On Presentations of Semigroup of Transformations Restricted by an Equivalence

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Abstract

For a non-empty set X denote the full transformation semigroup of X by $T(X)$. Let σ be an equivalence relation on X and $E(X, \sigma)$ denotes the semigroup (under composition) of all $\alpha : X \longmapsto X$, such that $\sigma \subseteq ker(\alpha)$. Semigroup of transformations with restricted equivalence occur when we take all transformations whose kernel is contained in some fixed equivalence, $E(X, \sigma)$. First, we found that $E(X, \sigma)$ is a disjoint union copies of two generating sets. Next, we discuss the presentations, acts, subacts, direct products and bilateral semidirect product of the semigroup of transformation with restricted equivalence $E(X, \sigma)$ and its application.

Keywords: Semigroups of Transformations; Generating Set; Presentations; Direct Product; Bilateral Semi-direct Product

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1 Introduction

In the algebraic study of semigroups, Transformation Semigroups play a role analogous to that of permutation groups in group theory. For a given set X, denoted by $T(X)$, the semigroup of full transformations on X, that is, the set of all functions from X to X with functions composition as the semigroup operation. The embeddability of every semigroup in the full transformation $T(X)$ is the central reason for its fundamental role in semigroup theory. A presentation of a semigroup is a concise method of defining a semigroup in terms of generators and relations (represents a semigroup as a homomorphic images of free semigroups). The advantage of presentations when compared to other means of defining semigroups (such as Cayley tables or transformation semigroups) is that it enables us to study a larger class of semigroups, including various infinite semigroups. Therefore, the most plausible class of semigroup that presents a better analysis through presentations are the Finitely Presented Semigroup. Evidently, in some special cases much information about a particular abstract structure can be derived from a given presentation [9, 26]. A classical example of this is Coxeter Presentations [10]. Semigroups are commonly represented either by presentations using abstract generators and defining relations or by a generating set which consists of a specific types of element, such as transformations, matrices or binary relations.

Let X be a nonempty set and $T(X)$ be the semigroup (under composition) of the full transformation from X into itself, fix an equivalence σ on the set X and define a subsemigroup of $T(X)$ called the semigroup of transformations restricted by an equivalence:

$$E(X, \sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \Rightarrow x\alpha = y\alpha\}, \tag{1}$$

where $ker(\alpha) = \{\alpha \in X : \forall\ X \times X,\ x\alpha = y\alpha\ \ \forall\ x, y \in X\}$

Semigroup of transformations with restricted range (kernel) occur when we take all transformations whose image (kernel) is contained in some fixed subset (equivalence). This study have been fruitful since it was first considered by Mendes-Goncalves and Sullivan in [18], where they considered some interesting properties such as regularity, Green's relations and ideals of $E(X, \sigma)$. Sun and Wang [19] proved that for any non-trivial set X, $E(X, \sigma)$ is right abundant but not left abundant. In 2017, Sawatraska and Namnak [20] described embeddability of $E(X, \sigma)$. Next, Han and Sun [21] investigated the natural partial order of $E(X, \sigma)$ using ordered relations defined via a composition of semigroups and determined the minimal and maximal elements with respect to the partial order. In 2019, Oluyori and Imam [22] considered the semigroup $E(X, \sigma)$ on two regular semigroups and proved that $E(X, \sigma)$ is completely regular but not an inverse semigroup on its regular part (the largest regular subsemigroup), $RE(X, \sigma)$ and completely characterized its starred ideals. Yan and Wang [23] characterized the Greens relations (with respect to a nonempty subset U of the set of idempotents) as a new method of partition due to Lawson [24] on $E(X, \sigma)$ where they proved among other results that the semigroup $E(X, \sigma)$ is right Ehreshmann and the regular part, $RE(X, \sigma)$ is orthodox (completely regular) if and only if the set X consists of at most two σ-classes. Some generalizations of $E(X, \sigma)$ were considered by [33] as they studied the semigroup of partial transformations with restricted kernel and image.

In algebra and computer science an *action* or *act* of a semigroup on a set is a rule which associate to each element of the semigroup transformation of the set in such a way that the product of two elements of the semigroup is associated with the composite of the two corresponding transformations. Basically, the idea of *act or action* on semigroup affirms that the elements of the semigroup are acting as transformations of the set. From algebraic point of view, semigroup actions are generalization of the notion of a group action in group theory. Also in computer science, it is closely related to *Automata* which is the models the state of the automaton and the action models transformations of the state in correspondence to its inputs. An important special case is a *Monoid action (or Monoid act)* in which the semigroup is a monoid act as the identity transformation of a set. We see from a category theoretic view that a monoid is category with one object and an act is a functor from that category to the category of sets which provides a generalization to monoid acts on objects in categories. Finite presentability of acts was first studied by Normak in [4] and it is well known to be a fundamental finiteness condition for the theory of monoid acts [11]. In 2019, Miller and Ruskuc [2] developed a systematic theory of presentation of acts over monoids considering presentations for quotients and subacts, where they deduce a number of finite presentability results.

The notion of a bilateral semidirect product of two semigroups as considered by Kunze in [36] can be strongly link to the ideas of automata theories (see [37] and [38]). Also in [39], Kunze showed that the semigroup of all order-preserving full transformations on a finite chain is a quotient of a bilateral semidirect product of two of its subsemigroups. These results as well as its applications to Formal Languages were also discussed by the author in [40]. In 1998, Lavers [41] gave conditions under which a bilateral semidirect product of two finitely presented monoids is itself finitely presented under some conditions by presentations. The authors in [42], developed a general method for the bilateral semidirect product of two free monoids defined by the associated presentations to these free monoids. They further apply this results to some monoids of transformations that preserve or reverse the order of finite chain. In a follow up paper in 2015 [43], the authors constructed the bilateral semidirect product of two proper submonoids of certain monoids of partial permutations.

The goal of this article is to study the presentations of the semigroup of transformations restricted by an equivalence, $E(X, \sigma)$ with respect to acts, subacts, direct and bilateral semidirect product. The rest of the paper is structured as follows. In Section 2 we present some preliminaries as a background to the paper. Section 3 is furnished with some examples with respect to the two generating sets of semigroup of transformations restricted by an equivalence, $E(X, \sigma)$ denoted by Ω_1 and Ω_2, where Ω_1 represent the set of elements whose fixed equivalence is contained in the kernel of α and Ω_2, the set of elements whose fixed equivalence σ is not contained in the kernel. Thus $E(X, \sigma)$ is a disjoint union of Ω_1 and Ω_2. In Section 4, we define the presentation of the semigroup of transformation restricted by equivalence and its finitely generated monoid $\mathcal{M}$ of $E(X, \sigma)$ where we state that the semigroup $E(X, \sigma)$ has a presentation $\langle A : \sigma \rangle$ via φ. Section 5 present some results on the finite generation and finite presentability. We showed in Section 6 that for any finitely generated monoid $\mathcal{M}$ of $E(X, \sigma)$ large subacts and small extensions inherits finite generation and finite presentability. Section 7 discusses the direct products and diagonal-act of two presentations in the monoid $\mathcal{M}$

of $E(X, \sigma)$ and showed that the diagonal $(A \times B) - act$ is finitely generated (finitely presented) if the diagonal $A - act$ and the diagonal $B - act$ are finitely generated (finitely presented). In Section 8 and 9, we construct decompositions of the semigroup of transformation with restricted equivalence, $E(X, \sigma)$ by means of bilateral semidirect products and quotients and its application on the semigroup $E(X, \sigma)$.

2 Preliminaries

We recall some basic definitions as a build up to our results on the the semigroup of transformations restricted equivalence $E(X, \sigma)$. For further details of both background and technicalities we refer the reader to the following texts ([8], [9], [10], [11], [12], [13], [14], [15], [16], [17]).

A *semigroup* is a set with an associative binary operation, typically denoted by juxtaposition. A *monoid* is a semigroup with an identity element. Unless otherwise specified, the identity of any monoid is denoted by 1. A *semigroup morphism* is a map $\varphi : S \longrightarrow T$, where S and T are semigroups, and $(xy)\varphi = (x\varphi)(y\varphi)$ for all x, $y \in S$. A monoid morphism is a semigroup morphism between monoids that additionally maps the identity to the identity. If S is a semigroup, then S^1 denotes the monoid completion of S. If X is a set, then we denote by $X^*[X^+]$ the free monoid [semigroup] on X. If $\sigma \subseteq X^* \times X^*[\sigma \subseteq X^+ \times X^+]$ then we denote by $\sigma^\sharp$ the congruence on $X^*[X^+]$ generated by R. A monoid $\mathcal{M}$ is said to be defined by a presentation $\langle X|\sigma\rangle$ if $\mathcal{M}$ is isomorphic to $X/\sigma^\sharp$, where $\sigma^\sharp$ denotes the smallest congruence on X^* containing σ.

