Modified Burr Xii Distribution: Properties and Applications

Abstract

Creating suitable, data-friendly models to interpret statistical data has been a goal of numerous researchers. Even new generalizations of models with several parameters are developed to handle complex data. The Burr XIII distribution has been utilized extensively in software reliability, failure censored plans, future observables prediction, ball-bearing loss, tax revenue, etc. Abdel-Ghaly et al.(1997), Wu and Yu (2005)), AL-Hussaini (2003), Shao Q. et al. (2004), Mead M. E. (2014). This article suggests changing the Burr XIII (MB XIII) distribution. The projected distribution is more adaptable and manageable than the Burr type XIII (B XIII) distribution and its parent. The proposed distribution's characteristics are deduced in some cases. To demonstrate its utility, we compare Burr XIII and its sub-model to real-world data sets, ball-bearing failure metrics, and actual tax revenue data (in 1000 million Egyptian pounds).

Keywords: CDF; PDF; MBXII; SF; HF.

1 Introduction

Burr [1] introduced a family of distributions for lifetime data. The burr family consists of twelve probability distributions named Burr-I, Burr-II, Burr-III, Burr-VII, burr-X and Burr XII, etc. Depending on their distribution parameter, this family has several shapes. Because they are more adaptable distributions for lifetime data, the Burr III and Burr XII are the most significant distributions in this family. Examples include The Burr XII distribution's parameters were estimated using point and interval estimators by Wang and Keats [2] using the maximum likelihood method. Software reliability was measured using the Burr type XII distribution by Abdel-Ghaly et al. in [3]. The Burr type XII distribution's statistical and probabilistic characteristics were also examined by Zimmer et al. [4] (1998), who also discussed how it related to other distributions used in reliability studies. Under three alternative loss functions, Moore and Papadopoulos [5] (2000) deduced Bayesian estimators of the parameter and the reliability function for the Burr type XII distribution. In their 2002 study, Ali Mousa and Jaheen [6] looked at the Bayesian estimate of the Burr distribution's parameters using gradually censored data. To test the shape parameter and generate a confidence range for the shape parameter of the Burr type XII distribution under the failure-edited plan, Wu and Yu [7] (2005) suggested pivotal quantities. The empirical estimators of reliability performances for the Burr XII distribution under the LINEX loss function were proposed by Li et al. [8] in 2007. The point estimators of the parameters for the Burr type XII distribution were derived by Wu et al. [9] using the maximum likelihood technique. AL-Hussaini and Ahmed [10] (2003), AL-Hussaini and Ahmed (2003), Kumar [11] (2016), Abdel-Hamid [12] (2009), and Singh and Shukla [13] investigated constant partially accelerated life tests for Burr XII distribution with progressive type-II censoring, and Burr type XII distribution with Marshall-Olkin [14] extension distributions with inverse moments. Researchers haven't paid much attention to Burr XIII's performance over distributions.

One of the transformations for the Burr family is inverse, which allows the Burr III to become the Burr XII with a Cumulative distribution function

$$F(\mathrm{x};\alpha,\beta)=1-(1+x^{\alpha})^{-\beta} \tag{1.1}$$

And the probability density function is:

$$f(x;\alpha,\beta)=\alpha\beta\frac{x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}}, \qquad \alpha,\beta>0, x>0 \tag{1.2}$$

Where α and β are the shape parameters. For α =1, it reduced in Pareto type II (Lomax) distribution (See Kleiber and Kotz,[15] 2003), and β = one it became Champernowne [16] distribution (see Champernowne, 1952). Burr family is also known as the Singh-Maddala [17] distributions, by Wingo [18] (1983 and 1993).

2 Modified Burr XII Distribution

2.1 PDF of MBXII distribution

The probability density function of any distribution is used to determine the probability at a specific point or interval. The Modified Burr XII distribution is defined as

$$F(y) = 1 - \left(1 + \gamma y^{\beta}\right)^{-\frac{\alpha}{\gamma}}; \qquad y > 0; \alpha, \beta, \lambda > 1 \tag{2.1.1}$$

with the probability density function

$$f(y) = \alpha\beta(y)^{\beta-1}\left[1 + \gamma(y)^{\beta}\right]^{\frac{-\alpha}{\gamma}-1}; 0 < y < \infty, \alpha, \beta, \gamma > 0 \tag{2.1.2}$$

Where α β and γ are the shape parameters.

2.2 Plot of the PDF and CDF of MBXII distribution

The PDF and CDF plot of MBXII in figure (1) shows that MBXII distribution is many stretchers and covers the exponential spectacles to the moderately positively skewed phenomena and gives bell shape, which means it also covers the symmetric phenomena.

