CONTRIBUTION TO THE NUMERICAL SOLUTION OF NONLINEAR HEAT TRANSFER EQUATION SUBJECT TO A BOUNDARY INTEGRAL SPECIFICATION

Abstract

The work present the numerical methods to solve the nonlinear heat equation subject to a boundary integral specification: Firsly the implicite Euler time discretisation reduce the partial differential equation to a second ordre boundary value problem. Then the third order finite difference scheme along with the $\frac{1'}{3}$s composite Simpson quadrature is used to produced a non linear. The approximation solution is obtained, using The multivariate Newton method.

KEYWORD: NONLINEAR HEAT TRANSFERT, BOUNDARY INTEGRAL EQUATION, IMPLICITE EULER METHODS, THIRD FINITE DIFFERENCE, NEWTON METHOD.
AM class: 34A05, 34A08, 42A10, 40A30, 65B10

1 INTRODUCTION

This paper consider the problems of obtaining numerical solution to the one-dimensional nonlinear unsteady heat conduction equation [4],[7] given by

$$\left\{ \begin{array}{rcl} \rho c_p \dfrac{\partial u}{\partial t} & = & \dfrac{\partial}{\partial x}\left(\kappa\left(u\right)\dfrac{\partial u}{\partial x}\right) \; a < x < b, \quad 0 < t \leq T \\ \int_a^b u\left(x,t\right)dt & = & \alpha\left(t\right), 0 < t \leq T \\ \frac{\partial u}{\partial x}(b,t) & = & \beta\left(t\right), 0 < t \leq T \end{array} \right. \tag{1}$$

where the unknown function $u(x,t)$ is the temperature at position x and time t, ρ is the density, c_p is the specific heat capacity at constant pressure et κ is the thermal conductivity of the media. We assume that c_p and ρ have constant values, but κ depends on the temperature u. By Differentiating the first equation in right hand side of (1), we get

$$\rho c_p \frac{\partial u}{\partial t} = \kappa\left(u\right) \frac{\partial^2 u}{\partial x^2} + \partial_u \kappa\left(u\right) \left[(\frac{\partial u}{\partial x})^2 \right] \tag{2}$$

When κ does not depend on u i.e $\partial_u \kappa\left(u\right) = 0$, then (2) is a linear (parabolic) partial differential equation. When $\partial_u \kappa\left(u\right) \neq 0$, then (2) is nonlinear. The boundary conditions for equation (2) are the second and third equation in (1) where $\alpha\left(t\right)$ and $\beta\left(t\right)$ are know. The initial condition is assumed to be of the form

$$u(x,0) = g\left(x\right), \ x \in [a,b] \tag{3}$$

The first boundary condtion in (1) is the non-local condition [9]-[10] and the second condition in (1) is the Neumann condtion for $x = b$.

2 METHODS OF RESOLUTION

Tis section describe successively the implicite Euler discretisation, the third order finite difference scheme and the iteration method of newton.

2.1 Implicite Euler discretisation

We first discretise the equation (2) in time, using a time step $\tau >$. The time line $t \geq 0$ is partition by equally mesh-points as:

$$\tau_n = n\tau, \ n = 0, 1, 2, ... \tag{4}$$

Using the implicite Euler scheme [8], [2], equation (2) is dicretised on the mesh grids (4)

$$\rho c_p \frac{u_n - u_{n-1}}{\tau} = \kappa\left(u_n\right) \frac{d^2 u_n}{dx^2} + \partial_u \kappa\left(u_n\right) (\frac{du_n}{dx})^2 \tag{5}$$

here $u_n = u_n\left(x\right)$ and $u_{n-1} = u_{n-1}\left(x\right)$ approximate the values of $u\left(x, t_n\right)$ and $u\left(x \ t_{n-1}\right)$ respectively. The equation (5) is the approximate of the partial differential equation (2). The error is $O(\tau)$, hence the approximation scheme is first-order accurate in time. This implicite method is stable.

Solving the equation (5), for the second spatial derivative of the temperature u_n, we obtain as in [2]:

$$\frac{d^2u_n}{dx^2} = \frac{\rho c_p\left(u_n - u_{n-1}\right)}{\tau\kappa\left(u_n\right)} - \frac{\partial_u\kappa\left(u_n\right)}{\kappa\left(u_n\right)}\left(\frac{du_n}{dx}\right)^2$$

or

$$\frac{d^2u_n}{dx^2} = \phi\left(u_n, v_n, u_{n-1}\right) \tag{6a}$$

where $v_n = \dfrac{du_n}{dx}$ is the gradient of the temperature u_n and ϕ the nonlinear fonction defined as

$$\phi\left(u_n, v_n, u_{n-1}\right) = \frac{\psi\left(u_n, v_n, u_{n-1}\right)}{\kappa\left(u_n\right)} \tag{7}$$

with

$$\psi\left(u_n, v_n, u_{n-1}\right) = \rho c_p\frac{u_n - u_{n-1}}{\tau} - \partial_u\kappa\left(u_n\right)v_n^2 \tag{8}$$

