2$. Since for each $s \in S$, we have that $1 - \frac{1}{2^{1-s}} \neq 0$, it follows that the zeroes of ζ coincide exactly with the zeroes of the function ζ^* given only by the alternating Dirichlet sum (Stein and Shakarchi [2010]) i.e.

$$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}. \tag{4}$$

We now make a digression and prove a preliminary proposition that will be needed.

Proposition 2.1. *For each $\sigma \in (0,1)$,*

$$\sup_{x\in(0,1)} \frac{x}{e^{\sigma x}-1} = \frac{1}{\sigma}. \tag{5}$$

Proof. We note that by applying the fundamental theorem of calculus, for any differentiable f, g on $[0,1)$ such that $g(x) \neq 0$ for each x and $f(0) = g(0) = 0$, we have that for each $x > 0$,

$$\frac{f(x)}{g(x)} = \frac{\int_{(0,x)} f'(y)dy}{\int_{(0,x)} g'(y)dy}. \tag{6}$$

Now we define $f(x) = x$ and $g(x) = e^{\sigma x} - 1$. Since, $f'(x) \leq \frac{g'(x)}{\sigma}$, from 6, we have that

$$\frac{f(x)}{g(x)} \leq \frac{1}{\sigma}. \tag{7}$$

Further, by applying L'Hospital's rule, it follows that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{1}{\sigma}$. Hence, we obtain 5. □

We now study the defined function ζ^*. Suppose we define the vector Z_0 and for each $s \in S$, we define the sequence $\{Z(s)_n\}_{n\in\mathbb{Z}^+}$ as follows

$$Z_0 = (1,0); \tag{8}$$

$$Z_n(s) = \frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}; \text{ for each } n \in \mathbb{Z}^+. \tag{9}$$

Hence, the function ζ^* may expressed as

$$\zeta^*(s) = Z_0 + \sum_{n=1}^{\infty} Z_n(s). \tag{10}$$

We may now prove the following proposition.

Proposition 2.2. *The series in 10 converges absolutely.*

Proof. Let $s = (\sigma, t) \in S$. Given the nature of the cos and sine functions and the henceforth established symmetry of ζ^* we have the $\zeta^*(\sigma, t) = -\zeta^*(\sigma, -t)$. Hence, we assume without loss of generality that $t > 0$. Let $n_0 \in \mathbb{Z}^+$ such that for each $n \geq n_0$, we have that $(\ln(2n+1) - \ln(2n))t \leq \frac{\pi}{2}$. This means that beyond the point n_0, the angle in radians traversed in $\mathbb{R}^2$ between the vectors $\frac{1}{(2n+1)^s}$ and $\frac{1}{(2n)^s}$ which is the value $(\ln(2n+1) - \ln(2n))t$ is at most $\frac{\pi}{2}$ and is hence, an acute angle.

Now, define the angle

$$\theta_n = (\ln(2n+1) - \ln(2n))t. \tag{11}$$

Geometrically, we may prove using trignometric relations, that

$$\|Z_n(s)\| = \|\frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}\| \tag{12}$$

$$= \sqrt{\left(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\right)^2 + \frac{2}{(2n+1)^\sigma (2n)^\sigma}(1-\cos(\theta_n))}. \tag{13}$$

Again, geometrically we may prove that distance $\|\frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}\|$ is at most the difference $\left(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\right)$ plus the arc length $\theta_n \frac{1}{(2n+1)^\sigma}$, which corresponds to the circle centered at zero and has radius $\frac{1}{(2n+1)^\sigma}$. This is obtained using the triangular inequality. The arc length is greater than the distance between the vector $\frac{1}{(2n+1)^s}$ and the unique point z on the line segment conv$\{0, \frac{1}{(2n)^s}\}$ such that $\|z\| = \frac{1}{(2n+1)^\sigma}$. Note that $\|z - \frac{1}{(2n)^s}\| = \left(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\right)$. Then, we apply the triangular inequality with the vectors $\frac{1}{(2n+1)^s}$, $\frac{1}{(2n)^s}$ and z. Hence, we obtain that for each $n \geq n_0$,

$$\|Z_n(s)\| \leq \left(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\right) + \theta_n \frac{1}{(2n+1)^\sigma}. \tag{14}$$

We now apply proposition 2.1, by setting $x = (\ln(2n+1) - \ln(2n))$, we obtain the inequality

$$\theta_n \frac{1}{(2n+1)^\sigma} \leq \frac{t}{\sigma}\left(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\right) \tag{15}$$

Hence, for each $n \geq n_0$, we have that

$$\|Z_n(s)\| \leq \left(1 + \frac{t}{\sigma}\right)\left(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\right). \tag{16}$$

Since the series $\sum_{n\geq 1}\left(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\right)$ converges absolutely, it follows from 16 that the series in 10 converges absolutely. □

Based on the proof of the above proposition, we define the function

$$F(\sigma) := \sum_{n\geq 1}\left(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\right). \tag{17}$$

We prove the following proposition.

