Original Research Article

EXISTENCE OF RANDOM ATTRACTORS FOR A STOCHASTIC STRONGLY DAMPED PLATE EQUATIONS WITH MULTIPLICATIVE NOISE

Abstract: In this article we study the asymptotic dynamics for a stochastic strongly damped generated plate equations with homogenous Neumann boundary conditions and multiplicative noise. First, we investigate the existence and uniqueness of solutions, bounded absorbing set, then the asymptotic compactness.
[1] [2]

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n (n=5)$ with the smooth boundary $\partial\Omega$.

Considered the following stochastic strongly damped plate equations with multiplicative noise:

$$\begin{cases} u_{tt} + \alpha\Delta^2 u_t + \Delta^2 u + \varepsilon u + g(u) = f(x) + cu \circ \dfrac{dW(x,t)}{dt}, \quad x \in \Omega, t \geq 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, t \leq 0, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial n}|_{\partial\Omega} = 0,\ t \geq 0, \end{cases} \tag{1.1}$$

where $\varepsilon, \alpha, c > 0$ are positive constants, $\triangle$ is the Laplacian with respect to the variable $x \in \mathbb{R}^5$, $u = u(x,t)$ is a real function on $\Omega \times [0, +\infty)$, where $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, $f(x) \in H_0^1(\Omega) \cap H^2(\Omega)$ are given external forces. $W(x,t)$ is an independent two sided real-valued wiener processes on probability space $(\Omega, \mathcal{F}, \mathbb{P})$,where

$$\Omega = \{\omega = (\omega_1,\ \omega_2, \ldots\ldots,\ \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\},$$

is endowed with compact open topology, $\mathbb{P}$ is the corresponding wiener measure, and $\mathcal{F}$ is the $\mathbb{P}-$completion of Borel $\sigma-$algebra on Ω. We identify $W(t)$ with $(W_1(t),\ W_2(t), \ldots\ldots,\ W_m(t))$, i.e.,

$$W(t) = (W_1(t),\ W_2(t), \ldots\ldots,\ W_m(t)),\ t \in \mathbb{R}.$$

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Define the time shift $(\theta_t)_{t\in\mathbb{R}}$ on Ω by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t),\ t \in \mathbb{R},\ \omega \in \Omega.$$

The nonlinear term g is C^1-function with $g(0) = 0$, that satisfies the following conditions:

(h_1) There exists constants $0 \leq p \leq 4, n \geq 5$, and $C_1 > 0$ such that

$$|g'(u)| \leq C_1(1 + |u|^p)\ , \forall\ u \in \mathbb{R}, \tag{1.2}$$

and

(h_2) There exists positive constants

$$\begin{cases} \lim\inf\limits_{|u|\to\infty} \dfrac{g(u)}{u} \geq 0\ , \forall\ u \in\ \mathbb{R}, \\ \lim\inf\limits_{|u|\to\infty} \dfrac{g(u)u - \mu_i G(u)}{u^2} \geq 0,\ \forall\ u \in \mathbb{R} \end{cases} \tag{1.3}$$

(h_3) There exists constants $k > 0$ and μ_1 such that $\forall\mu \in (0, \mu_1)$, there exists $\mu_i \in \mathbb{R} \hookrightarrow\to$ satisfying

$$\begin{cases} kG(u) - \mu u^2 + c_\mu \leq ug(u),\ \forall\ u \in \mathbb{R} \\ G(u) \geq \mu\|u\|^{P+2} + c_\mu\|u\|^2,\ \forall\ u \in \mathbb{R} \end{cases} \tag{1.4}$$

where $G(s) = \int_0^s g(r)dr$.

The asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics and the theory has been greatly developed over the last decade or so. In the deterministic case the global attractor, a compact invariant and attracting set, occupies a central position see, for example, Temam [5]. In this paper, we study random attractors of equation(1.1) when the forcing term is independent of time. In this case we introduce two parametric spaces to describe the dynamics of the equations: one is responsible for deterministic forcing and the other is responsible for stochastic perturbations. Existence and upper semi-continuity of the global attractor, pullback attractor (or kernel sections) for deterministic autonomous and non-autonomous dynamical systems were studied widely related with this problem (see,e.g.,[18, 20, 24, 25, 26, 29, 30, 36, 37]).

In order to study the corresponding random dynamical system, some authors have introduced a different notion of an attractor from the view of stochastic partial differential equations, for example, see Morimoto [19], L. Arnold [27], H. Crauel and F. Flandoli [17], J. Duan, K. Lu and B. Schmalfuß [23], J. Hale, X. Lin and G. Raugel [25], and T. Caraballo and J. Langa [34] studied the existence and the upper semi-continuity of attractors for deterministic and random dynamical systems, respectively. They obtained a general criteria for attractor to exist and be upper semi-continuous for non-autonomous stochastic evolution equations with the time-dependent external term and multiplicative noise, Wang[12] established a useful theory about the existence and upper semi-continuity of random attractors by introducing two parametric spaces and given some applications to non-autonomous stochastic reaction-diffusion equations and wave equations, see

also [11, 13, 21, 22, 33, 34, 35, 41] for more details.
Recently many results have been established on the dynamics of a variety of systems related to equation (1.1). The deterministic hyperbolic equations has been studied to possess global attractors despite being subsets of an infinite-dimensional phase space, are finite-dimensional objects, see[33, 38, 39] and references therein. For instance, A. Khanmamedov[6], G. Yue and C. Zhong [7] proved the existence of a global attractors for the linear damped plate equations with critical exponent, [20, 21, 22, 23, 24, 25] obtained the nonlinear damped, and Ma et al.[4] obtained the strongly damped plate equations with white noise.
Recently, many authors have established the existence of random attractors for other equations (see[8, 14, 15, 16, 31, 40]). Further, there are no result of random attractors for the equation (1.1). In this article, we study the existence of random attractors for system (1.1)-(1.2), there is difficult to prove compactness of the generated random dynamical system, but its asymptotic compactness can be proved by using the decomposition of solution method (see[33, 41]) and more accurate calculation.

This paper is organized as follows. In section 2, we recall some basic concepts and properties for general random dynamical systems. In section 3, we first provide some basic settings about (1.1) and show that it generates a random dynamical system in proper function space and existence and uniqueness of solutions. In section 4, we devote to uniform energy estimates on the solutions of (1.1) defined on $\mathbb{R}^5$ when $t \to \infty$ with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the random dynamical system associated with the equation. In section 5, we prove the existence of a random attractor.

2. Random dynamical systems

In this section we recall some basic concepts related to RDS and a random attractors for RDS (see [1, 2] for more details), which are important for getting our main results. To study the asymptotic behavior of the RDS determined by (1.1), we need to recall some difinitions and properties. Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel $\sigma-$algebra $\mathfrak{B}(X)$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. Suppose that the the mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))-$measurable and satisfies the following properties:
(i) $\phi(0, \omega)x \;=\; x;$
(ii) $\phi(s, \theta_t\omega) \circ \phi(t, \omega)x \;=\; \phi(s + t, \omega)x;$
for all $s, t \in \mathbb{R}^+,\; x \in X$ and $\omega \in \Omega$. Then ϕ is called RDS. Further more, ϕ si called a continuous RDS if ϕ is continuous with respect to $t \le 0$ and $\omega \in \Omega$.