Let X be an *alphabet* and denote by X^+ the free semigroup generated by X and by X^* the free monoid generated by X. A *monoid presentation* is an ordered pair $\langle X|\sigma\rangle$, where X is an alphabet and σ is a subset of $X^* \times X^*$. An element (u, v) of $X^* \times X^*$ is called a *relation* and it is usually represented by the equality $u = v$. Let $\mathcal{M}$ be a monoid. A (right) $\mathcal{M}$-act is a non-empty set A together with a map $A \times \mathcal{M} \Rightarrow A$, $\ (a, m) \mapsto am$ such that $a(mn) = (am)n$ and $a1 = a$ for all $a \in A$ and $m, n \in \mathcal{M}$. For instance, $\mathcal{M}$ itself is an M-act via right multiplication. It is a well-known fact that an equivalence relation partitions a set (group, semigroup) into a collection of equivalence classes. An equivalence relation σ on an $\mathcal{M}$-act A is an ($\mathcal{M}$-act) congruence on A if $(a, b) \in \rho$ implies $(am, bm) \in \rho$ for all $a, b \in A$ and $m \in \mathcal{M}$. Note that the congruences on the $\mathcal{M}$-act, $\mathcal{M}$ are precisely the right congruences on $\mathcal{M}$. An *M-act A* is finitely generated if there exists a finite subset X $\subseteq$ A such that $A = XS^1$, and A is finitely presented if it is isomorphic to a quotient of a finitely generated free M-act by a finitely generated congruence. Let x be an element in X. The *length of x with respect to* X is the minimum of the set of positive integers $\{n \mid s = x_1...x_n$, for some $x_1..., x_n \in X\}$, if s is not the identity or zero, otherwise. The presentation $\langle X, \sigma\rangle$ is letter-invariant if $x \neq 1 \in S$ and $x = y$ in S if and only if $x \equiv y$, for $x, y \in X$.

A *semigroup action* of S on X becomes a monoid act by adjoining an identity to the semigroup and requiring that it acts as the identity transformation on X. A semigroup act *S-act* is cyclic is if it generated by an element of a generating set. Let M be a monoid and X be a nonempty set if there is a mapping $\alpha : X \times M \longrightarrow X$, given by $(x \bullet s) \longmapsto x \bullet s := \alpha(x \bullet s)$, and such that (i) $x \bullet 1 = x$; (ii) $(x \bullet s) \bullet t = x \bullet st$, for all $x \in X$, for all $s, t \in M$. Let M be a monoid. For any two $M - acts$ A and B, the cartesian products $A \times B$ can be made into an $M - act$ by defining $(a, b)m = (am, bm)$ for all $(a, b) \in A \times B$ and $m \in M$, thus we refer to this as *Direct Product of* $A \times B$. By definition, the direct product of any two semigroups S and T, $(S \times T)$ is the system consisting of all ordered pairs of both elements (s, t), where $s \in S$ and $t \in T$, and the operation $(s, t)(s^{'}, t^{'}) = (ss^{'}, tt^{'})$. Clearly, the direct product of the $M - act$, $M \times M$ is the *Diagonal M-act.* The theory of monoid acts is simply the representations of monoids by transformations of sets. An act is *decomposable* if it can be written as the coproduct of two subacts, $A = B \cup C$. The rank of a semigroup S, denoted by $rank(S)$, is the minimum number of elements required to generate S, that is $rank(S) = min\{\mid X \mid: X \subseteq S, S = \langle X\rangle\}$. From rank we can determine the notion of growth of a semigroup by defining *R-sequence.* The *R-sequence* of a semigroup S is the sequence obtained by taking the rank of incremental direct products of S with itself that is $R(S) = (R(s), R(s)^2, R(s)^3, ...)$. Results in this direction abounds in literatures [[34], [35] and others]. A pseudovariety is a class of finite monoids closed under finite direct products, submonoids and homomorphic images. For monoids, presentations play a central role in word/decidability problems. Although, presentations of a number of classical monoids abound in literatures, the catalog is far from complete as computing presentations of a given monoid clarifies the structural complexity of a mathematical structure.

3 Generating Set of $E(X, \sigma)$

Well known examples abound in literatures of structural theorems for semigroups, which involve decomposing a semigroup into disjoint union of subsemigroups. For example, up to isomorphism, the *Rees Theorem* states that every completely simple semigroup is a *Rees Matrix Semigroup over a group* G and is thus a disjoint union copies of G (See [12], Theorem 3.3.1); every Clifford Semigroup is a strong semilattice of groups and as such a disjoint union of its maximal subgroups (See [12], Theorem.4.2.1); every commutative semigroup is a semilattice of archimedean commutative semigroups (See [13], Theorem 4.2.2).

The establish the generating set of $E(X, \sigma)$, we go by some few examples:

We consider a finite case for any set $X = \{a, b, c, d, e, f\}$.

Then the fixed equivalence σ is defined thus

$$\sigma = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d), (e, e), (e, f), (f, e), (f, f)\}$$

and a partition set X induced by σ

$$X/\sigma = \{\{a, b\}, \{c, d\}, \{e, f\}\}$$

Define a map α_1 thus

$$\alpha_1 = \begin{pmatrix} \{ab\} & \{cd\} & \{ef\} \\ c & b & d \end{pmatrix}$$

Recall that only element in the same kernel can be paired or related. Thus:

$$\begin{aligned} ker(\alpha_1) = &\{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), \\ &(d, c), (d, d), (e, e), (e, f), (f, e), (f, f)\} \end{aligned}$$

Comparing the elements in the fixed equivalence σ with $ker(\alpha_1)$. We see that $\sigma \subseteq ker(\alpha_1)$ and the map $\alpha_1 \in E(X, \sigma)$.

Let the set X and the fixed equivalence be defined as given in example 3.1. The define another partition for the set X induced by σ as follow:

$$X/\sigma = \{\{a, d\}, \{b, e\}, \{c, f\}\}$$

Define a map α_1 thus

$$\alpha_2 = \begin{pmatrix} \{ad\} & \{be\} & \{cf\} \\ a & d & e \end{pmatrix}$$

$$\begin{aligned} ker(\alpha_2) = &\{(a, a), (a, d), (d, a), (d, d), (b, b), (b, e), \\ &(e, b), (e, e), (c, c), (c, f), (f, c), (f, f)\} \end{aligned}$$

Comparing the fixed equivalence σ with $ker(\alpha_2)$, we see that $\sigma \nsubseteq ker(\alpha_2)$ and the map $\alpha_2 \ni E(X, \sigma)$. It is evident that for any set X, the semigroup of transformations restricted by an equivalence $E(X, \sigma)$ is generated by two generating set defined as follow: $\Omega_1 = \{\alpha \in T(X) : \sigma \subseteq ker(\alpha)\}$ whose kernel is contained in the equivalence and $\Omega_2 = \{\alpha \in T(X) : \sigma \nsubseteq ker(\alpha)\}$ whose kernel is not contained in the equivalence. Thus $E(X, \sigma)$ is a disjoint union of Ω_1 and Ω_2 (that is $E(X, \sigma) := \Omega_1 \cap \Omega_2 = \emptyset$). We establish this result from the foregoing

The semigroup of transformation restricted by an equivalence, $E(X, \sigma)$ is a disjoint union of the subsemigroups Ω_1 and Ω_2.

4 Presentations of $E(X, \sigma)$

Presentation theory provides a method of establishing results about infinite semigroups. Therefore, by presentation we derive information about an algebraic structure from a presentation of it. Summarily, we see that describing an algebraic structure in terms of its set of generators and defining relations is the presentation of that algebraic structure.

We define the presentation for the semigroup of transformation with restricted equivalence, $E(X, \sigma)$ for any elements $a_1, a_2, a_3 \in E(X, \sigma)$, we have:

$$E(X,\sigma) = \langle a_1, a_2, a_3 : a_1 \sim a_1, \quad a_1 \sim a_2 \Rightarrow a_2 \sim a_1, \quad a_1 \sim a_2 \wedge a_2 \sim a_3 \Rightarrow a_1 \sim a_3 \rangle \tag{2}$$

where $a_1, a_2, a_3 \in E(X,\sigma)$. More importantly in this research, we define the semigroup of transformation with restricted equivalence, $E(X,\sigma)$ as a monoid by adjoining an empty element ϵ, to form a monoid, therefore

$$\mathcal{M} = E(X,\sigma) \cup \epsilon \tag{3}$$

Throughout this paper, we assume for any finite set of X. Let the semigroup $E(X,\sigma)$ be finite and a monoid to which we adjoin an identity denoted by $\mathcal{M}$.