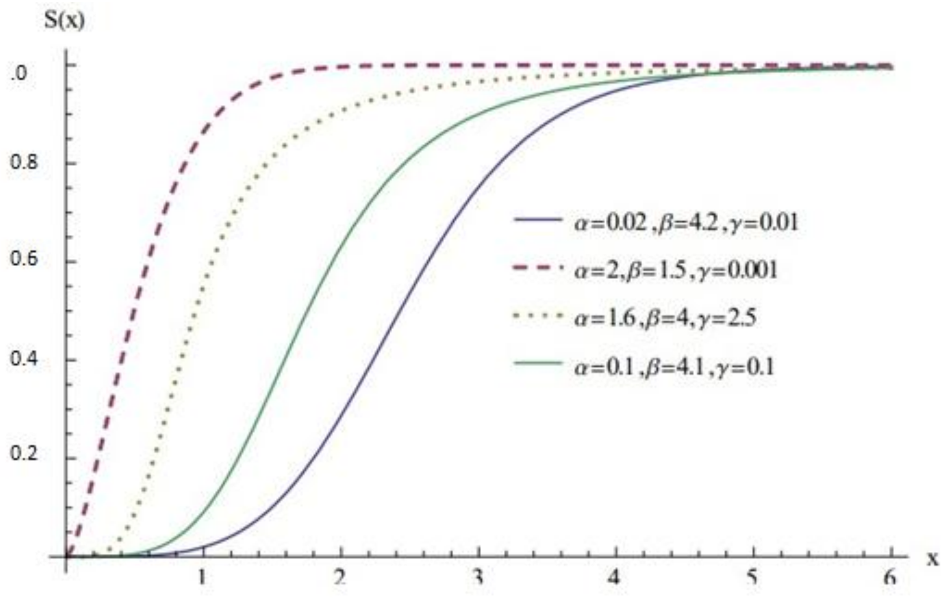

Fig. 1. CDF Plot of MBXII distribution

The different shapes of MBXII distribution's CDF are shown in Fig. (1) for different parametric values. It shows that as RV increases its CDF approaches one.

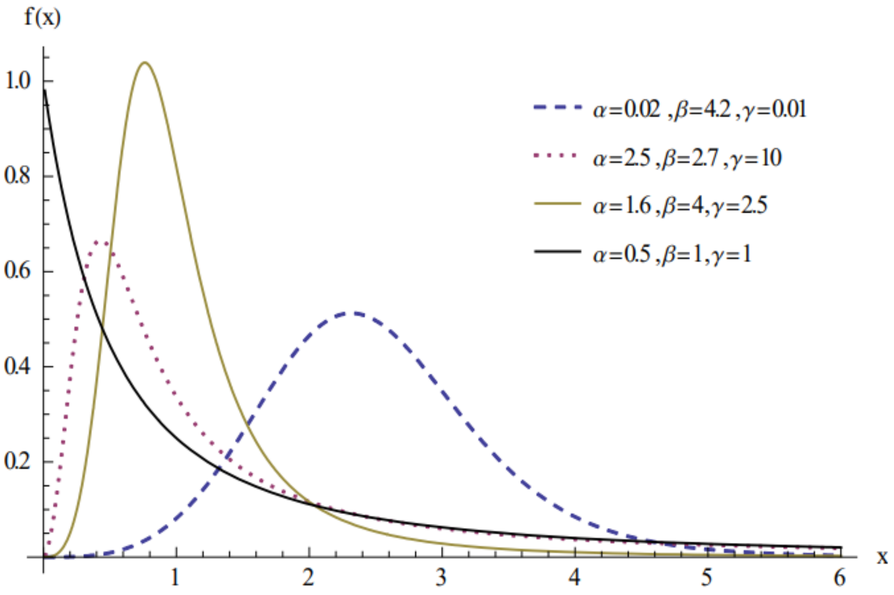

Fig. 2. Plots of PDF of MBXII Distribution

The PDF plot of MBXII in Fig. (2) shows that MBXII distribution is much stretcher and it covers the exponential spectacles to the moderately positively skewed spectacles and it also gives a bell shape which means it also covers the symmetric spectacles.

3 Properties of MB III Distribution

Some features of the MB XIII distribution have been derived from comprehending the range of applications for distribution. Moments, percentiles, random number generation, the mode, survival function, hazard function dependability, and the MB XIII distribution's characteristics are derived.

3.1 Moments of MBXII distribution

A moment is a specific quantitative measure used in mathematics and statistics of the shape of a set of points. It has to describe a probability distribution's center, dispersion, and profile (skewness and kurtosis). The following section will obtain the moments about the origin, central moments, and negative moments of MBXII distribution.

By definition of r^{th} moments about the origin for continuous distribution, we have

$$\mu_r' = E\left(y^r\right) = \int_y y^r f\left(y\right) dy = \frac{\alpha}{\gamma^{\frac{r}{\beta}+1}} B\left[A_r, B_r\right], r = 1, 2, ... \tag{3.1.1}$$

Where $B(A_r, B_r)$ is the Beta function, $A_r = \frac{r}{\beta}+1$, $B_r = \frac{\alpha}{\gamma} - \frac{r}{\beta}$.