Definition 2.2. A mapping $\Phi(t, \tau\;, \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$, if for all $\tau \in \mathbb{R}$,$\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied :
i) $\Phi(t, \tau, \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}, \mathcal{B}(\mathbb{R}))$ measurable mapping,

ii) $\Phi(0,\tau,\omega,x)$ is identity on X,
iii) $\Phi(t+s,\tau,\omega,x)=\Phi(t,\tau+s,\theta_s\omega,x)\circ\Phi(s,\tau,\omega,x)$,
iv) $\Phi(t,\tau,\omega,x): X\to X$ is continuous.
Definition 2.3. A set-valued mapping $B:\Omega\to 2^X$ is called a random closed set if $B(\omega)$ is closed, nonempty, and $\omega\mapsto d(x,B(\omega))$ is measurable for all $x\in X,\ \omega\in\Omega$. A random set $\mathcal{B}:=\{B(\omega)\}_{\omega\in\Omega}$ is said to tempered if

$$\lim_{t\to\infty} e^{-\eta t}d(B(\theta_{-t}\omega))=0,$$

for a.e. $\omega\in\Omega$ and all $\eta>0$, where $d(B):=\sup_{x,y\in B}\ d(x,y)$.
Definition 2.4. Let $\mathcal{D}$ be a collection of random subset of X and $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, then K is called an absorbing set of $\Phi\in\mathcal{D}$, if for all $\tau\in\mathbb{R},\omega\in\Omega$ and $B\in\mathcal{D}$, there exists, $T=T(\tau,\omega,B)>0$ such that

$$\Phi(t\ ,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega))\subseteq K(\tau,\omega)\ ,\ \forall\ t\ \geq T.$$

Definition 2.5. Let $\mathcal{D}$ be the collection of all tempered random sets in X, and a random set $\mathcal{A}:=A\{(\omega)\}_{\omega\in\Omega}\in X$ is called a random attractor for the RDS ϕ if P-a.s.
(i) $\mathcal{A}$ is a random compact set, i.e. $A(\omega)$ is nonempty and compact for a.e. $\omega\in\Omega$ and $\omega\mapsto d(x,A(\omega))$ is measurable for every $x\in X$;
(ii) $\mathcal{A}$ is $\phi-$invariant, i.e. $\phi(t,\omega,A(\omega))\ =\ A(\theta_t\omega)$, for all $t\geq 0$ and a.e. $\omega\in\Omega$;
(iii) $\mathcal{A}$ attracts every set in X, i.e. for all bounded (and non-random) $B\subset X$,

$$\lim_{t\to\infty} dist(\phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)),A(\omega))=0,\quad a.e.\ \omega\in\Omega.$$

Lemma 2.6. Let ϕ be a continuous random dynamical system on E over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$. Suppose that there exists a random compact set $\{K(\omega)\}_{\omega\in\Omega}$ which absorbs every bounded non-random set $B\in\mathcal{D}$. Then, the set

$$\mathcal{A}=\{A(\omega)\}_{\omega\in\Omega}\ =\overline{\cup_{B\subset X}\Lambda_B(\omega)},$$

is a global attractors for ϕ, where the union is taken over all bounded $B\subset X$, and $\Lambda B(\omega)$ is the $\omega-$limits set of B given by

$$\Lambda_B(\omega)=\bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}(\phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)),A(\omega))},\ \omega\in\Omega.$$

3. Existence and uniqueness of solutions

In this section, we study the existence and uniqueness of solution for the system (1.1) on bounded set of $\Omega\subset\mathbb{R}^n,(n=5)$. From now on, it is assumed that conditions $(i)-(iii)$ hold, the space E and the probability space $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ are defined as in Section 1. Let $A=\Delta^2$ with Neumann boundary condition on Ω, then $D(A)=\{u\in H^4(\Omega)\cap H_0^2(\Omega):\frac{\partial u}{\partial n}|_{\partial\Omega}=0\}$. Clearly, A is a self-adjoint and positive linear operator with eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$:

$$0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_i\leq\cdots,\lambda_i\to+\infty\quad(i\to+\infty).$$