Theorem 4.1 *Let $\mathcal{M}$ be the monoid of $E(X,\sigma)$, with the presentation $\langle A|\sigma\rangle$ then $\mathcal{M} \approxeq A^*/\sigma^\sharp$ where $\sigma^\sharp$ is the congruence on A^* generated by σ*

Proof. The result is clear from the definition of presentation. □

Remark 4.2 *It is clear from the last result that if a monoid surmorphism (surjective homomorphism) $A^* \longmapsto \mathcal{M}$, with kernel $\sigma^\sharp$. If ϕ is such surmorphism, the next result follows naturally.*

Corollary 4.3 *The semigroup $E(X,\sigma)$ has a presentation $\langle A|\sigma\rangle$ via ϕ where $A \subseteq \mathcal{M}$*

Proof. It is clear from remark φ is a surjective homomorphism, thus it remains to show that $ker\varphi = \sigma^\sharp$. First, we prove inclusion by showing that, $\sigma \subseteq ker(\varphi)$. In doing this we prove that φ preserves each relation from σ. Let a, $b \in A$, under the relation σ such that $\phi(ab) = \phi(a)\phi(b)$.

For the reverse inclusion, let $(a,b) \in ker\varphi$, such that $a, b \in A$ and $\varphi(a) = \varphi(b)$. Thus $a \sim b$.

□

Theorem 4.4 *[6] Every finitely generated semigroup is the quotient semigroup of a finitely generated free semigroup*

The next result is similar to [7] though in particularly for the presentation of $E(X,\sigma)$ as defined in equations (2) and (3)

Theorem 4.5 *Let $\mathcal{M}$ be the monoid of the semigroup $E(X,\sigma)$ defined by the monoid presentation $\langle A|\sigma\rangle$. Then there exists a semigroup presentation which defines $\mathcal{M}$. Moreover, if the monoid presentation is finite, then the semigroup presentation is finite.*

5 Finite Generation and Finite Presentabiliy of $E(X,\sigma)$

This section and the next is strongly motivated by the general fact in algebra that for any semigroup which can be decomposed into a disjoint union of subsemigroups, how does the properties of these subsemigroups influence the semigroup S (See [46], [47], [48]). From the result in corollary 3.3, we see that $E(X,\sigma)$ is a disjoint copies of two generating set Ω_1 and Ω_2. On this premise, we prove some results for the monoid $\mathcal{M}$ of $E(X,\sigma)$ where $\Omega_1 \cup \Omega_2 = E(X,\sigma) \subseteq \mathcal{M}$. Let $\Omega_1 = A$ and $\Omega_2 = B$ and $C = A \cup B$. Throughout this section we aim to prove some results in particular for $E(X,\sigma)$ where $C = A \cup B$ is an $M - act$ of $E(X,\sigma)$ with subacts, A and B. We prove our result in general setting for conditions when $A \cap B$ is either empty or potentially nonempty to show when $\mathcal{M}$ is finitely generated and finitely presented.

Lemma 5.1 *For any finitely generated $\mathcal{M}$. Let A and B be subacts of the union of the M-act and $C = A \cup B$. If A and B are finitely generated, then C is finitely generated.*

Proof. The proof is quite straight forward. Suppose $A = \langle\Omega_1\rangle$ and $B = \langle\Omega_1\rangle$, then $C = \langle A \cup B\rangle$. □

Theorem 5.2 *Let $\mathcal{M}$ be a finitely generated monoid of $E(X,\sigma)$ and $C = A \cup B$ be an $\mathcal{M}-act$ with A and B subacts of C. Suppose that $A \cap B$ is empty or finitely generated. If C is finitely generated, then both A and B are finitely generated.*

Proof. Let U be the generating set for any nonempty intersection of the set A and B (that is $A \cap B$). The result is straight forward when $A \cap B = \emptyset$, by this have an empty generating set U (that is $U = \emptyset$). Conversely, if $A \cap B \neq \emptyset$, then $A \cap B = \langle U \rangle$ for any finite set U.

Now, suppose that $C = \langle P \rangle$. let $Q = P \setminus B$ and $X = Q \cup U$. For any $a \in A$, if $a \in A \setminus B$, then $a = qm$ for some $q \in Q$ and $m \in \mathcal{M}$. If $a \in A \cap B$, then $a = um$, for some $u \in U$ and $m \in \mathcal{M}$. By this, we have that for any finite P, A is finitely generated and analogously, B is also finitely generated. □

Remark 5.3 *At this juncture, we see by the results in this section that two possibilities exist for $A \cap B$. The first reason is obvious from the fact that $E(X, \sigma)$ is generated by two disjoint sets, ω_1 and ω_2, thus $A \cap B = \emptyset$. The other reason is that when a monoid acts on a set the identity element acts as the identity of the function, thus $A \cap B \neq \emptyset$.*

Theorem 5.4 *Let $\mathcal{M}$ be a finitely generated monoid of $E(X, \sigma)$. Let A and B be disjoint $M-acts$. The $A \cup B$ is finitely generated if and only if A and B are finitely generated.*

Proof. Recall that by (Theorem 5.2) that A, $B \in \mathcal{M}$ are defined as the subacts of the union $A \cup B$. Also, if A and B are finitely generated, then the union $A \cup B$ is finitely generated. □

Theorem 5.5 *Let $\mathcal{M}$ be a finitely generated monoid and $C = A \cup B$ be a momoid act with A and B subacts of C. Where A and B be presented as $\langle X|\sigma_1 \rangle$ and $\langle Y|\sigma_2 \rangle$ respectively. If $A \cap B$ is finitely generated, then U is the generating set of $A \cap B$ (i.e. $A \cap B = \langle U \rangle$); otherwise the generating set $U = \emptyset$. For each $u \in U$, take $\rho_x(u) \in F_x$ and $\rho_y(u) \in F_y$ which both represent u in C, we define a set*

$$\alpha = \{\rho_x(u) = \rho_y(u) : u \in U\}$$

Then $C = \langle X, Y : \sigma_1, \sigma_2, \alpha \rangle$

Proof. Let $p_1, p_2 \in F_\xi$ such that $p_1 = p_2 \in C$, where $\xi = X \cup Y$. Suppose $p_1, p_2 \in F_X$, then the equality $p_1 = p_2$ is a consequence of σ_1. Similarly, if $p_1, p_2 \in F_Y$, then $p_1 = p_2$ is a consequence of σ_2.

Now suppose $p_1 \in F_X$ and $p_2 \in F_Y$. Let $c = um$, where $u \in U$ and $m \in M$ for any u and $m \in A \cap B$ that both p_1 and p_2 represent. It is clear that since $p_1 = \rho_x(u)m \in A$, it is consequence of the relation σ_1. Analogously, we have that is consequent to the relation σ_2, $p_2 = \rho_y(u)m \in B$. By the foregoing we can establish that by the repeated application of α, we can obtain $\rho_y(u)m$ from $\rho_x(u)m$. Therefore, the equality $p_1 = p_2$ is a consequence of σ_1, σ_2 and α. □

Theorem 5.6 *For any finitely generated $\mathcal{M}$. Let A and B be any disjoint M-act. Then $A \cup B$ is finitely presented if and only if A and B are finitely presented.*

Proof. Suppose that $\mathcal{M}$ is finitely generated such that A, $B \in \mathcal{M}$. Let A and B be M-acts and finitely presented. It follows from (Theorem 5.4) that the union $A \cup B$ is finitely presented if A and B are finitely presented. □

Now the next result gives the sufficient condition for the union of the subacts, $C = A \cup B$ to be finitely presented, though not in general [2] (Ex. 5.7 and Theorem 5.8).

Corollary 5.7 ([2], Theorem 5.9) *Let $\mathcal{M}$ be a finitely presented monoid. Let $C = A \cap B$ be a monoid act with A and B subacts of C. suppose that $A \cap B = \emptyset$ or $A \cap B = \langle U \rangle$. If A and B are finitely presented, then C is finitely presented.*

6 Results on Acts and Subacts of $E(X,\sigma)$

For the remainder of this section, we consider a case where we have a subact A with finite complement in an $\mathcal{M}-act$, $(A \subseteq B \setminus A)$ which is analogous to *large subsemigroups* within the semigroup framework (See [3], [18] and [22]). In [3], the author show that various finiteness properties inherited by both large subsemigroups and small extensions (as briefly discussed in the later part of the last section). In the same vein, Miller and Ruskuc [2] showed that for any finitely generated monoids $\mathcal{M}$, finite generation is inherited by both large subacts and small extensions while only small extension inherits finite presentability. Having established the foregoing, we will show that for any the finitely generated monoid $\mathcal{M}$ of $E(X,\sigma)$ that large subacts and small extensions both inherits finite generation and finite presentability.