The equation $(6a)$ with boundary conditions

$$\int_a^b u_n\left(x\right)dx = \alpha\left(t_n\right) \tag{9}$$

$$\frac{du_n\left(b\right)}{dx} = \beta\left(t_n\right) \tag{10}$$

constitutes a nonlinear boundary values problems for the unknown u_n. Given the known value u_{n-1} , the problem can be solved, by some numerical technique for nonlinear problems. Starting from the initial condition u_0, we can solve reccursively $(6a)$ for $n = 1, 2, ...$

2.2 Finite difference method discretisation

We use the finite difference method (FDM) [5] ,[10], [12] for the solution of the problem $(6a)$. We divide the intervals $[a, b]$ into N equal subintervals with space mesh $h = \dfrac{b-a}{N}$ such that

$$x_i = a + ih,\ i = 0, 1, 2, ..., N \tag{11}$$

To approximate the space derivative in the equation $(6a)$ to third-order accuracy at some general point x on the uniform mesh (11), we assume that it may be replaced by the five point formula [11] :

$$\frac{d^2u_n\left(x_i\right)}{dx^2} \simeq \frac{1}{12h^2}\left(11u_{n,i-1} - 20u_{n,i} + 6u_{n,i+1} + 4u_{n,i+2} - u_{n,i+3}\right), i = 1, ..., N-2 \tag{12}$$

$$\frac{d^2u_n\left(x_{N-1}\right)}{dx^2} \simeq \frac{1}{12h^2}\left(u_{n,i-3}-6u_{n,i-2}+26u_{n,i-1}-4u_{n,i}+21u_{n,i+1}-u_{n,i+2}\right) \quad (13)$$

$$\frac{d^2u_n\left(x_N\right)}{dx^2} \simeq \frac{1}{12h^2}\left(2u_{n,i-4}-11u_{n,i-3}+24u_{n,i-2}-14u_{n,i-1}+10u_{n,i}-9u_{n,i+1}\right) \quad (14)$$

We set $x = x_i$ in $(6a)$ and $u_n\left(x_i\right), v_n\left(x_i\right),\ u_{n-1}\left(x_i\right)$ has been replaced with their approximations $u_{n,i}, v_{n,i},\ u_{n-1,i}$ with

$$v_{n,i} = \frac{u_{n,i+1}-u_{n,i-1}}{2h} \quad (15)$$

Then the equation $(6a)$ can be formulated as

$$\frac{1}{12h^2}\left(11u_{n,i-1}-20u_{n,i}+6u_{n,i+1}+4u_{n,i+2}-u_{n,i+3}\right) = \phi\left(u_{n,i}, v_{n,i}, u_{n-1,i}\right) \quad (16)$$

for $i = 1, ..., N-2$

$$\frac{1}{12h^2}\left(u_{n,i-3}-6u_{n,i-2}+26u_{n,i-1}-4u_{n,i}+21u_{n,i+1}-u_{n,i+2}\right) = \phi\left(u_{n,i}, v_{n,i}, u_{n-1,i}\right) \quad (17)$$

for $i = N-1$

$$\frac{1}{12h^2}\left(2u_{n,i-4}-11u_{n,i-3}+24u_{n,i-2}-14u_{n,i-1}+10u_{n,i}-9u_{n,i+1}\right) = \phi\left(u_{n,i}, v_{n,i}, u_{n-1,i}\right) \quad (18)$$

For $i = N$. Assuming N even, the non local boundary condition (9) can be discretised by the composite quadrature rule of Simpson $\frac{1}{3}$, using the values of unknown at the grids points in (11) to get:

$$\int_a^b u_n\left(x\right)dx = \frac{h}{3}\left[u_{n,0}+u_{n,N}+4\sum_{j=1}^{\frac{N}{2}}u_{2j-1}+2\sum_{j=1}^{\frac{N}{2}-1}u_{2j}\right] = \alpha\left(t_n\right)$$

or

$$h\sum_{j=0}^{N}c_ju_{n,j} = \alpha\left(t_n\right) \quad (19)$$

where

$$c_0 = c_N = \frac{1}{3},\quad c_{2j} = \frac{2}{3},\quad j = 1, 2, ..., \frac{N}{2}-1,\quad c_{2j-1} = \frac{4}{3},\quad j = 1, 2, ..., \frac{N}{2}$$