Let $E=H_0^2(\Omega)\times L^2(\Omega)$, which is a separable Hilbert space endowed with the usual norm

$$\|Y\|_{H_0^2\times L^2}=(\|\Delta u\|^2+\|v\|^2)^{\frac{1}{2}}\quad\text{for}\quad Y=(u,v)^\top,\tag{3.1}$$

where $\|\cdot\|$ denotes the usual norm in $L^2(\Omega)$ and $\top$ stands for the transposition. it could be defined the powers A^r of A for $r \in \mathbb{R}$. The space $V_{2r} = D(A^r)$ is the Hilbert space with the standard inner product and norm, respectively

$$((\cdot,\cdot))_{D(A^r)} = (A^r\cdot, A^r\cdot),\ \|\cdot\|_{D(A^r)} = \|A^r\cdot\|, ((u,u)) = \int_\Omega \Delta u \Delta v dx, \quad \|\Delta u\| = ((u,u))^{\frac{1}{2}},$$

$\forall u, v \in H_0^2(\Omega)$. Especially, (u,v) and $\|\cdot\|$ denote the $L^2(\Omega)$ inner product and norm respectively, $(u,u) = \int_\Omega uvdx$, $\|u\| = (u,u)^{\frac{1}{2}}$ $\forall u, v \in L^2(\Omega)$. Thus, the injection $D(A^r) \hookrightarrow D(A^s)$ is compact if $r > s$. Then by the generalized Poincaré inequality, there holds

$$\|u\|_r^2 \geq \lambda_0 \|u\|_s^2 \quad \text{Where} \quad \lambda_0 > 0 \quad \text{is the first eigenvalue of} \quad A.$$

For the purpose, is to convert the problem (1.1) into a deterministic system with random parameters but without noise terms, and then show that it generates random dynamical system.

Due to Ornstein-Uhlenbeck process deducing by the Brownian motion, which hold the *Itô* differential equation

$$dz + \alpha z dt = dW(t), \tag{3.2}$$

and hence the solution is given by

$$\theta_t\omega(s) = \omega(t+s) - \omega(t),$$
$$z(\theta_t\omega) = -\alpha \int_\infty^0 e^s(\theta_t\omega)(s)ds, \quad s,t \in \mathbb{R}, \quad \omega \in \Omega. \tag{3.3}$$

It is known from [3, 28, 33] the random variable $|z(\omega)|$ is tempered and there is a θ_t-invariant set $\bar{\Omega} \subseteq \Omega$ of full P measure such that for every $\omega \in \bar{\Omega}$, $t \mapsto z(\theta_t\omega)$ is continuous in t and

$$\lim_{t\to\infty} e^{-\alpha t}|z(\theta_{-t}\omega)| = 0,\ \forall\ \alpha > 0,\ \omega \in \bar{\Omega}. \tag{3.4}$$

Equation (3.3) has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein-Uhlenbeck process (see [1, 2, 8, 33] for more details). For convenience, in the following, written as $\bar{\Omega}$ as Ω.

Lemma 3.1.(see [3],[38]) For the Ornstien-Uhlenbeck process $z(\theta_t\omega)$ in (3.3), so that it becomes

$$\begin{cases} \lim\limits_{t\to\pm\infty} \dfrac{|z(\theta_t\omega)|}{|t|} = 0, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^0 z(\theta_s\omega)ds = E[z(\theta_s\omega)] = 0, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^0 z(\theta_s\omega)ds = E[z(\theta_s\omega)] = \dfrac{1}{\sqrt{\pi\delta}}, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^0 |z(\theta_s\omega)|^2ds = E[|z(\theta_s\omega)|^2] = \dfrac{1}{2\delta}, \end{cases} \tag{3.5}$$

by (3.5), there exists $\mathrm{T}_1(\omega) > 0$ such that for all $t \geq \mathrm{T}_1(\omega)$,

$$\int_{-t}^{0} z(\theta_s\omega)ds < \frac{2}{\sqrt{\pi\delta}}t, \quad \int_{-t}^{0} |z(\theta_s\omega)|^2 ds < \frac{1}{2\delta}t. \tag{3.6}$$