A *Subact* B of an $\mathcal{M}-act$. Let B be a large subset in A and A is said to be a small extension of B. For a subact B with a finite complement in the $\mathcal{M}-act$ A is finite. We investigate the finite generation and finite presentability of the $\mathcal{M}-act$ and a large subact B of A. The following results on subacts is similar to [2] and [11]

Theorem 6.1 *Let M be a finitely generated monoid of $E(X,\sigma)$. Let A be an $\mathcal{M}-act$ and B be a large subact of A. If B is finitely generated, then A is finitely generated*

Proof. The proof is straight forward. For any finite set X. Suppose the large subact B is generated by the set X, then the $\mathcal{M}-act$ A is generated by $X \cup (A \setminus B)$. □

Theorem 6.2 *Let $\mathcal{M}$ be a finitely generated monoid of $E(X,\sigma)$. Let A be an $\mathcal{M}-act$ and B be a large subact of A. If A is finitely generated then B is finitely generated.*

Proof. Since A is the disjoint union of its subacts B and $A \setminus B$ (similar to Ω_1 and Ω_2 as defined for $E(X,\sigma)$ in section 3). It is clear from the result (theorem 5.4), that the finite generation of B is consequence of A □

Remark 6.3 *We have shown from the results stated above that for any finitely generated monoid of $E(X,\sigma)$, $\mathcal{M}$ that finite generation is inherited by both large subacts and small extensions. Now we show if every small extension and large subact of every finitely presented $\mathcal{M}-act$ is finitely presented.*

Theorem 6.4 *For any finitely presented monoid $\mathcal{M}$ of $E(X,\sigma)$. The following are equivalent:*

i) every finite $\mathcal{M}-act$ is finitely presented

ii) every small extension of the finitely presented $\mathcal{M}-act$ is finitely presented.

Proof. We prove this result in two part. For the forward case (i)⇒ (ii). Let A be a small extension of a finitely presented $\mathcal{M}-act$ B. Recall that A/B is finite and thus we assume it is finitely presented.

For the backward case, (ii)⇒ (i). For any finite $\mathcal{M}-act$, A, we take a finitely presented $\mathcal{M}-act$, B disjoint from A. Recall by Theorem 5.4 and Theorem 5.6 that $A \cup B$ is finitely presented. □

7 Direct Product of $E(X,\sigma)$

This section is strongly motivated by the work of [1], [8], [9] and [32]. Using the generating set and presentation of $E(X,\sigma)$, we construct the direct product $A\times B$, which leads to the characterisations of the monoids $\mathcal{M}$ that have the property that, for any two $\mathcal{M}-acts$ A and B, the direct product $A\times B$ is finitely generated (resp. finitely presented) if and only if both A and B are finitely generated (resp. finitely presented). Taking some cue from general algebra of groups and semigroups (see for example [8], [9]). East in [32] affirmed that, presentations for direct products of semigroups are not well-behaved as that of groups or monoids. Therefore we consider the presentations of the direct product of the monoid of $E(X,\sigma)$, $\mathcal{M}$ as defined in section 3.

Theorem 7.1 *For any monoid $\mathcal{M}$ of the semigroup $E(X, \sigma)$ with two presentations defined as $A = \langle X_1 : \sigma_1 \rangle$ and $B = \langle X_1 : \sigma_1 \rangle$, the direct product $A \times B = \langle X_1 \cup X_2 : \sigma_1 \cup \sigma_2 \cup \sigma_3 \rangle$, where σ_3 consists of all relations*

Proof. Let $\mathcal{M}$ be a monoid. For any presentations defined as $A = \langle X_1 : \sigma_1 \rangle$ and $B = \langle X_1 : \sigma_1 \rangle$ $\in \mathcal{M}$, the direct product $A \times B$ is preserved by definition. □

Theorem 7.2 *Let A and B be any presentations in the monoid $\mathcal{M}$ of the semigroup $E(X, \sigma)$. If $A = \langle X : \sigma_1 \rangle$ and $B = \langle X_1 : \sigma_1 \rangle$, the direct product $A \times B$ is finitely presented if and only A and B are finitely presented.*

Proof. In this proof, it remain to show that the presentations A and B are homomorphic images of the direct product $A \times B$. □

Theorem 7.3 *For any monoid $\mathcal{M}$, the property $\mathcal{Q}$ is preserved in direct products if, for any two $\mathcal{M} - acts$ A and B, the direct product $A \times B$ has property $\mathcal{Q}$ if and only if both A and B have property $\mathcal{Q}$.*

Proof. Let $\mathcal{M}$ be a monoid and $A = \langle X_1 : \sigma_1 \rangle$ and $B = \langle X_1 : \sigma_1 \rangle \in \mathcal{M}$. It follows by (Theorem 7.1) that the direct product $A \times B$ is preserved □

Theorem 7.4 *Let $\mathcal{M}$ be the monoid that preserve finite generation (resp. finitely presentability) in direct products then the diagonal M-act $\mathcal{M} \times \mathcal{M}$ is finitely generated (resp. finitely presented).*

Proof. Let $\mathcal{M}^*$ and $\mathcal{M}_{\circledast} \in E(X, \sigma)$ be two monoids with presentations defined on them as follows:
For $\mathcal{M}^*$, we have presentations $A = \langle X_1 : \sigma_1 \rangle$ and $B = \langle X_1 : \sigma_1 \rangle$ and
For $\mathcal{M}_{\circledast}$, we have presentations $A = \langle X_1 : \sigma_1 \rangle$ and $B = \langle X_1 : \sigma_1 \rangle$. It is clear by (Theorem 7.2) that the cartesian product of $\mathcal{M}^*$ and $\mathcal{M}_{\circledast}$ are diagonal $M - act$ and are finitely generated and finitely presented. □

Next we state a general result by Gallagher (2005) to lay the foundation for subsequent results on diagonal act.

Theorem 7.5 ([44], Theorem 4.1.5, Corollary 4.1.9) *Let X be any infinite set, and let M be any of the of the following transformation monoids on X:*

- *$\mathcal{B}_{\mathcal{X}}$ (the monoid of binary relations);*
- *$\mathcal{T}_{\mathcal{X}}$ (the full transformation monoid);*
- *$\mathcal{P}_{\mathcal{X}}$ (the monoid of partial transformations);*
- *$\mathcal{F}_{\mathcal{X}}$ (the monoid of finite full one-to-one transformation).*

Then the diagonal $\mathcal{M} - act$ is a free cyclic $\mathcal{M} - act$ and hence finitely presented

Theorem 7.6 ([45], Lemma 2.2) *Let $\mathcal{M}$ be a monoid, let $\mathcal{N}$ be a submonoid of $\mathcal{M}$, and suppose that $\mathcal{M} \setminus \mathcal{N}$ is an ideal of $\mathcal{M}$. If the diagonal $\mathcal{M} - act$ is generated by a set $U \times U$, then the diagonal $\mathcal{N}$-act is generated by the set $V \times V$ where $V = U \cap \mathcal{N}$. In particular, if the diagonal $\mathcal{M}$-act is finitely generated, then the diagonal $\mathcal{N}$-act is finitely generated.*

Theorem 7.7 *Let M be a monoid of the semigroup $E(X, \sigma)$. The diagonal $M - act$ is finitely generated if and only if there exist a generating set of the form $Q \times Q$ for some finite subset $Q \subset M$.*

Proof. The proof is consequent to (Theorem 7.2 - Theorem 7.4) □

Our next result shows that the monoid property that the diagonal act is finitely generated is preserved by direct products.