The negative moments of the MBXII distribution are defined as:

$$E\left(y^{-r}\right) = \frac{\alpha}{\gamma^{-\frac{r}{\beta}+1}} .B\left[\left(-\frac{r}{\beta}+1\right),\left(\frac{\alpha}{\gamma}+\frac{r}{\beta}\right)\right] \tag{3.1.2}$$

The mean and variance of MB XIII distribution are given as

$$E(y)=\frac{\alpha}{\gamma^{\frac{r}{\beta}+1}},B\left[\left(\frac{1}{\beta}+1\right),\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)\right] \tag{3.1.3}$$

$$Var(y)=\frac{\alpha}{\gamma^{\frac{2}{\beta}+1}}B\left[\left(\frac{2}{\beta}+1\right),\left(\frac{\alpha}{\gamma}-\frac{2}{\beta}\right)\right]-\frac{\alpha}{\gamma}B\left[\left(\frac{1}{\beta}+1\right),\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)\right]^2 \tag{3.1.4}$$

The r^{th} central moments of the MBXII distribution are defined as

$$\mu_r=\sum_{j=0}^{r}\binom{r}{j}(-1)^j\frac{\alpha^{j+1}}{\gamma^{r+j+1}}.B(A_1,B_1)^j.B(A_{r-j},B_{r-j}) \tag{3.1.5}$$

The Cumulants of the MBXII distribution is

$$K_r=\frac{\alpha}{\gamma^{\frac{r}{\beta}+1}}B(A_r,B_r)-\sum_{j=1}^{r-1}\binom{r-1}{j-1}k_j\frac{\alpha}{\gamma^{\frac{r-j}{\beta}+1}}B(A_{r-j},B_{r-j}) \tag{3.1.6}$$

The model is

$$y=\left(\frac{\beta-1}{\gamma+\alpha\beta}\right)^{\frac{1}{\beta}} \tag{3.1.7}$$

The coefficient of skewness is:

$$\beta_1=\frac{\sum_{j=0}^{3}\binom{3}{j}(-1)^j\frac{\alpha^{j+1}}{\gamma^{3+j+1}}.B(A_1,B_1)^j.B(A_{3-j},B_{3-j})}{\left[\sum_{j=0}^{2}\binom{2}{j}(-1)^j\frac{\alpha^{j+1}}{\gamma^{2+j+1}}.B(A_1,B_1)^j B(A_{2-j},B_{2-j})\right]^{\frac{3}{2}}} \tag{3.1.8}$$

and the coefficient of kurtosis is:

$$\beta_2=\frac{\sum_{j=0}^{4}\binom{4}{j}(-1)^j\frac{\alpha^{j+1}}{\gamma^{4+j+1}}B(A_1,B_1)^j\frac{\alpha}{\gamma^{4-\frac{j}{\beta}+1}}B(A_{4-j},B_{4-j})}{\left[\sum_{j=0}^{2}\binom{2}{j}(-1)^j\frac{\alpha^{j+1}}{\gamma^{2+j+1}}.B(A_1,B_1)^j.B(A_{2-j},B_{2-j})\right]^2} \tag{3.1.9}$$

The behavior of skewness and kurtosis of MBXII distribution for different values of the parameter $\alpha,\beta\,\gamma$ are bellowed in Figs. 3. (a), 3 (b), and 3 (c), respectively.

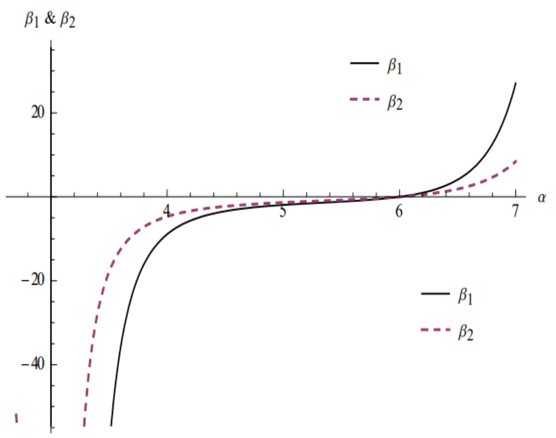

Fig. 3. (a) Plots of Skewness and Kurtosis w.r.t α

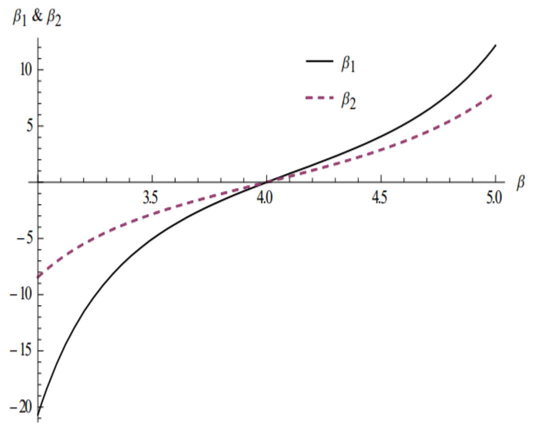

Fig. 3. (b) Plots of Skewness and Kurtosis w.r.t β

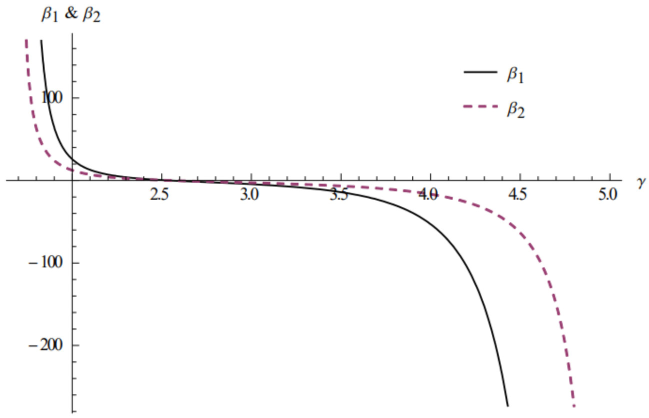

Fig. 3. (c) Plots of Skewness and Kurtosis w.r.t γ

Where different values α show in Fig. 3 (a) that MBXII distribution is positively skewed and mesokurtic, Fig. 3.(b) has silently negatively skewed towards the symmetric and leptokurtic view for different values of β and at last, Fig. 3 (c) shows that MBXII distribution has a straight line and platykurtic shape for different value of γ, where other parameters remain constant.