It is convenient to reduce (1.1) to evaluation equation of first order in time, let $v = u_t + \varepsilon u - cuz(\theta_t\omega)$, then as it is

$$\begin{cases} \dfrac{du}{dt} = v - \varepsilon u + cuz(\theta_t\omega), \\ \dfrac{dv}{dt} = (\varepsilon - \alpha A)v - (\varepsilon - \alpha A + A + \mu)u - g(u), \\ \qquad - (v - 2\varepsilon u + cuz(\theta_t\omega) + (A-1)\alpha u)z(\theta_t\omega) + f(x), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) = u_1(x) + \varepsilon u_0(x) - cu_0(x)z(\theta_t\omega), \end{cases} \tag{3.7}$$

Let

$$Y = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad L = \begin{pmatrix} \varepsilon I & -I \\ \varepsilon I - \alpha A + A + \mu & -\varepsilon I + \alpha A \end{pmatrix},$$

and

$$Q(t,\omega,Y) = \begin{pmatrix} cuz(\theta_t\omega) \\ -g(u) - (v - 2\varepsilon u + cuz(\theta_t\omega) + (A-1)\alpha u)z(\theta_t\omega) + f(x) \end{pmatrix},$$

Then problem (3.7) has the simple matrix form

$$Y' + LY = Q(t,\omega,Y) \tag{3.8}$$

it is defined

$$\psi_1 = u, \quad \psi_2 = \frac{du}{dt} + \varepsilon u - cuz(\theta_t\omega), \tag{3.9}$$

where ε is a given positive constant, then the system (3.7), can be written as the following equivalent system with random coefficients, in E

$$\begin{cases} \dfrac{d\psi_1}{dt} = \psi_2 - \varepsilon\psi_1 + c\psi_1 z(\theta_t\omega), \\ \dfrac{d\psi_2}{dt} = (\varepsilon - \alpha A)\psi_2 - (\varepsilon - \alpha A + A + \mu)\psi_1 - g(\psi_1), \\ \qquad - (\psi_2 - 2\varepsilon\psi_1 + c\psi_1 z(\theta_t\omega) + (A-1)\alpha\psi_1)z(\theta_t\omega) + f(x), \\ \psi_1(x,0) = u_0(x), \quad \psi_2(x,0) = v_0(x) = u_1(x) + \varepsilon u_0(x) - cu_0(x)z(\theta_t\omega), \end{cases} \tag{3.10}$$

then, the random differential equation (3.10) can be written as the following vector form

$$\begin{cases} \psi\prime + L\psi = Q(\psi,t,\omega), \\ \psi_0 = (\psi_1(x,0), \psi_2(x,0)) = (u_0(x), u_1(x) + \varepsilon u_0(x) - cu_0(x)z(\theta_t\omega))^\top, \end{cases} \tag{3.11}$$

whereas

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad L = \begin{pmatrix} \varepsilon I & -I \\ \varepsilon I - \alpha A + A + \mu & -\varepsilon I + \alpha A \end{pmatrix},$$

and

$$Q(\psi,t,\omega)=\begin{pmatrix} c\psi_1 z(\theta_t\omega) \\ -g(\psi_1)-(\psi_2-2\varepsilon\psi_1+c\psi_1 z(\theta_t\omega)+(A-1)\alpha\psi_1)z(\theta_t\omega)+f(x) \end{pmatrix}.$$

In line with [5, 10], it is known that the operator L in (3.11) is the infinitesimal generator of C_0-semigroup e^{Lt} of contractions on E for $t>0$, and also generates a C_0-semigroup e^{-Lt} of contractions on E. By the assumptions (h_2) and the embedding relation $H_0^2(\Omega)\hookrightarrow L^{10}(\Omega)$, it is easy to check $Q(\psi,t,\omega):E\to E$ is locally Lipschitz continuous with respect to φ for each $\omega\in\Omega$, by the classical semigroup theory concerning the (local) existence and uniqueness solution of evolution differential equation [10], have the following theorem.