Theorem 7.8 *Let A and B be two monoids in the semigroup $E(X, \sigma)$. Then the diagonal $(A \times B) - act$ is finitely generated if and only if both the diagonal $A - act$ and the diagonal $B - act$ are finitely generated*

Proof. For the necessary case. Suppose the diagonal $A - act$ and the diagonal $B - act$ are generated by the finite sets $U \times U$ and $V \times V$ respectively. Then it remain to show that the diagonal $(A \times B) - act$ is generated by $(U \times V) \times (U \times V)$. For any (a_1, b_1), $(a_2, b_2) \in A \times B$. Then
$(a_1, a_2) = (u_1 = u_2)a$ for some $u_1, u_2 \in U$ and $a \in A$ and
$(b_1, b_2) = (v_1 = v_2)a$ for some $v_1, v_2 \in V$ and $b \in B$

Therefore, we have that

$$((a_1, b_1), (a_2, b_2)) = ((u_1, v_1)(u_2, v_2))(a, b)$$

For the sufficiency case. Suppose the diagonal $(A \times B) - act$ is generated by $U^* \times U^*$, where U^* is the projection of U to A. For any $a_1, a_2 \in A$, and $b_1, b_2 \in B$, we have that

$$((a_1, b_1), (a_2, b_2)) = ((p_1, q_1)(p_2, q_2))(a, b)$$

.

for some $(p_1, q_1), (p_2, q_2) \in U$ and $(a, b) \in A \times B$. Hence, we have that

$$(a_1, a_2) = (p_1, p_2)a \in \langle U^* \times U^* \rangle$$

Thus, we have proved that $A \times A$ is finitely generated. Analogously, we can prove for $B \times B$ and by that complete the proof □

Remark 7.9 *Now for the finite presentability, since the semigroup of transformation restricted by an equivalence, $E(X, \sigma)$ is well known to as a subsemigroup of the full transformation $\mathcal{T}_{\mathcal{X}}$, we recall from (Theorem 7.5) that the diagonal $\mathcal{M} - act$ is finitely presented if $\mathcal{M}$ is any of the monoids of binary relations, full transformations, partial transformations and finite full one-to-one transformations for the infinite case. Thus we prove in what follows that the monoid property of the diagonal act is finitely presented and can be inherited by substructures and extensions in some special cases,and is also preserved by direct products.*

Theorem 7.10 *For any monoid, $\mathcal{M}$ of the semigroup $E(X, \sigma)$. Let A be a submonoid of $\mathcal{M}$ and suppose that $\mathcal{M} \setminus A$ is an ideal of $\mathcal{M}$. If the diagonal $\mathcal{M} - act$ is finitely presented, then the diagonal $A - act$ is finitely presented.*

Proof. Let $\mathcal{M} \times \mathcal{M}$ be defined by the finite presentation $\langle Y \times Y : \sigma \rangle$. Recall from [Theorem 7.6], that $A \times A = \langle X \times X \rangle$, where $X = Y \cap A$. Hence, let $B = X \times X$ and $\sigma' = \sigma \cap (F_B \times F_B)$ and show that $A \times A$ is defined by the finite presentation $\langle B : \sigma' \rangle$.

From the foregoing it is clear that $A \times A$ satisfies the relation σ'. Let $a_1, a_2 \in F_B$ be such that equality holds that is $a_1 = a_2$ in $A \times A$. Thus it remains to show that the equality $a_1 = a_2$ is a consequence of the relation σ'. Recall that $a_1 = a_2 \in \mathcal{M} \times \mathcal{M}$, by this it is clear that there exist an R-sequence connecting a_1 and a_2. Since $M \setminus A$ is an ideal of M, every element of M in the sequence is contained in A. Thus, $a_1 = a_2$ is a consequence of σ'. This completes the proof □

8 Bilateral Semidirect Decomposition of $E(X, \sigma)$

This section is strongly motivated by the works of Kunze [39] and most recently by Fernandes and Quinteiro on full transformations semigroup (see [42]) and others. We further the investigation on acts, subacts and quotient of the finitely generated monoid of the semigroup of transformation restricted by an equivalence. But first we begin with the construction of the bilateral semidirect products using presentations of semigroup of transformation restricted by an equivalence. We present a general technique to obtain a bilateral semidirect decomposition of a monoid in terms of its submonoids.

Let S and T be two semigroups. Let

$$\delta : T \longrightarrow \mathcal{T}(S)$$

$$u \longmapsto \delta_u : S \longrightarrow S$$

$$s \longrightarrow u \bullet s$$

be an *antihomomorphism of semigroups* (i.e. $(uv) \bullet s = u \bullet (v \bullet s)$, for $s \in S$ and $u, v \in T$) and let

$$\varphi : S \longrightarrow \mathcal{T}(T)$$

$$s \longmapsto \varphi_s : T \longrightarrow T$$

$$u \longrightarrow u^s$$

be a homomorphism of semigroups (i.e. $(u^{sr} = u^s)^r$. For $s, r \in S$ and $u \in T$) such that:

(SPR) $(uv)^s = u^{v \bullet s} v^s$, for $s \in S$ and $u, v \in T$ (*Sequential Processing Rule*); and

(SCR) $u \bullet (sr) = (u \bullet s)(u^s \bullet r)$, for $s, r \in S$ and $u \in T$ (*Serial Composition Rule*)

Thus, we say δ is a $left\ action$ of T on S and φ is a *right action* of S on T. In 1992, Kunze [39] proved that the set of the product S and T (i.e. $S \times T$) is a semigroup with respect to multiplication in this wise:

$$(s, u)(r, v) = (s(u \bullet r), u^r v)$$

for $s, r \in S$ and $u, v \in T$. Thus we simply denote this semigroup $S \times T$ with respect to δ and φ as $S \bowtie T$ which we call *Bilateral Semidirect Product* of S and T associated with δ and φ

8.1 Constructing Bilateral Semidirect Products of $E(X, \sigma)$

Let A and B be two alphabets. Suppose we defined actions on the elements satisfying

$$b \bullet a \in A \cup \{1\}, \quad 1 \bullet a = a, \quad b \bullet 1 = 1 \quad , 1 \bullet 1 = 1 \ldots (1)$$

and

$$b^a \in B^*, \quad b^1 = b, \quad 1^a = 1, \quad , 1 \bullet 1 = 1, \ldots (2)$$

for $a \in A$ and $b \in B$. The first, inductively on the length of $u \in B^+$, define

$$(ub) \bullet a = u \bullet (b \bullet a) \ldots (3)$$

and

$$(ub)^a = u^{b \bullet a} b^a, \ldots (4)$$

for $a \in A \cup \{1\}$ and $b \in B$. Secondly, inductively on the length of $s \in A^+$, define

$$u \bullet (as) = (u \bullet a)(u^a \bullet s) \ldots (5)$$

and

$$u^{as} = (ua)^s, \ldots (6)$$

for $u \in B^*$ and $a \in A$. Thus, we have well defined mappings

$$\delta : B^* \Rightarrow T(A^*)$$

$$u \mapsto \delta_u : A^* \longrightarrow A^*$$

$$s \mapsto u \bullet s$$

and

$$\varphi : A^* \mapsto T(B^*)$$
$$s \mapsto \varphi_s : B^* \mapsto B^*$$
$$u \longmapsto u^s$$

The dual of (1) - (6) above is thus:

Having defined actions of and on the letter satisfying

$$b \bullet a \in A^*, \quad 1 \bullet a = a, \quad b \bullet 1 = 1 \quad , 1 \bullet 1 = 1 \ldots (7)$$

and

$$b^a \in B \cup \{1\}, \quad b^1 = b, \quad 1^a = 1, \quad , 1^1 = 1, \ldots (8)$$

for $a \in A$ and $b \in B$. The first, inductively on the length of $s \in A^+$, define

$$b^{as} = (b^a)^s \ldots (9)$$

and

$$b \bullet (as) = (b \bullet a)(b^a \bullet s) \ldots (10)$$

and

for $a \in A$ and $b \in B \cup \{1\}$ and secondly, inductively on the length of $u \in B^+$, define

$$(ub)^s = u^{b \bullet s} b^s, \ldots (11)$$

and

$$(ub) \bullet s = u \bullet (b \bullet s) \ldots (12)$$

for $u \in A^*$ and $a \in B$.

Lemma 8.1 *For a finitely generated monoid $\mathcal{M}$ of the semigroup $E(X, \sigma)$. Let $s, t \in A^*$ and $u, v \in B^*$. The following holds:*

(a) $1 \bullet s = s$ *and* $1^s = 1$;

(b) $u \bullet 1 = 1$ *and* $u^1 = u$

Proof. The proof is in two parts: For $| S | \leq 1$, by this it is clear that equalities follow naturally from (1) and (2) from the above construct. Proceeding by induction, the length of s. Suppose that $| S | > 1$ and let $a \in A$ and $s' \in A^+$ such that $s = as'$. Since $1 \leq | s' | < | s |$. Inductively, we have $1 \bullet s' = s'$ and $1^{s'} = 1$, where $1 \bullet s = 1 \bullet (as') = (1 \bullet a)(1^a \bullet s') = a(1 \bullet s') = as' = s$, by (5) and by applying (6), we have

$$1^s = 1^{as'} = (1^a)^{s'} = 1^{s'} = 1$$

For the second part, For $| u | \leq 1$, equalities holds from (3) and (4). Thus, by induction on the length of u. Suppose $| u | > 1$, and let $b \in B$ and $u' \in B^+$ such that $u = bu'$. Thus by induction hypothesis, since $1 \leq | u^1 | < | u |$, we have that $u^1 \bullet 1 = u$ by (1) and $(u^1) = u$ by (2) □

The following are well-known results from [42] as necessary results to establish our contribution on the Monoid acts of the semigroup of transformations restricted by an equivalence, $E(X, \sigma)$. The next result follows naturally from the definition of the *Sequence Processing Rule (SPR)* and *Sequence Composition Rule (SCR)*.