3.2 Quantile function and random numbers generation

In general, a distribution function is a non-decreasing function. The generalized inverse of the quantile function of F is provided by $F^{-1}(\lambda) = \inf\{x : F(x) \geq \lambda\}$. It is left continuous for the quantile function. [0, 1] is its range. The MB XIII distribution's quantile function is

$$y = \left[\frac{(1-\lambda)^{-\frac{\gamma}{\alpha}} - 1}{\gamma}\right]^{\frac{1}{\beta}}$$

The equivalent random number of MBXII dispersed can be found if λ it is deemed uniformly distributed. Simply substituting λ with the supplied equation, for example, $\lambda = 0.5$ the median of the MBXII distribution, any percentile can be generated.

3.3 Survival and hazard function of MBXII distribution

The survival function of the MBXII distribution is

$$s(\mathrm{y}) = \left(1+\gamma y^{\beta}\right)^{-\frac{\alpha}{\gamma}}; \qquad \alpha,\beta,\gamma>0, \mathrm{y}>0 \tag{3.3.1}$$

The failure rate of MBXII distribution is

$$h(y) = \frac{\alpha\beta\left(y\right)^{\beta-1}}{\left(1+\gamma y^{\beta}\right)} \, or \, \frac{\alpha\beta\left(y\right)^{\beta}}{y\left(1+\gamma y^{\beta}\right)}; \qquad \alpha,\beta,\gamma>0, y>0 \tag{3.3.2}$$

Forbes et al. [19] (2010) defined the inverse of the hazard rate function as Mill's ratio and denoted by m(y). The Mill's percentage of MBXII distribution is given as:

$$m(y) = \frac{\left(1+\gamma y^{\beta}\right)}{\alpha\beta\left(y\right)^{\beta-1}} \, or \, \frac{y\left(1+\gamma y^{\beta}\right)}{\alpha\beta\left(y\right)^{\beta}}; \qquad \alpha,\beta,\gamma>0, y>0 \tag{3.3.3}$$

The sf and hf of the proposed distribution for different values of parameters are shown in Figures (4) and (5). It shows that as MBXII distribution increases, the chance of survival decreases, the survival function advances to zero, and different shapes of the hazard function.

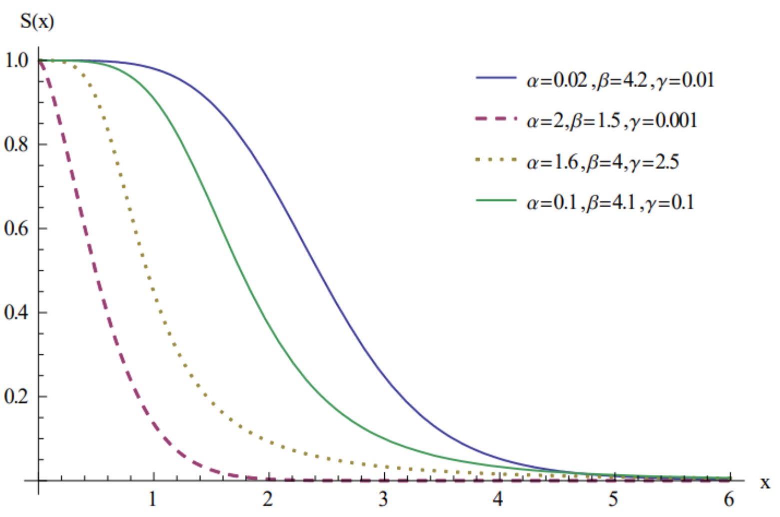

Fig. 4. Plots of sf of MBXII Distribution

The survival function of the proposed distribution for different values of parameters is shown in Fig. (4). It shows that as MBXII distribution increases the chance of survival decreases, survival function advances to zero.

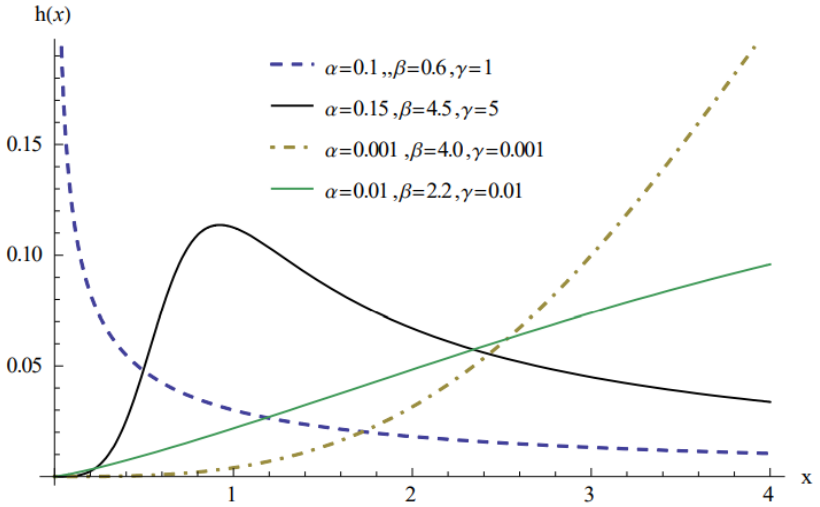

Fig. 5. Plots of hf of MBXII Distribution

Fig. (5) shows the possible shapes of the hazard function of MBXII distribution on the the base of different parameter values.