Theorem 3.1. Assume that h_1-h_3 hold, for each $\omega\in\Omega$ and for any $\psi_0\in E$, there exists $T>0$ such that (3.11) has a unique mild function $\psi(\cdot,\omega,\psi_0)\in C([0,+\infty);E)$ such that $\psi(0,\omega,\psi_0)=\psi(0)$ satisfies the integral equation

$$\psi(t,\omega,\psi_0)=e^{-Lt}\psi_0(\omega)+\int_0^t e^{L(t-s)}Q(\psi(s,\omega,\psi_0),\theta_s\omega,s)ds. \tag{3.12}$$

However, $\psi(t,\omega,\psi_0)$ is jointly continuous into (ψ_0) and measurable in ω.
From Theorem 3.1, it is known that for P-a.s. each $\omega\in\Omega$, then the following results hold for all T >0
(i)- if $\psi_0(\omega)\in E$ then, $\psi(\cdot,\omega,\psi_0)\in C([0,+\infty;E)$,
(ii)- $\psi(t,\omega,\psi_0)$ is jointly continuous into t and measurable in $\psi_0(\omega)$,
(iii)- the solution mapping of (3.11) satisfies the properties of RDS.

it is noticed that a unique solution $\psi(\cdot,\omega,\psi_0)$ of (3.11) could defined a continuous random dynamical system over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$. Hence the solution mapping

$$\begin{aligned}&\bar{\Phi}(t,\omega):\mathbb{R}\times\Omega\times E\mapsto E, t\geq 0,\\ &\psi(0,\omega)=(u_0(\omega),v_0(\omega),)^\top\mapsto(u(t,\omega),v(t,\omega),)^\top=\psi(t,\omega),\end{aligned} \tag{3.13}$$

generates a random dynamical system. Moreover,

$$\Phi(t,\omega):Y_0=\psi(0,\omega)+(0,cuz(\theta_0\omega))^\top\mapsto Y(t,\omega,Y_0)=\psi(t,\omega,\psi_0)+(0,cuz(\theta_t\omega))^\top, \tag{3.14}$$

where $Y_0=(u_0,u_1)^\top$ and $\psi_0=(u_0,u_1+cuz(\theta_t\omega))^\top$. Then $\Phi(t,\omega)$ is a continuous random dynamical system associated with the problem (3.8) on E. $\Phi(t,\omega)$ has the following relation with $\bar{\Phi}(t,\omega)$

$$\bar{\Phi}(t,\omega)=R(\theta_t\omega)\Phi(t,\omega)R^{-1}(\theta_t\omega) \tag{3.15}$$

whereas $R(\theta_t\omega):(a,b)^\top\mapsto(a,b-cuz(\theta_t\omega)^\top$ is a homeomorphism of E.
it is also defined the following transformation.

$$\varphi_1=u=\psi_1,\ \varphi_2=u_t+\varepsilon u, \tag{3.16}$$

similar to (3.11), it is got that

$$\begin{cases}\varphi'+H(\varphi)=Q_\varepsilon(\varphi,t,\omega)\\ \varphi_0(x,0)=(u_0,v_0)^\top=(u_0(x),u_1(x)+\varepsilon u_0(x))^\top,\end{cases} \tag{3.17}$$

whereas

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad H(\varphi) = \begin{pmatrix} v - \varepsilon u \\ (\varepsilon - \alpha A + A + \mu)u - (\varepsilon - \alpha A)v \end{pmatrix},$$

and

$$Q_\varepsilon(\varphi, \omega, t) = \begin{pmatrix} 0 \\ cuz(\theta_t\omega) - g(u) + f(x) \end{pmatrix}.$$

it introduced the isomorphism $\mathrm{T}_\varepsilon\varphi = (\varphi_1, \varphi_2 - \varepsilon\varphi_1)^\top, \quad \varphi = (\varphi_1, \varphi_2)^\top \in E$ which has inverse isomorphism $T_{-\epsilon}\varphi = (\varphi_1, \varphi_2 + \varepsilon\varphi_1)^\top$ it follows that (θ, φ) with mapping