Theorem 8.2 *Let $\mathcal{M}$ be a finitely generated monoid of $E(X, \sigma)$ with submonoids A^* and B^*. Let $p, q \in A^*$ and $u, v \in B^*$. Then:*

(a) $u \bullet (pq) = (u \bullet p)(u^p \bullet q)$;

(b) $(uv)^p = u^{v \bullet p} v^p$

Proof. Suppose $p = 1 = q$, then equality is obvious by 8.1(b). For any $p,\ q \in A^+$, we proceed by induction on the length of p. Now, if $|p| = 1$, the equality holds for the expression (5) as defined in the construct. Therefore, let $p = ap'$ with $a \in A$ and $p' \in A^+$. Hence $1 \leq |p'| < |p|$, we have that

$u \bullet (pq) = u \bullet (ap'q)$
$= u \bullet a)(u^a \bullet (p'q)) \ldots$ (by (5) as defined in the construct)
$= (u \bullet a)(u^a \bullet p')((u^a)^{p'} \bullet q) \ldots$ (by induction hypothesis)
$= (u \bullet (ap'))(u^{ap'} \bullet q) \ldots$ (by (5) and (6) as defined in the construct)
$= (u \bullet p)(u^p \bullet q)$

To prove the second part, we show that $(uv)^a = u^{v \bullet a} = u^{v \bullet a} v^a$, for $a \in A \cup \{a\}$. If $u = 1$, equality follows from (2) as defined in the construct. Now, since $v \bullet a \in A \cup \{1\}$. Now for $|u| \geq 1$, we proceed by induction on the length of u. Suppose $v = 1$, equality is obvious by (1) and (2) and $|v| = 1$, by (4). Let $v = v'b$ with $v' \in B^+$ and $b \in B$. Then as $1 \leq |v'| < |v|$ and $b \bullet a \in A \cup \{1\}$, we have

$(uv)^a = (uv'b)^a$
$= (uv')^{b \bullet a} b^a \ldots$ (by (4) as in the construct)
$= u^{v' \bullet (b \bullet a)} v'^{b \bullet a} b^a \ldots$ (by Induction hypothesis)
$= u^{(v'b) \bullet a} (v'b)^a$ (by (3) and (4) as in the construct)
$= u^{v \bullet a} v^a$

Next, we show that equality follows for any $p \in A^*$ by induction hypothesis with the length of p. Recall that for $|p| \leq 1$, take $p = ap'$ where $a \in A$ and $p' \in A^*$. Since $1 \leq |p'| < |p|$ and $v \in a \in A \cup \{1\}$, we have

$(uv)^p = (uv)^{ap'}$
$= ((uv)^a)^{p'}$ (as defined by (6))
$= (u^{v \bullet a} v^a)^{p'}$ (for case $|p| = 1$)
$= (v^{v \bullet a})^{va \bullet p'} (v^a)^{p'}$ (by induction hypothesis)
$= u^{(v \bullet a)(v^a \bullet p')} v^{ap'}$ (by (6) and lemma 1.1(b))
$= u^{v \bullet (ap')} v^{ap'}$ (by (5))
$= u^{v \bullet p} v^p$ $\square$

Theorem 8.3 *For a finitely generated monoid $\mathcal{M}$ of $E(X, \sigma)$. Let $p,\ q \in A^*$ and $u,\ v \in B^*$ be submonoids of $\mathcal{M}$. Then:*

(a) $(uv) \bullet s = u \bullet (v \bullet p)$;

(b) $u^{pq} = (u^s)^r$

Proof. (a) Suppose that $(uv) \bullet a = u \bullet (v \bullet a)$ for $a \in A \cup \{1\}$, by induction on the length of v. By $|v| \leq 1$, equality holds from (1) and (3) in the construct above. Now, for $|v| > 1$. Let $b \in B$ and $v' \in B^+$ be such that $v = v'b$. Since $1 \leq |v'| < |v|$ and $b \bullet a \in A \cup \{1\}$ by (3) and by induction hypothesis, we have

$$(uv) \bullet a = (uv'b) \bullet a = (uv') \bullet (b \bullet a) = u \bullet (v' \bullet (b \bullet a)) = u \bullet ((v'b) \bullet a) = u \bullet (v \bullet a)$$

Taking the induction on the length of p. For $|p| > 1$. Let $a \in A$ and $p' \in A^+$ such that $p = ap'$. Since $1 \leq |p'| < |p|$, we have that

$(uv) \bullet p = (uv) \bullet (ap')$
$= ((u \bullet v) \bullet a)((uv)^a \bullet p')$
$= ((u^{v \bullet a}))((u^{v \bullet a} \bullet v^a) \bullet p')$ (by the case $|p| = 1$ and SPR)
$= (u \bullet (v \bullet a))(u^{v \bullet a} \bullet (v^a \bullet p'))$ (by induction hypothesis)
$= u \bullet ((v \bullet a)(v^a \bullet p'))$ (by SCR)
$= u \bullet (v \bullet p)$

(b) Now suppose $p = 1$ and $q = 1$, equality holds from 8.1(b). For $p, q \in A$. We proceed by induction on the length of p. If $|p| = 1$, equality holds in (6). Let $p = ap^{'}$, with $a \in A$ and $p^{'} \in A^{+}$. Since $1 \leq |p^{'}| < |p|$, we have that

$$u^{pq} = u^{ap^{'}q} = (u^{a})^{p^{'}q} = ((u^{a})^{p^{'}q} = (u^{ap^{'}})^{q} = (u^{p})^{q}$$

Using (6) in the second and fourth expressions and taking the induction hypothesis on the third expression our result is complete. □

The next two results are from the definition of (1) - (6) and (7) - (12) respectively

Lemma 8.4 *Let the mapping δ and φ represents the unique left action of B^* on A^* and right action of A^* on B^* respectively, extending the actions of letters defined on letters.*

Lemma 8.5 *Let the mappings as defined in (7) - (12) be the unique left action of B^* on A^* and right action of A^* on B^* preserve the actions of the letters defined on it.*

Let δ be a left action of B^* on A^* and φ be a right action of A^* on $B^* \subseteq \mathcal{M}$. We say the mapping δ (resp. φ) preserves letters if the action of a letter on a letter is a letter or the empty word or in otherword if it satisfies (1) and (8) as defined above.

Theorem 8.6 *Let σ_1 be set of relations on A^* and σ_2 be set of relations on B^*. Let S and T be the monoid defined by the presentations $\langle A \mid \sigma_1 \rangle$ and $\langle B \mid \sigma_2 \rangle$ respectively. Then the action δ (resp. φ)* preserves *the presentation $\langle A \mid \sigma_1 \rangle$ and $\langle B \mid \sigma_2 \rangle$ if $b \bullet s = b \bullet r$ in S (resp., $b^s = b^r$ in T) for all $(s = r) \in \sigma_1$ and $b \in B$ and $u \bullet a = v \bullet a$ in S (resp., $u^a = v^a$ in T) for all $(u = v) \in \sigma_2$ and $a \in A$. Let $z \in A^*$ and w_1, $w_2 \in B^*$ be such that $w_1 = w_2$ in T. Then, we have $w_1 \bullet z = w_2 \bullet z$ in S and ${w_1}^z = {w_2}^z$ in T.*

Proof. The proof is consequent to Theorem 8.2 and 8.4 □

The next result gives a well-defined bilateral semidirect product induced by the actions of φ and δ.

Theorem 8.7 *If a left action of $B^{\star}$ on $A^{\star}$ and a right action of $A^{\star}$ on $B^{\star}$ preserve letters and preserve the letter-invariant presentations $\langle A|\sigma_1$ and $\langle B|\sigma_2$, this induces left action of T on S and a right action of S on T.*

Proof. The result is consequent to Theorem 8.2 and Theorem 8.4 □

As a recap, Let $\mathcal{M}$ be a finitely generated monoid and let S and T be two submonoids of $\mathcal{M}$. Let A and B be the set of generators of S and T respectively. we say the left action of T on S (resp., S on T) *preserves* A(resp. B) if $b \bullet a \in A \cup \{1\}(resp., b^a \in B \cup \{1\})$, for $a \in A$ and $b \in B$. Also, it is clear that if the left action preserves A then, $u \bullet a \in A \cup \{1\}$, for $a \in A$ and $u \in T$.