3. 4 Order statistics of MBXII distribution

By definition, we suppose that $Y_1, Y_2, Y_3, \ldots, Y_n$ are n jointly distributed variables, and the order statistics of these variables $Y_{1:n}, Y_{2:n}, Y_{3:n}, \ldots, Y_{n:n}$ are obtained by arranging them $Y_i's$ in non-decreasing order. i.e. $Y_{1:n} \le Y_{2:n} \le Y_{3:n} \le \ldots \le Y_{n:n}$ .It is often to suppose that the probability distribution $Y_i's$ should be identical and independent; the density function and moments of r^{th} order statistic of MBXII Distribution are computed, respectively.

$$f(y_i) = \frac{n!\alpha\beta}{(i-1)!(n-i)!} \cdot \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j y^{\beta-1} \left[1+\gamma y^{\beta}\right]^{-\frac{\alpha(n+i-j-1)}{\gamma}-1} \tag{3.4.1}$$

And

$$E\left(y_{(i)}^r\right) = \frac{n!\alpha}{(i-1)!(n-i)!\gamma^{\frac{r}{\beta}+1}} \cdot \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j .B\left[\left(\frac{r}{\beta}+1\right), \left(-\frac{\alpha(i+j)}{\gamma} - \frac{r}{\beta} - 2\right)\right] \tag{3.4.2}$$

If $i=1$ in the equation in equation (3.4.1), the PDF of the smallest order statistic of MBXII distribution is obtained as:

$$f(y_1) = \frac{n!\alpha\beta}{(n-1)!} \cdot (-1)^j y^{\beta-1} \left[1+\gamma y^{\beta}\right]^{-\frac{\alpha(n-j)}{\gamma}-1} \tag{3.4.3}$$

If $i=n$ in equation (3.4.1), the PDF of the minimum order statistic of MBXII distribution is obtained as:

$$f\left(y_2\right)=\frac{n!\alpha\beta}{(n-1)!}\cdot\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\, y^{\beta-1}\left[1+\gamma y^{\beta}\right]^{-\frac{\alpha(2n-j-1)}{\gamma}-1} \tag{3.4.4}$$

3.5 Reliability of MBXII distribution

A device's strength to function under specific stress until it wears out is sometimes termed reliability. One feature of several gadget designs is a strength that can withstand stress while working continuously. Numerous scientific disciplines, including engineering, medicine, sociology, etc., have explored the relationship between stress and strength. It focuses on estimating probabilities of the form $P(x, y, z)$ etc. The name of the "stress-strength" or "reliability" models is given to this collection of probabilistic models.

Let X_1 and X_2 be a random strength and random stress; respectively, both variables are assumed to be independently distributed. Thus reliability (R) is given as:

$$R=\frac{\alpha_2}{\alpha_1+\alpha_2}, \qquad \alpha_1+\alpha_2>0 \tag{3.5.1}$$

The reliability, R is defined as

$$R=P(X_2<X_1)=\int_0^{\infty} f_1(y)F_2(y)\,dy \tag{3.5.2}$$

After substituting the values from equations (2.1.1) and (2.1.2)

$$R=\frac{\alpha_1}{\gamma}\left|\frac{z^{-\frac{\alpha_1}{\gamma}}}{-\frac{\alpha_1}{\gamma}}\right|_1^{\infty}-\frac{\alpha_1}{\gamma}\left|\frac{z^{-\frac{\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}}}{-\frac{\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}}\right|_1^{\infty} \tag{3.5.3}$$

The reliability of the component of MBXII distribution is given as:

$$\frac{\alpha_1}{\gamma}\left|\frac{z^{-\frac{\alpha_1}{\gamma}}}{-\frac{\alpha_1}{\gamma}}\right|_1^{\infty}-\frac{\alpha_1}{\gamma}\left|\frac{z^{-\frac{\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}}}{-\frac{\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}}\right|_1^{\infty}=\frac{\alpha_2}{\alpha_1+\alpha_2} \tag{3.5.4}$$

$\alpha_1 \,\&\, \alpha_2$ both can be negative as well as positive

If stress and strength are equal, the reliability of the component of MBIIIG distribution is 0.5.