$$\bar{\Phi}_\varepsilon(t, \omega) = T_\varepsilon \bar{\Phi}(t, \omega) T_{-\varepsilon} : \varphi_0 \mapsto \varphi(t, \omega, \varphi_0) \tag{3.18}$$

is a random dynamical system associated with (3.16), whereas $\varphi_0 = (u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_0\omega))^\top$ and $T_\varepsilon : (a, b)^\top \mapsto (a, b + \varepsilon a)^\top$ is a isomorphism of E. Notice that all above random dynamical systems $\Phi(t, \omega), \bar{\Phi}(t, \omega), \bar{\Phi}_\varepsilon(t, \omega)$ are equivalent. In this article, the existence of random attractor for RDS will be studied Φ based on Theorem 3.1.

4. Uniform estimates of solutions

In this section, the existence of a random absorbing set for the RDS will be shown that $\varphi\left(t, \omega, \varphi_0(\omega)\right), t \geq 0$ in the space E, and uniform estimaties on the solutions of (3.11) defined on $\mathbb{R}^n$ (n=5). For this purpose, it introduced a new Hilbert space E, that is, $(\varphi, \tilde{\varphi})_E = \gamma(A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}u_2) + (v_1 v_2)$ and $\|\varphi\|_E = (\varphi, \varphi)_E^{\frac{1}{2}}$ for any $\varphi = (u_1, v_1)^\top, \ \tilde{\varphi} = (u_2, v_2)^\top \in E$, whereas γ is chosen as

$$\gamma = \frac{4 + \alpha\lambda_1 + \beta_1}{4 + 2(\alpha\lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1}, \tag{4.1}$$

It is clearly that, the norm $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H_0^2 \times L^2}$ of E.

Lemma 4.1 For any $\varphi = (u, v)^T \in E$, it follows that

$$(H(\varphi), \varphi)_E \geq \frac{\varepsilon}{2}\|\varphi\|_E^2 + \frac{\varepsilon}{4}\|u\|_2^2 + \frac{\alpha}{2}\|v\|^2.$$

Proof Let $\varphi(t) = (u(t), v(t))^T$ and $H(\varphi) \in E$, it is obtained

$$\begin{aligned}(H(\varphi), \varphi)_E =& \|\varphi\|_E^2 - \varepsilon\|u\|_2^2 + (\alpha - \varepsilon)\|v\|^2 - \varepsilon(\alpha - \varepsilon)(u, v) \\ \geq& \varepsilon\|u\|_2^2 + (\alpha - \varepsilon)\|v\|^2 - \varepsilon(\alpha - \varepsilon)(u, v) \\ =& \frac{\varepsilon}{2}\|\varphi\|_E^2 + \varepsilon\|u\|_2^2 + \frac{\alpha}{2}\|v\|^2.\end{aligned}$$

Lemma 4.2 Suppose that $(h_1) - (h_3)$ hold. Then there exists a random variable $r_1(\omega) > 0$ and a bounded ball $B_0(t, \omega) \subset E$, centered at 0 with random radius $r_0(\omega) > 0$, $B_E(0, r_0(\omega) \in \mathcal{D}(E)$, such that for any bounded non-random set $B \subset \mathcal{D}(E)$, there exists a deterministic $T = T(t, \omega, B) \geq 0$, such that the solution

$\varphi(t,\omega;\varphi(\omega))$ of (3.17) with initial value $(u_0, u_1+\varepsilon u_0, \eta_0)^T \in B$ satisfies, for $P-a.s.\ \omega\in\Omega$,

$$\|\varphi(t,\omega;\varphi(0,\omega))\|_E \le r_0^2(\omega),\ t\ge T(B).$$

Proof For any $\omega\in\Omega$, $t\ge 0$, suppose that $\varphi(t)=(u(t),v(t))\in E$ be mild solution of (3.17), taking the inner product $(\cdot,\cdot)_E$ of (3.17) with $\varphi(t)=(u,v)=(u,u_t+\varepsilon u-cuz(\theta_t\omega))^\top$, it reached