Proposition 2.3. *Suppose that* $s = (\sigma, t) \in S$ *such that* $t \in [-\frac{\pi}{2\ln(3/2)}, \frac{\pi}{2\ln(3/2)}]$. *If*

$$\left(1 + \frac{t}{\sigma}\right)F(\sigma) < 1, \tag{18}$$

then $\zeta(s) \neq 0$.

Proof. The proof follows from the proof of proposition 2.2. Assume, without any loss of generality that $t > 0$. Since $t \leq \frac{\pi}{2\ln(3/2)}$, this means that for each $n \geq 1$, the angle θ_n is acute. Hence, the upper bound in 16 applies. Hence, we have that

$$\sum_{n\geq 1}\|Z_n(s)\| \leq \left(1 + \frac{t}{\sigma}\right)F(\sigma) < 1. \tag{19}$$

Suppose for contradiction $\zeta(s) = 0$. Then, we get that $\|Z_0\| = \|\sum_{n\geq 1} Z_n(s)\| = 1$. Hence, we get from 19 that $1 > \sum_{n\geq 1} \|Z_n(s)\| \geq \|\sum_{n\geq 1} Z_n(s)\| = 1$, which is a contradiction. □

This implies the following proposition.

Proposition 2.4. *If* $s = (\sigma, t) \in [\frac{1}{2}, 1) \times \left(\frac{1-\sqrt{2}}{2}, \frac{\sqrt{2}-1}{2}\right)$*, then* $\zeta(s) \neq 0$.

Proof. Follows from proposition 2.3 since $F(\sigma) \leq \frac{1}{2^\sigma}$. □

The expectations representation In Basu [2022], it was shown that identifying zeroes of the riemann zeta function may be interpretated as a problem of identifying a zero expectation random vector. This may be expressed as follows. If a series $\sum_{n\geq 1} z_n$ is absolutely convergent, then we define on the set of positive integers $\mathbb{Z}^+$, a probability measure $\mu(\{n\}) := \frac{\|z_n\|}{\sum_{m\geq 1}\|z_m\|}$ and random vector $X(n) := z_n/\|z_n\|$. Then, we have that the expectation $\mathbb{E}_\mu[X] = 0$ if and only if $\sum_{n\geq 1} z_n = 0$. In this paper, we are interested in the sequence pertaining to 9 i.e $z_1 := Z_0$ and $z_n := Z_{n-1}(s)$ for each $n \geq 2$. The vector $Z_n(s)$ may be conveniently represented via geometry in polar form when θ_n is acute, as for the defined angle

$$\hat{\theta}_n := \tan^{-1}\left(\frac{\frac{1}{(2n+1)^\sigma}\sin(\theta_n)}{\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\cos(\theta_n)}\right), \tag{20}$$

we have that

$$Z_n(s) = \|Z_n(s)\|(\cos(-\ln(n)t + \hat{\theta}_n), \sin(-\ln(n)t + \hat{\theta}_n)). \tag{21}$$

Proposition 2.3 in this paper and Proposition 2.4 from Basu [2022] hence can be obtained by an application of a more general proposition, which we next prove. For any $z \in \mathbb{R}^2$, denote as $\theta(z) \in [0, 2\pi]$, the angle in radians, traversed by the vector z in its polar form.

Proposition 2.5. *Let* μ *be a probability measure on the unit circle* $\mathbb{S}^1$*. Then,*

$$\mathbb{E}_\mu[z] \neq 0 \text{ if there exist numbers } 0 \leq \theta' \leq \theta'' \leq 2\pi \text{ such that } \theta' - \theta \leq \frac{\pi}{2} \text{ and}$$
$$\mu(\{z : \theta' \leq \theta(z) \leq \theta''\}) > \frac{1}{1 + \cos\left(\frac{\theta''-\theta'}{2}\right)}. \tag{22}$$

Proof. Suppose that 22 holds. Then, by geometry, essentially there is enough weight on the arc $A = \{z : \theta' \leq \theta(z) \leq \theta''\}$ so that the expectation is non-zero. The conditional expectation $\mathbb{E}_\mu[z|A]$ is at least distance $\cos\left(\frac{\theta''-\theta'}{2}\right)$ away from 0 and the conditional expectation $\mathbb{E}_\mu[z|\mathbb{S}^1 \backslash A]$ is at most distance 1 away from 0. □

The above proposition may be applied to obtaining concentration bounds by computing the expected angle $\mathbb{E}_\mu[\theta(z)]$ traversed. One may be able to obtain probability lower bounds such as 22 possibly by re-defining the angle as traversed from an origin other than $(1,0)$ (see also Kuipers and Niederreiter [2012] for a similar problem).