3.6 Cumulants of MBXII distribution

By definition of r^{th} Cumulants for the continuous variable, we have

$$k_r=\mu_r'-\sum_{i=1}^{r-1}\binom{r-1}{j-1}k_i\mu_{r-j}' \tag{3.6.1}$$

The Cumulants of the MBXII distribution is

$$=\frac{\alpha}{\gamma^{\frac{r}{\beta}+1}}B\left(A_r,B_r\right)-\sum_{j=1}^{r-1}\binom{r-1}{j-1}k_j\frac{\alpha}{\gamma^{\frac{r-j}{\beta}+1}}B\left(A_{r-j},B_{r-j}\right) \tag{3.6.2}$$

3.7 Entropy of MBXII distribution

Entropy is the measure of uncertainty; a larger entropy value means greater uncertainty. In this section, measures of uncertainty of MBXII distribution have been derived by definition:

If $X \sim MBXII\left(\alpha,\beta,\gamma\right)$, then Renyi entropy of order θ is given as:

$$I_R\left(\theta\right)=\frac{1}{1-\theta}\log\left[I\left(\theta\right)\right] \tag{3.7.1}$$

By definition of Renyi entropy, we have

$$I\left(\theta\right)=\int_R f^{\theta}\left(y\right)dy \quad , \qquad \theta>0 \tag{3.7.2}$$

By using equation (3.2.2), we get.

$$I\left(\theta\right)=\int_0^{\infty}\left(\alpha\beta\right)^{\theta}y^{\theta(\beta-1)}\left[1+\gamma y^{\beta}\right]^{\theta\left(-\frac{\alpha}{\gamma}-1\right)}dy \tag{3.7.3}$$

Let $\gamma y^{\beta}=z$

$$=\alpha^{\theta}\beta^{\theta-1}\left(\frac{1}{\gamma}\right)^{\frac{\theta(\beta-1)}{\beta}+1+\left(\frac{1+\beta}{\beta}\right)}\int_0^{\infty}z^{\frac{\theta(\beta-1)}{\beta}-\frac{\beta-1}{\beta}}\left(1+z\right)^{\theta\left(-\frac{\alpha}{\gamma}-1\right)}dz$$

Again $z=\frac{w}{1-w}$, we get.

$$=\alpha^{\theta}\beta^{\theta-1}\left(\frac{1}{\gamma}\right)^{\frac{\theta(\beta-1)}{\beta}+1+\left(\frac{1+\beta}{\beta}\right)}\int_0^{1}\left(\frac{w}{1-w}\right)^{\frac{\theta(\beta-1)}{\beta}-\frac{(\beta-1)}{\beta}}\left(1+\frac{w}{1-w}\right)^{\theta\left(-\frac{\alpha}{\gamma}-1\right)}\frac{dw}{\left(1-w\right)^2}$$

$$I\left(\theta\right)=\alpha^{\theta}\beta^{\theta-1}\left(\frac{1}{\gamma}\right)^{\frac{\theta(\beta-1)}{\beta}+\left(\frac{1+\beta}{\beta}\right)+1}.B\left[\left(\frac{\theta\left(\beta-1\right)}{\beta}-\frac{\left(\beta-1\right)}{\beta}+1\right),\left(-\left(\theta+1\right)+\frac{\left(\theta-1\right)}{\beta}-\theta\left(-\frac{\alpha}{\gamma}-1\right)+1\right)\right],\alpha,\beta,\gamma,\theta>0 \tag{3.7.4}$$

For $\left(-\left(\theta+1\right)+\frac{\left(\theta-1\right)}{\beta}-\theta\left(-\frac{\alpha}{\gamma}-1\right)+1\right)<0$ and the function $\frac{\left(\theta-1\right)}{\beta}\ \&\ \frac{\alpha}{\gamma}$ should not be integers.

4 Estimation of the Parameters of MBXII Distribution

In this section, the estimation of MBXII Distribution parameters can be derived using the maximum likelihood technique.

Suppose that $x_1, x_2, x_3, x_4, ..., x_n$ are random samples of size n drawn from MBXII distribution having $\alpha, \beta\ \gamma$ and as unknown parameters. The Likelihood Function (LF) of the distribution, by using the equation (3.17.1.1), is defined as:

$$L(\alpha, \beta, \gamma; y) = \prod_{i=1}^{n} f(\alpha, \beta, \gamma; y) \tag{4.1}$$

Parameters of MBXII Distribution can be obtained by substituting equation (2.1.1) in equation (4.1).

$$L(\alpha, \beta, \gamma; y) = \prod_{i=1}^{n}\left(\alpha\beta y^{\beta-1}\left[1+\gamma y^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}\right) = (\alpha\beta)^{n} . \sum_{i=1}^{n} y^{\beta-1} . \sum_{i=1}^{n}\left(1+\gamma y^{\beta}\right)^{-\frac{\alpha}{\gamma}-1}$$

Applying natural log on both sides

$$= n\ln(\alpha\beta) + (\beta-1)\sum_{i=1}^{n}\ln(y) - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left(1+\gamma y^{\beta}\right)$$

Differentiate concerning α

$$= n\frac{\partial}{\partial(\alpha)}\ln(\alpha\beta) + (\beta-1)\sum_{i=1}^{n}\frac{\partial}{\partial(\alpha)}\ln(y) - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\frac{\partial}{\partial(\alpha)}\ln\left(1+\gamma y^{\beta}\right)\ \frac{n}{\alpha} - \frac{1}{\alpha}\sum_{i=1}^{n}\ln\left(1+\gamma y^{\beta}\right) = 0 \tag{4.2}$$