Resolving the equation (15) for $u_{n,0}$ give

$$u_{n,0} = \frac{1}{hc_0}\left(\alpha\left(t_n\right)-h\sum_{j=1}^{N}c_ju_{n,j}\right)$$

$$= \frac{\alpha\left(t_n\right)}{hc_0} - \frac{1}{c_0}\sum_{j=1}^{N} c_j u_{n,j} \tag{20}$$

For $i = 1$ and using the equation (16), we get

$$\frac{1}{12h^2}\left(11u_{n,0} - 20u_{n,1} + 6u_{n,2} + 4u_{n,3} - u_{n,4}\right) = \phi\left(u_{n,1}, v_{n,1}, u_{n-1,1}\right) \tag{21}$$

Inserting (20) equation in (21,) permit to obtain

$$\frac{-\left(20 + \frac{11c_1}{c_0}\right)u_{n,1} - \left(6 - \frac{11c_2}{c_0}\right)u_{n,2} + \left(4 - \frac{11}{c_0}c_3\right)u_{n,3} - \left(1 + \frac{11}{c_0}c_4\right)u_{n,4} - \frac{11}{c_0}\sum_{j=5}^{N} c_j u_{n,j} + \frac{11\alpha\left(t_n\right)}{hc_0}}{12h^2} = \phi\left(u_{n,1}, v_{n,1}, u_{n-1,1}\right) \tag{22}$$

The Neuman boundary condition can be writed as

$$\frac{du_{n,N}}{dx} = \beta\left(t_n\right) \tag{23}$$

Using the ghost point at x_{N+1},the equation (**??**) became

$$\frac{u_{n,N+1} - u_{n,N-1}}{2h} = \beta\left(t_n\right) \tag{24}$$

$$u_{n,N+1} = u_{n,N-1} + 2h\beta\left(t_n\right)$$

For $i = N$, the equation can be written as

$$\frac{2u_{n,N-4} - 11u_{n,N-3} + 24u_{n,N-2} - 14u_{n,N-1} + 10u_{n,N} - 9u_{n,N+1}}{12h^2} = \phi\left(u_{n,N}, v_{n,N}, u_{n-1,N}\right) \tag{25}$$

Inserting the value of $u_{n,N+1}$ from (23) in (24) we obtain

$$\frac{2u_{n,N-4} - 11u_{n,N-3} + 24u_{n,N-2} - 23u_{n,N-1} + 10u_{n,N} - 18h\beta\left(t_n\right)}{12h^2} = \phi\left(u_{n,N}, v_{n,N}, u_{n-1,N}\right) \tag{26}$$

The equations (16), (21) and (26) constitute un nonlinear algebric equation to be resolved with nonlinear techniques. The equation (16) approximates the equation ($6a$) with error of $O\left(h^3\right)$ .The equation (26) approximates the Neuman boundary condition (10) with error of $O(h^2)$.The composite Simpson rule (19) approximates the non local condition (9) with error of $O(h^4)$. Hence the over all algebric system approximate the nonlinera continue boundary value problem with a of high order accurate in space.

2.3 Derivatives for the Newton Method

Let us define

$$\phi\left(u,v;w\right)=\frac{\psi\left(u,v;w\right)}{\kappa\left(u\right)}$$

and introduce the notation $\phi=\phi\left(u,v;w\right)$, $\psi=\psi\left(u,v;w\right)$. Denoting also the derivatives by $q=q\left(u,v;w\right)$, $p=p\left(u,w\right)$,we get

$$q=\frac{\partial\phi}{\partial u}=\frac{1}{\kappa\left(u\right)}\left[\frac{\partial\psi}{\partial u}-\phi\partial_u\kappa\left(u\right)\right]$$

$$p=\frac{\partial\phi}{\partial v}=\frac{1}{\kappa\left(u\right)}\frac{\partial\psi}{\partial v} \tag{27}$$

where

$$\frac{\partial\psi}{\partial u}=\frac{\rho c_p}{\tau}-\partial^2_{uu}\kappa\left(u\right)v^2 \tag{28}$$

$$\frac{\partial\psi}{\partial v}=-2\partial_u\kappa\left(u\right)v \tag{29}$$

SOLVING THE NONLINEAR SYSTEMS BY NEWTON METHOD

The non linear algebric equation correspondant to(16)-(19)-(26) can be write as fallow:

$$\left\{\begin{array}{rcl}
-\alpha_1u_{n,1}+\alpha_2u_{n,2}+\alpha_3u_{n,3}+\alpha_4u_{n,4}+\sum\limits_{j=5}^{N}\alpha_ju_{n,j}+\dfrac{11\alpha\left(t_n\right)}{hc_0}-12h^2\phi_{n,1} & = & 0\\
11u_{n,i-1}-20u_{n,i}+6u_{n,i+1}+4u_{n,i+2}-u_{n,i+3}-12h^2\phi_{n,i} & = & 0\\
u_{n,N-4}-6u_{n,N-3}+26u_{n,N-2}-5u_{n,N-1}+21u_{n,N}-2h\beta\left(t_n\right)-12h^2\phi_{n,N-1} & = & 0\\
2u_{n,N-4}-11u_{n,N-3}+24u_{n,N-2}-23u_{n,N-1}+10u_{n,N}-18h\beta\left(t_n\right)-12h^2\phi_{n,N} & = & 0
\end{array}\right. \tag{30}$$

where

$$\phi_{n,i}=\phi\left(u_{n,i},v_{n,i},u_{n-1,i}\right),\ 1\leq i\leq N \tag{31}$$

$$c_0=c_N=\frac{1}{3},\ c_{2j}=\frac{2}{3},j=1,2,...,\frac{N}{2}-1,\ \ c_{2j-1}=\frac{4}{3},\ \ j=1,...,\frac{N}{2} \tag{32}$$

$$\alpha_1=-20-\frac{11c_1}{c_0},\alpha_2=6-\frac{11c_2}{c_0},$$

$$\alpha_3=4-\frac{11}{c_0}c_3,\alpha_4=-1-\frac{11}{c_0}c_4,\alpha_N=-11,$$

$$\alpha_j=-\left\{\begin{array}{l}-44\ \text{ when } j \text{ odd, } j=5\leq j<N\\ -22\ \text{ when } j \text{ even, } j=6\leq j<N\end{array}\right.$$

The systems of non-linear equations (30) can be written as

$$\mathbf{G}_n\left(\mathbf{u}_n\right)=\left[\begin{array}{c} G_{n;1}\left(\mathbf{u}_n\right) \\ G_{n;2}\left(\mathbf{u}_n\right) \\ \ldots \\ G_{n;N}\left(\mathbf{u}_n\right) \end{array}\right]=\left[\begin{array}{c} 0 \\ 0 \\ \ldots \\ 0 \end{array}\right], \qquad \mathbf{u}_n=\left[\begin{array}{c} u_{n,1} \\ u_{n,2} \\ \ldots \\ u_{n,N} \end{array}\right] \tag{33}$$

with components

$$G_{n,1}\left(\mathbf{u}_n\right)=\alpha_1 u_{n,1}+\alpha_2 u_{n,2}+\alpha_3 u_{n,3}+\alpha_4 u_{n,4}+\sum_{j=5}^{N}\alpha_j u_{n,j}+\frac{11\alpha\left(t_n\right)}{hc_0}-12h^2\phi_{n,1}$$

$$G_{n;i}\left(\mathbf{u}_n\right)=11u_{n,i-1}-20u_{n,i}+6u_{n,i+1}+4u_{n,i+2}-u_{n,i+3}-12h^2\phi_{n,i}, i= \; 2,...,N-2 \tag{34}$$

$$G_{n,N-1}\left(\mathbf{u}_n\right)=u_{n,N-4}-6u_{n,N-3}+26u_{n,N-2}-5u_{n,N-1}+21u_{n,N}-2h\beta\left(t_n\right)-12h^2\phi_{n,N-1} \tag{35}$$

$$G_{n,N}\left(\mathbf{u}_n\right)=2u_{n,N-4}-11u_{n,N-3}+24u_{n,N-2}-23u_{n,N-1}+10u_{n,N}-18h\beta\left(t_n\right)-12h^2\phi_{n,N} \tag{36}$$

Assume that the exact solution is $\mathbf{u}_n^{(e)}$. Suppose that the initial estimate of the solution is, $\mathbf{u}_n^{(0)}$. A Talor expansion of first order is

$$\mathbf{G}_n\left(\mathbf{u}_n^{(e)}\right)=\mathbf{G}_n\left(\mathbf{u}_n^{(0)}\right)+\frac{\partial\mathbf{G}_n}{\partial\mathbf{u}_n}\left(\mathbf{u}_n^{(0)}\right)\Delta\mathbf{u}_n$$

with $\Delta\mathbf{u}_n=\mathbf{u}_n^{(e)}-\mathbf{u}_n^{(0)}$ Using

$$\mathbf{G}_n\left(\mathbf{u}_n^{(e)}\right)=0$$

gives

$$\frac{\partial\mathbf{G}_n}{\partial\mathbf{u}_n}\left(\mathbf{u}_n^{(0)}\right)\Delta\mathbf{u}_n=-\mathbf{G}_n\left(\mathbf{u}_n^{(0)}\right)$$