The main idea here is to divide an absolutely convergent infinite sum $\sum_{n\geq 1} z_n$ into two parts $\sum_{n\in E} z_n$ and $\sum_{n\notin E} z_n$ and show that the two parts don't add up in some sense i.e. either the absolute values of the two parts are different; they are aligned at different angles; or one part bears higher weight on one half of a hyperplane than the other. We demonstrate this last idea as follows.

Suppose $\sum_{j\in J} z_j$ is a finite sum of vectors and suppose $q \neq 0$ is a hyperplane. Define $J^+ = \{j \in J : q.z_j > 0\}$ and $J^- = \{j \in J : q.z_j < 0\}$. Suppose that there exist sequences of pairwise disjoint sets $\{J_k^+\}_{k=1}^K \subseteq J^+$ and $\{J_k^-\}_{k=1}^K \subseteq J^-$ such that $\cup_{k=1}^K J_k^- = J^-$ and $q.(\sum_{j\in J_k^+} z_j + \sum_{j\in J_k^-} z_j) > 0$ for each $k \in \{1, ..., K\}$. Then, $\sum_{j\in J} z_j \neq 0$ since $q.(\sum_{j\in J} z_j) > 0$.

We prove the next proposition.

Proposition 2.6. *Let $t = \frac{\pi}{\ln(2)}$, then there exists $\sigma' \in (0,1)$ such that for each $\sigma \in (\sigma', 1)$, we have that $\zeta(\sigma, t) \neq 0$.*

Proof. Let $\sigma' \in (0,1)$ be such that

$$(1/2)^\sigma > (1/4)^\sigma + (1/7)^\sigma \text{ and } \frac{1 + \frac{\pi}{\sigma \ln(2)}}{8^\sigma} < 1. \tag{23}$$

for each $\sigma \in (\sigma', 1)$. Let $s = (\sigma, t)$ where $\sigma \in (\sigma', 1)$. Define the following vectors in $\mathbb{R}^2 : z_n := \frac{(-1)^{n+1}}{n^s}$ for $n \leq 7$ and define $z_8 := \sum_{n\geq 4} Z_n(s)$. Hence, $\zeta^*(s) = (\sum_{n=1}^7 z_n) + z_8$.

Now define the hyperplane $q = (1,0)$. Then, we have as strictly positive the values $q.z_1, q.z_2, q.z_3, q.z_5, q.z_6 > 0$. We also have $q.z_4, q.z_7 < 0$. Suppose that $q.z_8 < 0$. Now define $J = \{1, 2, ..., 8\}$ and the collections $\{\{1,2\}\}$ in J^+ and $\{\{4,7,8\}\}$ in J^-. Since $\|z_8\| \leq 1$ (by applying the upper bound in proposition 2.3 and 23) we may then show that $q.(z_1 + z_2 + z_4 + z_7 + z_8) > 0$ by applying 23. If on the other hand we have $q.z_8 \geq 0$, then we may define the collections $\{\{2\}\}$ in J^+ and $\{\{4,7\}\}$ in J^- and applying 23 show that $q.(z_2 + z_4 + z_7) > 0$. These facts may be proved geometrically by studying the defined points in $\mathbb{R}^2$ and the unit circle $\mathbb{S}^1$. □

References

Pathikrit Basu. On the riemann hypothesis. *Asian Journal of Probability and Statistics*, 20(4):51–56, 2022.

Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*. Springer, 2008.

Daniel Bump, Kwok-Kwong Choi, Pär Kurlberg, and Jeffrey Vaaler. A local riemann hypothesis, i. *Mathematische Zeitschrift*, 233(1):1–18, 2000.

J Brian Conrey. The riemann hypothesis. *Notices of the AMS*, 50(3):341–353, 2003.

Godfrey H Hardy and John E Littlewood. The zeros of riemann's zeta-function on the critical line. *Mathematische Zeitschrift*, 10(3):283–317, 1921.

Godfrey Harold Hardy. Sur les zéros de la fonction ζ (s) de riemann. *CR Acad. Sci. Paris*, 158(1914):1012, 1914.

Lauwerens Kuipers and Harald Niederreiter. *Uniform distribution of sequences*. Courier Corporation, 2012.

Jeffrey C Lagarias. An elementary problem equivalent to the riemann hypothesis. *The American Mathematical Monthly*, 109(6):534–543, 2002.

H von Mangoldt. Zur verteilung der nullstellen der riemannschen funktion........ *Mathematische Annalen*, 61:1–19, 1905.

Dave Platt and Tim Trudgian. The riemann hypothesis is true up to 3· 10 12. *Bulletin of the London Mathematical Society*, 53(3):792–797, 2021.

Bernhard Riemann. On the number of primes less than a given magnitude. *Monatsberichte der Berliner Akademie*, pages 1–10, 1859.

Elias M Stein and Rami Shakarchi. *Complex analysis*, volume 2. Princeton University Press, 2010.