Differentiate with respect to β

$$\frac{\partial\left[\ln\left\{L(\alpha, \beta, \gamma; y)\right\}\right]}{\partial(\beta)} = n\frac{\partial}{\partial(\beta)}\ln(\alpha\beta) + (\beta-1)\sum_{i=1}^{n}\frac{\partial}{\partial(\beta)}\ln(y) - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\frac{\partial}{\partial(\beta)}\ln\left(1+\gamma y^{\beta}\right)$$

$$0 = \frac{n}{\beta} + \sum_{i=1}^{n}\ln(y) - (\alpha+\gamma)\sum_{i=1}^{n}\frac{\beta y^{\beta-1}}{\left(1+\gamma y^{\beta}\right)} \tag{4.3}$$

Differentiate with respect to γ

$$\frac{\partial\left[\ln\left\{L(\alpha, \beta, \gamma; y)\right\}\right]}{\partial(\gamma)} = n\frac{\partial}{\partial(\gamma)}\ln(\alpha\beta) + (\beta-1)\sum_{i=1}^{n}\frac{\partial}{\partial(\gamma)}\ln(y) - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\frac{\partial}{\partial(\gamma)}\ln\left(1+\gamma y^{\beta}\right)$$

$$= 0 + 0 - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\frac{y^{\beta}}{\left(1+\gamma y^{\beta}\right)} - \frac{\alpha}{\gamma^{2}}\sum_{i=1}^{n}\ln\left(1+\gamma y^{\beta}\right)$$

$$\left[(\alpha+\gamma)\sum_{i=1}^{n}\frac{y^{\beta}}{\left(1+\gamma y^{\beta}\right)}+\frac{\alpha}{\gamma}\sum_{i=1}^{n}\ln\left(1+\gamma y^{\beta}\right)\right]=0 \tag{4.4}$$

The above equations (4.2), (4.3) and (4.4) are nonlinear equations that are used to find the maximum likelihood estimates of unknown parameters $\alpha, \beta\ \gamma$.

5 Applications of Modified BURR XII Distribution

In this section, the applications of the proposed model, MBXII distribution, have been discussed on two real-life data sets. Moreover, a comparison has been made between the MBXII distribution, Weibull distribution, and Burr XII distribution. As a result, it has been found that the given proposed distribution is better fitted than the other two distributions based on -2 Log-likelihood (-2LL), AIC submitted by Akaike [20] (1974), BIC introduced by Schwarz [21] (1978) and Consistent Akaike information criteria by Bozdogan [22] (1987). The relatively good fit is confirmed by the smaller values of the above measures.

5.1 Applications on measurements of failure of ball bearings

Introduced distribution along with Weibull and Burr XII distribution has been fitted on the real-life data set, the measurements of failure of the ball bearings. The data set is measured in millions of revolutions originally published by Lieblein and Zelan (1956) and was used as an example by Shao Q. et al. [23]. Data set is 17.88,28.92,33.00,41.52,41.12,45.60,48.48,51.84,51.96,54.12,55.56,67.80,68.64,68.64,68.88,84.12,93.12,98.64, 105.12,105.84,127.92,128.04,173.40

The following table provides the data set's descriptive statistics.

Table 1. Data set's descriptive statistics

Data	Observations	Maximum	Minimum	Mean	SD	Skewness	Kurtosis
Failure of Ball Bearings	23	173.40	17.88	72.22	37.49	1.008	0.926

The maximum likelihood estimates of unknown parameters of MBXII, Weibull, Burr XII with four parameters and Burr XII with two parameters distributions, and information criteria are given in Table 2.

Table 2. Maximum likelihood estimates and information criteria

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	$-2\log$	AIC	BIC	CAIC
MBXII	0.001	2.083	1.36×10^{-7}	-	227.40	233.40	231.487	234.487
Weibull	2.102	81.88	-	-	231.05	235.05	234.086	236.086
BXII (4 Para)	10.23	0.08	0.397	2.34	240.08	248.08	245.523	249.522
BXII (2 Para)	0.013	18.98	-	-	302.39	306.39	305.426	307.426

Table 2 shows that the values of -2log, AIC, BIC and CAIC are lesser for MBXII compared to Weibull, BXIII (4 Para.) and Burr XII (2 Para.) distribution; MBXII distribution fits better for the failure of ball bearings data. The graph of the given distributions is bellowed.