This system of linear equations leads to new approximation $\mathbf{u}_n^{(1)}=\mathbf{u}_n^{(0)}+\Delta\mathbf{u}_n$. The Newton-Raphson Algorithm with the jacobian matrix $L_n^{(k)}$ is then

$$L_n^{(k)}\left(\mathbf{u}_n^{(k-1)}\right)\Delta\mathbf{u}_n^{(k)}=-\mathbf{G}_n\left(\mathbf{u}_n^{(k-1)}\right) \tag{37}$$

$$L_n^{(k)}\left(\mathbf{u}_n^{(k-1)}\right)=\frac{\partial\mathbf{G}_n}{\partial\mathbf{u}_n}\left(\mathbf{u}_n^{(k-1)}\right) \tag{38}$$

$$\mathbf{u}_n^{(k)}=\mathbf{u}_n^{(k-1)}+\Delta\mathbf{u}_n^{(k)} \tag{39}$$

Starting by some initial guess $\mathbf{u}_n^{(0)}$, the equation (**??**) can be solved by the Newton iterative method:

$$\mathbf{u}_n^{(k+1)} = \mathbf{u}_n^{(k)} - \ L_n^k \left(\mathbf{u}_n^{(k)}\right)^{-1} \mathbf{G}_n \left(\mathbf{u}_n^{(k)}\right), \ k = 0, 1, 2, ... \tag{40}$$

with the matrix L_n^k in the form

$$L_n^{(k)} = \frac{\partial \mathbf{G}_n}{\partial \mathbf{u}_n} = \begin{bmatrix} \dfrac{\partial \mathbf{G}_{n,1}}{\partial \mathbf{u}_{n,1}} & \dfrac{\partial \mathbf{G}_{n,1}}{\partial \mathbf{u}_{n,2}} & ... & \dfrac{\partial \mathbf{G}_{n,1}}{\partial \mathbf{u}_{n,N}} \\ \dfrac{\partial \mathbf{G}_{n,2}}{\partial \mathbf{u}_{n,1}} & \dfrac{\partial \mathbf{G}_{n,2}}{\partial \mathbf{u}_{n,2}} & ... & \dfrac{\partial \mathbf{G}_{n,2}}{\partial \mathbf{u}_{n,N,}} \\ ... & ... & ... & \dfrac{\partial \mathbf{G}_n}{\partial \mathbf{u}_n} \\ \dfrac{\partial \mathbf{G}_n, N}{\partial \mathbf{u}_{n,1}} & \dfrac{\partial \mathbf{G}_{n,N}}{\partial \mathbf{u}_{n,2}} & ... & \dfrac{\partial \mathbf{G}_{n,N}}{\partial \mathbf{u}_{n,N}} \end{bmatrix} \tag{41}$$

The elements of jacobian, are

$$L_n^k\left(1,1\right) = \alpha_1 - 12h^2 q_{n,1}^{(k)}, \ L_n^k\left(1,2\right) = \alpha_2 - 6hp_{n,2}^{(k)}, \ L_n^k\left(1,3\right) = \alpha_3, \ L_n^k\left(1,4\right) = \alpha_4$$

$$L_n^k\left(1,i\right) = \begin{cases} \ \ -44 \ \text{ when } i \text{ odd} \\ -22 \ \text{ when } i \ \text{ even} \end{cases}$$

$$L_n^{(k)}\left(i,i\right) = -20 - 12h^2 q_{n,i}^{(k)}, i = 2, 3, ..., N-1 \tag{42}$$

$$L_n^{(k)}\left(i,i-1\right) \ = \ 11 + -6hp_{n,i}^{(k)},$$

$$\begin{aligned} L_n^{(k)}\left(i,i+1\right) &= -6 - -6hp_{n,i}^{(k)}, \ \ L_n^{(k)}(i,i+2) = 4, \\ L_n^{(k)}(i,i+3) &= -1; \end{aligned} \tag{43}$$

$$\begin{aligned} L_n^{(k)}\left(N-1,N-1\right) &= 5 - 12h^2 q_{n,N-1}^{(k)}, \\ L_n^{(k)}\left(N-1,N-2\right) &= 26 + 6hp_{n,N-1}^{(k)}, \\ L_n^{(k)}\left(N-1,N\right) &= 21 - 6hp_{n,N-1}^{(k)} \end{aligned}$$

$$L_n^{(k)}\left(N-1,N-4\right) = 1; \quad L_n^{(k)}\left(N-1,N-3\right) = -6$$

$$L_n^{(k)}\left(N,N\right) = 10 - 12h^2 q_{n,N}^{(k)}; \quad L_n^{(k)}\left(N,N-1\right) = -23 + 6hp_{n,N}^{(k)};$$

$$L_n^{(k)}\left(N,N-2\right) = 24; \qquad L_n^{(k)}\left(N,N-3\right) = -11; \qquad L_n^{(k)}\left(N,N-4\right) = 2$$

where

$$q_{n,i}^{(k)} = q\left(u_{n,i}^{(k)}, v_{n,i}^{(k)}; u_{n-1,i}\right) \quad p_{n,i}^{(k)} = p\left(u_{n,i}^{(k)}, v_{n,i}^{(k)}\right). \tag{44}$$