Theorem 8.8 *Let M be a monoid for two submonoids S and $T \in M$. Let A and B be sets of generators of the submonoids, S and T respectively. For $a \in A$ and $b \in B$, be such that $ba = (b \bullet a)b^a \in M$. If either the left action is preserves A or the right action preserves. For any $s \in S$ and $u \in T$, then $us = (u \bullet s)u^s \in M$*

Proof. For any $s \in S$ and $u \in T$. We proceed by induction hypothesis on the length of s that if $|S| = 0$, equality holds. Also for $|S| = 1$, $ua = (u \bullet a)u^a$, for $a \in A$ and $u \in T$. Similarly, for $|u| = 0$ and $|U| = 1$, equality holds. Now let $u \in T$ be such that $|U| = k$. Then $u = bv$, for some $b \in B$ and $v \in T$ with length $k = 1$. Let $a^{'} = v \bullet a \in A \cup \{1\}$. Hence for $1 \leq |u| < k$,

$$ua = b(va) = b((v \bullet a)v^a) = (ba^{'})b^{a^{'}}v^a = ((bv) \bullet a)(bv)^a = (u \bullet a)u^a$$

Similarly, for any $s \in S$ with length n such that $1 \leq |s| < n$. let $u \in T$. Then $s = ra$, for some $a \in A$ and $r \in S$ with length $n - 1$. Let $v = u^r \in T$. Then

$$us = (ur)a = ((u \bullet a)u^r)a = (u \bullet r)(v \bullet a)v^a = (u \bullet r)(u^r \bullet a)(u^r)^a = (u \bullet (ra))u^{va} = (u \bullet s)u^s$$

the result is complete by SCR condition □

Theorem 8.9 *Let S and T be two submonoids of the monoid M generated by A and B respectively. Let bilateral semidirect product of S and T, $S \bowtie T$, be such that either the left action preserves A or the right action preserves B. If $A \cup B$ generates the monoid M and $ba = (b \bullet a)b^a$ in M, for $a \in A$ and $b \in B$, then M is a homomorphic image of $S \bowtie T$.*

Proof. To show that the mapping

$$\varphi : S \bowtie T \longrightarrow M$$

$$(s, u) \longmapsto su$$

is a surjective homomorphism, we show that the map φ is a homomorphism. Let $(s, u)(r, v) \in S \bowtie T$. Then

$$(s, u)\varphi(r, v)\varphi = surv = s(u \bullet r), u^r v = (s(u \bullet r), u^r v)\varphi = ((s, u)(r, v))\varphi$$

Clearly from the second result of (theorem 8.8). For the second part, to show that the map φ is onto. Let $x \in M$, since $M = \langle A \cup B\rangle$, for some $s_1 \cdots s_k \in S$ and $u_1, \cdots, u_k \in T$ we have that $x = s_1u_1 \cdots s_ku_k \in T$. Assume that k is the least positive integer for which such a decomposition exist. Let $k \geq 2$, by **??**, we have that

$$x = s_1u_1, \cdots s_{k-1}(u_{k-1}s_k)u_k = s_1u_1 \cdots s_{k-1}(u_{k-1} \bullet s_k)u_{k-1}^{sk}u_{k-1}$$

Clearly k is not minimal by our result. Therefore, $k = 1$ as required. □

Theorem 8.10 *Let M be a monoid and let S and T be two submonoids of M generated by A and B respectively. Let $S \star T$ be a semidirect product of S and T. For any $a \in A$ and $b \in B$. If $M = \langle A \cap B\rangle$ and $ba = (b \bullet a)b \in M$ then M is a homomorphic image of $S \star T$*

Proof. The result follows from Theorem 8.8 and Theorem 8.9 □

9 Application

This section is the application of the results on the previous sections to the semigroup of transformations restricted by an equivalence $E(X, \sigma)$. From Section 3, we see that the semigroup of transformations restricted by an equivalence is a union of two generating sets, $\Omega_1 := \{\alpha \in T(X) : \sigma \subseteq ker(\alpha)\}$ and $\Omega_2 := \{\alpha \in T(X) : \sigma \nsubseteq ker(\alpha)\}$. Now in this section we construct the bilateral semidirect decomposition of the monoid of $E(X, \sigma)$. First, notice that Ω_1 and Ω_2 are isomorphic monoids (similar to the semigroup, $\mathcal{O}_n$ discussed by [42]). The mapping from Ω_1 onto Ω_2 which maps each transformation $a \in \Omega_1$ in the transformation Ω_2 is an isomorphic monoid. For $i \in \{1, ...n - 1\}$ Therefore let $A = \{a_1, ...a_{n-1}\}$ and $B = \{b_1, ...b_{n-1}\}$. For any alphabets A and B of the generating sets of Ω_1 and Ω_2 respectively.
Let $n \in \mathbb{N}$. We define the set of relations σ^+ for Ω_1 as follows:

- $a_i^2 = a_i$, for $1 \leq i \leq n - 1$
- $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$, for $1 \leq i \leq n - 2$ and
- $a_ia_j = a_ja_i$, for $1 \leq i \leq n - 1$ and $| i - j |\geq 2$

and for σ^-, the second set of relations

- $b_i^2 = b_i$, for $1 \leq i \leq n - 1$
- $b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}$, for $1 \leq i \leq n - 2$ and
- $b_ib_j = b_jb_i$, for $1 \leq i \leq n - 1$ and $| i - j |\geq 2$

Thus the monoid Ω_1 and Ω_2 can be defined by the presentations $\langle A \mid \sigma^+\rangle$ and $\langle B \mid \sigma^-\rangle$ respectively. Also, the monoid of $E(X, \sigma)$ can be defined in terms of the union of the generating sets Ω_1 and Ω_2 that is, $\Omega_1 \cup \Omega_2$ and the set of equivalence relations $\sigma = \sigma^+ \cup \sigma^-$ as follows:

- $a_ib_i = b_ia_{i-1}$, for $2 \leq i \leq n - 1$,
- $b_ia_i = a_ib_{i+1}$, for $1 \leq i \leq n - 2$,
- $a_ib_i = b_i$, for $1 \leq i \leq n - 1$,

- $b_i a_i = a_i$, for $1 \leq i \leq n-1$,
- $b_j a_i = a_i b_j$, for $1 \leq i, j \leq n-1$ and $j \ni \{i, i+1\}$,
- $a_{n-1}a_{n-2}a_{n-1} = a_{n-1}a_{n-2}$ and
- $b_1 b_2 b_1 = b_1 b_2$,

is defined by $\langle \Omega_1 \cup \Omega_2 \mid \sigma \rangle$. Therefore, by Prop. 1.4 and 1.5 we can consider the left action δ of B^* on A^* and the right action φ of A^* on B^* that extends the action of and on the letters:

$$b_j \bullet a_i = \begin{cases} 1, & \text{if j = i + 1;} \\ a_i, & \text{Otherwise.} \end{cases}$$

and

$$b_j^{a_i} = \begin{cases} 1, & \text{if j = i;} \\ b_j, & \text{Otherwise.} \end{cases}$$

for $1 \leq i, j \leq n-1$. From the fore-going, it is clear that both the presentations $\langle A \mid \sigma^+ \rangle$ and $\langle B \mid \sigma^- \rangle$ for the monoids of the generating set are letter-irredundant as the maps representing the left action of B^* on A^* and the right action of A^* on B^* δ and φ respectively preserve letters (words). By aforestated, these results follow naturally:

Theorem 9.1 *Let δ be the left action of B^* on A^* and φ the right action of A^* on B^*. Then the actions of both maps δ and φ preserve the presentations $\langle A \mid \sigma^+ \rangle$ and $\langle B \mid \sigma^- \rangle$*

Proof. To prove this result we state the following relations:
Case (i) For $1 \leq j \leq n-1$

$$\begin{cases} b_j \bullet a_i^2 = b_j \bullet a_i, & \text{for } 1 \leq \text{i} \leq \text{n-1 ;} \\ b_j \bullet (a_i a_{i+1} a_i) = b_j \bullet (a_{i+1} a_i a_{i+1}) = b_j \bullet (a_{i+1} a_i), & \text{for } 1 \leq \text{i} \leq \text{n-2;} \\ b_j \bullet (a_i a_k) = b_j \bullet (a_k a_i), & \text{for i} \leq \text{i, k} \leq \text{n-1 and } \mid i-k \mid \geq 2; \\ b_j^{a_i^2} = b_j^{a_i}, & \text{for } 1 \leq \text{i} \leq \text{n-1;} \\ b_j^{a_i a_{i+1} a_i} = b_j^{a_{i+1} a_i a_{i+1}} = b_j^{a_{i+1} a_i}, & \text{for } 1 \leq \text{i} \leq \text{n-2;} \\ b_j^{a_i a_k} = b_j^{a_k a_i}, & \text{for } 1 \leq \text{i,k} \leq \text{n-1 and } \mid i-k \mid \geq 2. \end{cases}$$