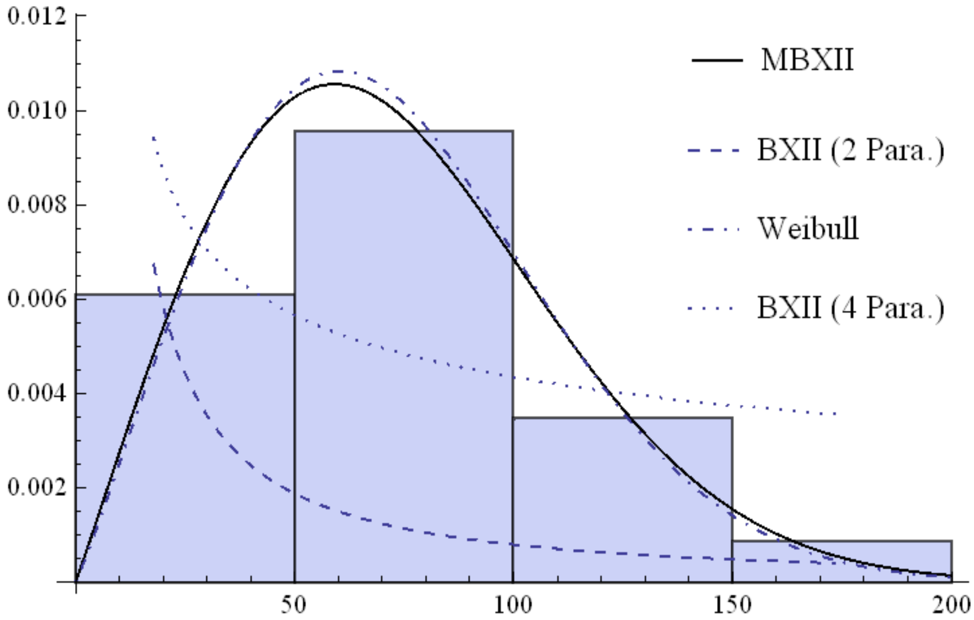

Fig. 6. Empirical Histogram and Fitted Distributions for Failure of Ball Bearing

5.2 Application of actual taxes revenue data

A second application of the proposed distribution in actual taxes revenue data (in 1000 million Egyptian pounds) along with exceptional cases Weibull and Burr XII (2 Para.) distribution. Mead M. E. [6] used this data as an example. Data is given below 5.99, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6 ,19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8

Please mention reference numbers 7 to 29 inside the text.

The descriptive statistics for the data set are presented in the following table.

Data	Observations	Minimum	Maximum	Mean	SD	Skewness	Kurtosis
Actual Taxes Revenue	59	4.10	39.20	13.4881	8.052	1.651	2.569

Table 3. Maximum Likelihood Estimates and Information Criteria

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$-2\log$	AIC	BIC	CAIC
MBXII	0.0001	3.989	0.000135	379.36	385.36	384.671	387. 671
BXII (2 Para)	1.840	15.31	-	394.76	398.76	398.30	400. 30
Weibull	0.047	8.608	-	514.46	518.46	518.001	520.001

The values of -2log, AIC, BIC, and CAIC shown in table 3 are minimum for MBXII distribution compared to its exceptional cases: Weibull and BXII distribution. This means that MBXII is a better-fit distribution for actual taxes revenue data (in 1000 million Egyptian pounds) data. We have used Mathematica version 9 software for obtaining the required results. The graph of the given distributions is bellowed.

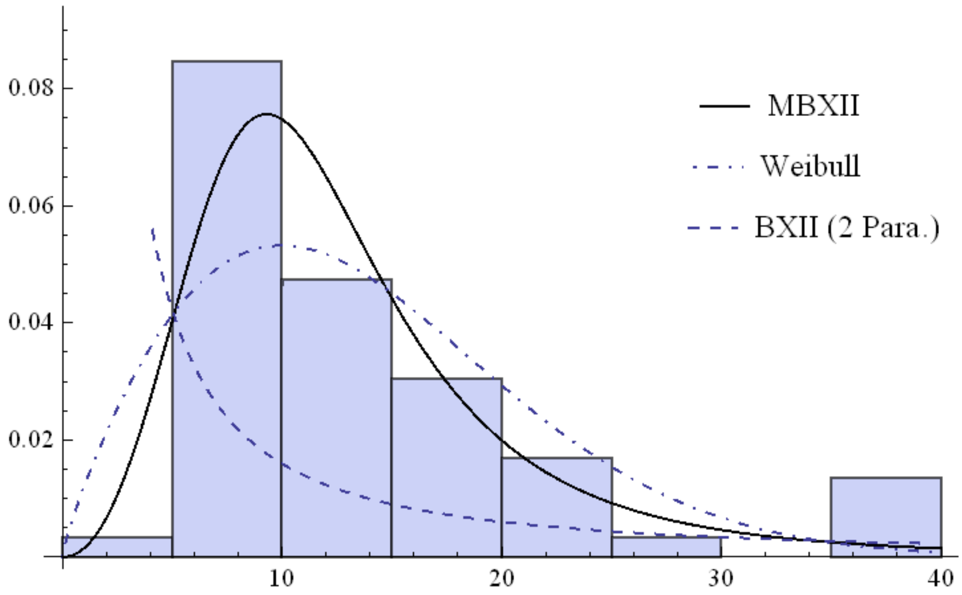

Fig. 7. Empirical Histogram and Fitted Distributions for Actual Taxes Revenue

6 Conclusion

In this study, we established the MBXII distribution's numerous statistical features. We fitted the suggested distribution to an actual data set to demonstrate how it could be used in practice and explained that it offers the most excellent fit when contrasted to its sub-models.

A very small effort to propose a new distribution named is “Modified Burr XII distribution”. The projected distribution with three shape parameters is more flexible and tractable as it may contain extremely skewed to symmetrical depending on parametric values.

For checking the failure rate and uncertainty of the proposed distribution we derived the hazard rate function and entropy respectively, the hazard function decreases after increasing initially. We computed the other Statistical properties of the defined distribution such as mean, median, mode, moments, central moments, cumulants, reliability, quantile function and order statistics, etc., and also the MLE method is used to calculate the unknown parameters.

Competing Interests

Authors have declared that no competing interests exist.

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