Iteration (39) is one step(two-level) iteration. Given an initial guess $\mathbf{u}_n^{(0)}$, we can calculate next approximation $\mathbf{u}_n^{(k+1)}$, $k = 0, 1, 2, ...$, using (39). The limiting vector $\mathbf{u}_n = \lim\limits_{k\to+\infty}\left(\mathbf{u}_n^{(k+1)}\right)$ is a solution to the nonlinear system (**??**), il the sequence is convergent. The iteration process is ended when

$$\left\|\mathbf{u}_n^{(k+1)} - \mathbf{u}_n^{(k)}\right\| < \varepsilon \tag{45}$$

This inequality is called a stopping criteria. Generally, we use as $\mathbf{u}_n^{(0)}$, the solution $\mathbf{u}_{n-1}$ found in previous step.

3 COMPUTER EXPERIMENT

We consider the problem defined on the interval $[1,\ 3]$.The density ρ and the heat capacity c_p are constant, but κ depends on the temperature ([2]) as

$$\kappa = \kappa_0 \exp\left(\chi u\right) \tag{46}$$

We choose $\rho = 1$, $c_p = 1$, $\kappa_0 = .1$. The boundary condtions are

$$\int_1^3 u\left(x,t\right) dx = \frac{5}{3} \tag{47}$$

$$\frac{\partial u\left(3,t\right)}{\partial x} = \frac{3}{2} \tag{48}$$

The initial temperature profile is ([2])

$$u\left(x,0\right) = 2 - \frac{x-1}{2} + \left(x-1\right)\left(x-3\right),\ x \in [1,\ 3] \tag{49}$$

We solve the partial differential equation (1)with boundary condition (47) and(48) and the initial condition (49) by the method described in this paper. The step size is choosing to be $\tau = 0.5$ in the$0 \leq t \leq 15$.We discretised the interval $[1,\ 3]$ with $N = 41$ mesh-points, i.e $h = 0.05$. The equation is solved for $\chi = 0.0,\ 0.05, 0.5, 1.0, 1.5, 2.0$

Below, the first figure in the left hand represent the profil of heat transfer when the parameter χ=0 for all time. The figure in the right hand is the profil of heat transfer coresponding to the final time $t = 15$ This situation arise when $\frac{\partial u}{\partial t} = 0$, the case of stability. All others must converge toward this staedy case. What is happen is that, as the parameter χ grows, the corresponding picture converge slowly to wards the point $u(x_i, 15) = 10, i = 1, 2, ..., N$