Case (ii) For $1 \leq i \leq n-1$

$$\begin{cases} b_j^2 \bullet a_i = b_j \bullet a_i, & \text{for } 1 \leq \text{i} \leq \text{n-1 ;} \\ (b_j b_{j+1} b_j) \bullet a_i = (b_{j+1} b_j b_{j+1}) \bullet a_i = (b_{j+1} b_j) \bullet a_i, & \text{for } 1 \leq \text{i} \leq \text{n-2;} \\ (b_j b_k) \bullet a_i = (b_k b_j) \bullet a_i, & \text{for } 1 \leq \text{j,k} \leq \text{n-1 and } \mid j-k \mid \geq 2; \\ b_j^{a_i^2} = b_j^{a_i}, & \text{for } 1 \leq \text{i} \leq \text{n-1;} \\ b_j^{a_i a_{i+1} a_i} = b_j^{a_{i+1} a_i a_{i+1}} = b_j^{a_{i+1} a_i}, & \text{for } 1 \leq \text{i} \leq \text{n-2;} \\ b_j^{a_i a_k} = b_j^{a_k a_i}, & \text{for } 1 \leq \text{i,k} \leq \text{n-1 and } \mid i-k \mid \geq 2 \end{cases}$$

The proof for both cases are analogous, so here we present the first case (i). We consider this by examining the various conditions of case (i):
Suppose $1 \leq j \leq n-1$. Thus, for $1 \leq i \leq n-1$,
$b_j \bullet a_i^2 = (b_j \bullet a_i)(b_j^{a_i} \bullet a_i) \bullet a_i) =$

$$b_j \bullet a_i^2 = (b_j \bullet a_i)(b_j \bullet a_i) = \begin{cases} 1(b_j \bullet a_i), & \text{if j = i+1;} \\ a_i(1 \bullet a_i), & \text{if j = i;} \\ a_i(b_j \bullet a_i), & \text{Otherwise.} \end{cases}$$

$$= \begin{cases} 1, & \text{if j = i+1;} \\ a_i^2, & \text{if j = i;} \\ a_i^2, & \text{Otherwise.} \end{cases}$$

$$= \begin{cases} 1, & \text{if j = i+1;} \\ a_i, & \text{Otherwise.} \end{cases} = b_j \bullet a_i;$$

For $1 \leq i \leq n-2$,

$$b_j \bullet (a_i a_{i+1} a_i) = (b_j \bullet a_i)(b_j^{a_i} \bullet (a_{i+1}a_i)) = \begin{cases} 1(b_j \bullet (a_{i+1}a_i)), & \text{if j = i+1;} \\ a_i(a_{i+1}a_i), & \text{if j = i;} \\ a_i(b_j \bullet (a_{i+1}a_i)), & \text{Otherwise.} \end{cases}$$

$$= \begin{cases} (b_j \bullet a_{i+1})(b_j^{a_{i+1}} \bullet a_i), & \text{if j = i+1;} \\ a_i a_{i+1} a_i, & \text{if j = i;} \\ a_i(b_j \bullet a_{i+1})(b_j^{a_{i+1}} \bullet a_i), & \text{Otherwise.} \end{cases}$$

$$= \begin{cases} a_{i+1}(1 \bullet a_i), & \text{if j = i+1;} \\ a_i a_{i+1} a_i, & \text{if j = i;} \\ a_i(b_j \bullet a_{i+1})(b_j \bullet a_i), & \text{otherwise.} \end{cases}$$

$$= \begin{cases} a_{i+1}a_i, & \text{if j = i+1;} \\ a_i a_{i+1} a_i, & \text{if j = i;} \\ a_i 1 a_i, & \text{if j = i+2;} \\ a_i a_{i+1} a_i, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} a_i, & \text{if j = i+2;} \\ a_{i+1}a_i, & \text{otherwise.} \end{cases} = (b_j \bullet a_{i+1})(b_j^{a_{i+1}} \bullet a_i) = b_j \bullet (a_{i+1}a_i)$$

Analogously, $b_j \bullet (a_{i+1}a_i a_{i+1}) = b_j \bullet (a_{i+1}a_i)$; for $1 \leq i, k \leq n-1$ and $\mid i-k \mid \geq 2$,

$$b_j(a_i a_k) = (b_j \bullet a_i)(b_j^{a_i} \bullet a_k) = \begin{cases} 1(b_j \bullet a_k), & \text{if j = i+1;} \\ a_i a_k, & \text{if j = i;} \\ a_i(b_j \bullet a_k), & \text{otherwise.} \end{cases}$$

$$= \begin{cases} a_k, & \text{if j = i+1;} \\ a_i, & \text{if j = k+1;} \\ a_i a_k, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} a_k, & \text{if j = i+1;} \\ a_i, & \text{if j = k+1;} \\ a_k a_i, & \text{otherwise.} \end{cases} = b_j \bullet (a_k a_i);$$

for $1 \leq i \leq n-1$,

$$b_j^{a_i^2} = (b_j^{a_i})^{a_i} = \begin{cases} 1, & \text{if j = i;} \\ b_j, & \text{otherwise.} \end{cases} = b_j^{a_i};$$

for $1 \leq i \leq n-2$,

$$b_j^{a_i a_{i+} a_i} = ((b_j^{a_i})^{a_{i+1}})^{a_i} = \begin{cases} 1, & \text{if j = i or j=i+1;} \\ b_j, & \text{otherwise.} \end{cases} = (b_j^{a_{i+1}})^{a_i} = b_j^{a_{i+1} a_i};$$

Analogously, this result applies to $b_j^{a_{i+1}a_i a_{i+1}} = b_j^{a_{i+1}a_i}$;

Finally, for $1 \leq i, k \leq n-1$ and $\mid i-k \mid \geq 2$,

$$b_j^{a_i a_k} = (b_j^{a_k})^{a_k} = \begin{cases} 1, & \text{if j = i or j=k;} \\ b_j, & \text{otherwise.} \end{cases} = (b_j^{a_k})^{a_i} = b_j^{a_k a_i}$$

This completes the proof. □

The next result is obvious from Theorem 1.8

Theorem 9.2 *Let δ be the left action of B^* on A^* and right action φ of A^* on B^*, a well defined bilateral semidirect product for the generating sets of the semigroup $E(X, \sigma)$ $\Omega_1 \bowtie \Omega_2$ holds.*

Proof. The result is consequent from Lemma 8.4 - 8.5 and Theorem 9.1 □

Theorem 9.3 *The union of the monoid of the semigroup $E(X, \sigma)$, Ω is a homomorphic image of $\Omega_1 \bowtie \Omega_2$*

Proof. Let Ω be the pseudovariety of monoids generated by $\{\Omega_n | n \in \mathbb{N}\}$ and let J be the pseudovariety of monoids generated by Ω_1 and Ω_2. It is clear that J is the pseudovariety of $\mathcal{J}$-trivial monoids which are the synctatic monoids of the piecewise testable language. □

Corollary 9.4 *Let Ω be the pseudovariety of monoids generated by the generating sets Ω_1 and Ω_2. Then $\Omega \subseteq J \bowtie J$.*

Proof. The proof is obvious from Theorem 9.3 □

10 Conclusion

In this paper, we studied the presentations of the semigroup of transformations restricted by an equivalence. We have shown that semigroup of transformations restricted by an equivalence, $E(X, \sigma)$ is generated by two generating sets denoted by Ω_1 and Ω_2. Next we define the presentation of the semigroup of transformation restricted by equivalence and its finitely generated (presented) monoid M of $E(X, \sigma)$ where we state that the semigroup $E(X, \sigma)$ has a presentation $\langle A : \sigma \rangle$ via φ. Also we state some results on the finite generation and finite presentability, acts, subacts and direct products were presented. Finally, we construct decompositions of the Semigroup of Transformation with restricted equivalence, $E(X, \sigma)$ by means of bilateral semidirect products and quotients and its application similar to the work of [42]. Having established the foregoing. We will like to suggest some open problems.

Open Problem

From the study, we suggest the following directions:

- To find the connection of the semigroup of transformations with restricted equivalence, $E(X, \sigma)$ with Rees Index.
- Describe the finite complete rewriting system and finite derivative type in the case of semigroup of transformations restricted by an equivalence, $E(X, \sigma)$.
- Wreath Product of the semigroups of transformations with restricted equivalence.

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