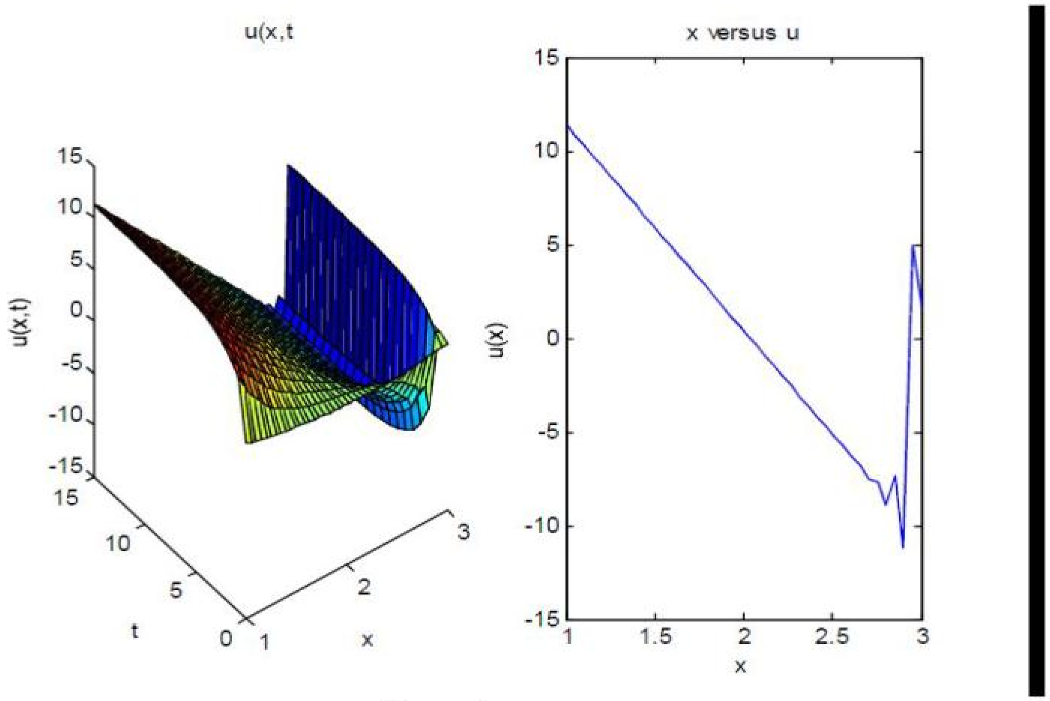

Figure for $\chi = 0$

Fig 1: Profile of heat transfer when the parameter x=0 for all time

Below the first figure in the left hand represent the profil of heat transfer when the parameter $\chi = 0.5$ for all time. The figure in the right hand is the profil of heat transfer corresponding to the final time $t = 15$.

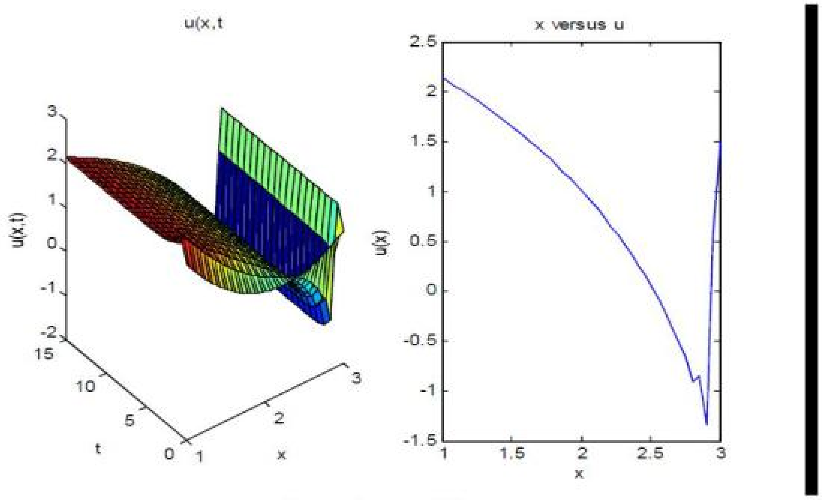

Figure for $\chi = 0.5$

Fig 2: Profile of heat transfer when the parameter x=0.5 for all time

The first figure in the left hand represent the profil of heat transfer when the parameter $\chi = 1$ for all time. The figure in the right hand is the profil of heat transfer coresponding to the final time $t = 15$.

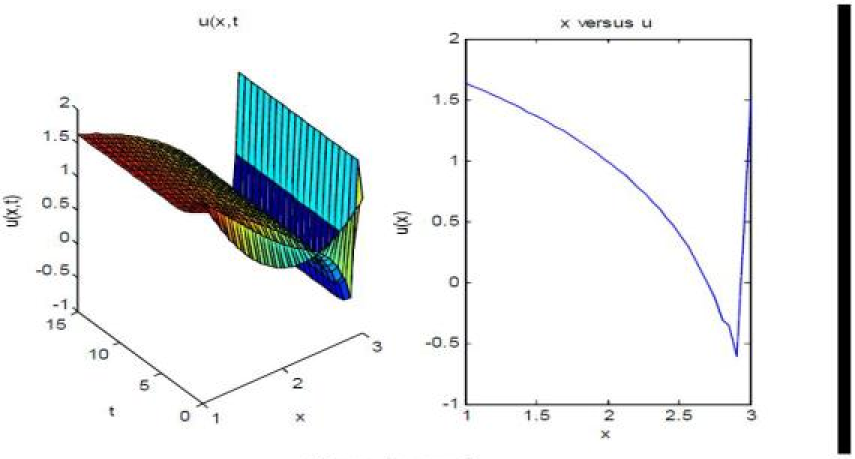

Figure for $\chi = 1$

Fig 3: Profile of heat transfer when the parameter x=1 for all time

Below, the first figure in the left hand represent the profil of heat transfer when the parameter $\chi = 2$ for all time. The figure in the right hand is the profil of heat transfer coresponding to the final time $t = 15$.

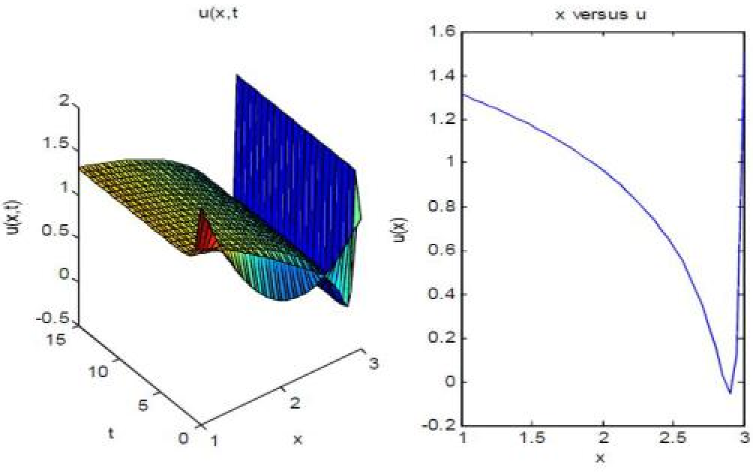

Figure for $\chi = 2$

Fig 4: Profile of heat transfer when the parameter x=2 for all time

Below, the first figure in the left hand represent the profil of heat transfer when the parameter $\chi = 1.5$ for all time The figure in the right hand is the profil of heat transfer corresponding to the final time $t = 15$

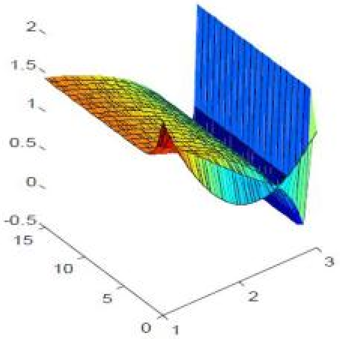

Figure for $\chi = 1.5$

Fig 5:Profile of heat transfer when the parameter x=1.5 for all time

Below, the first figure in the left hand represent the profil of heat transfer when the parameter $\chi = 0.05$ all time. The figure in the right hand is the profil of heat transfer corresponding to the final time $t = 15$

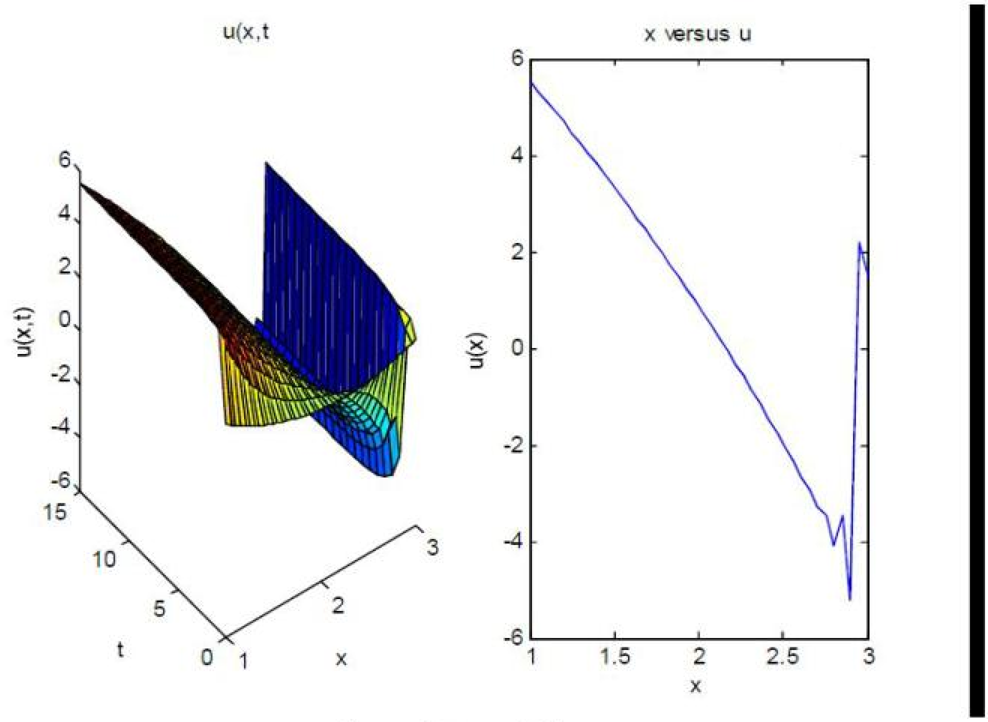

Figure for $\chi = 0.05$

Fig 6: Profile of heat transfer when the parameter x=0.5 for all time

4 Conclusion

This paper considered non linear heat transfer with non local boundary condition and temperature dependent thermal conductivity. The one-dimensional unsteady heat conduction equation was solved numerically by using implicit time-discretization and third order FDM. The boundary integral specification was computed using the 1/3 composite Simpson quadrature and Newton method provided the solution of the arising nonlinear two-point boundary value problems. The results obtained by the numerical computer experiments are consistent with the expected experimental datta. The proposed method is stable, unlike its explicit counterpart.

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