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_E^2+(H(\varphi,\varphi))_E=(Q(\varphi,\omega,t),\varphi), \tag{4.2}$$

By Lemma 4.1, as resulted

$$(H(\varphi,\varphi))_E=\frac{\varepsilon}{2}\|\varphi\|^2+\varepsilon\|u\|_2^2+\frac{\alpha}{2}\|v\|^2, \tag{4.3}$$

let us estimate the right hand side of (4.2)

$$\begin{aligned}(Q(\varphi,\omega,t),\varphi) = ((cuz(\theta_t\omega),u)) + (cu(\varepsilon-\alpha A)z(\theta_t\omega),w) + (c^2uz^2(\theta_t\omega),w)\\ +(cvz(\theta_t\omega),w)-(g(u),w)+(f(x),w)\end{aligned} \tag{4.4}$$

By the Cauchy-Schwartz inequality, it is found that

$$((cuz(\theta_t\omega),u))\le |c||z(\theta_t\omega)|\|u\|_2^2, \tag{4.5}$$

$$\varepsilon(cuz(\theta_t\omega),w)\le\varepsilon|c||z(\theta_t\omega)|\|u\|\|w\|\le\frac{\varepsilon|c||z(\theta_t\omega)|}{2\sqrt{\lambda_0}}(\|u\|_2^2+\|w\|^2), \tag{4.6}$$

$$(c^2uz^2(\theta_t\omega),w)\le|c|^2|z(\theta_t\omega)|^2\|u\|\|w\|\le\frac{c^2}{2\sqrt{\lambda_0}}|z(\theta_t\omega)|^2(\|u\|_2^2+\|w\|^2), \tag{4.7}$$

$$(cwz(\theta_t\omega),w)\le|c||z(\theta_t\omega)|\|w\|^2, \tag{4.8}$$

$$(f(x),w)\le\frac{2}{\alpha}\|f(x)\|^2+\frac{\alpha}{8}\|w\|^2, \tag{4.9}$$

$$\alpha(c\Delta uz(\theta_t\omega),\Delta w)\le\alpha|c||z(\theta_t\omega)|\|u\|\|w\|\le\frac{\alpha\sqrt{\lambda_0}|c||z(\theta_t\omega)|}{2}(\|u\|_2^2+\|w\|^2), \tag{4.10}$$

here it is estimated a nonlinear term (4.4), by $(h_2),(h_3)$ and the Hölder inequality, it is deduced that

$$\begin{aligned}(g(u),w)&=(g(u),u_t+\varepsilon u-cuz(\theta_t\omega))\\ &=\tfrac{d}{dt}\textstyle\int_U G(u)dx+\varepsilon\,(g(u),u)-cuz(\theta_t\omega)(g(u),u)\,.\end{aligned} \tag{4.11}$$

Due to (3.3), (h_1), *and* (h_2) and poincarè inequality, there exists positive constant μ_1,μ_2 such that

$$(g(u),u)-k\tilde{G}(u)+\mu_1\|u\|_2^2+\mu_2\ge 0, \tag{4.12}$$

it follows from(1.4) for each given $\mu_3,\mu_4>0$

$$(g(u),u)\le\mu_3\|u\|_2^2+\mu_4, \tag{4.13}$$

$$\begin{aligned}(g(u),w)&\ge\frac{d}{dt}\int_U G(u)dx+\varepsilon kG(u)-\varepsilon(\mu_1\|u\|_2^2+\mu_2)-|c||z(\theta_t\omega)|(\mu_3\|u\|_2^2+\mu_4).\\ &=\frac{d}{dt}\int_U G(u)dx+\varepsilon kG(u)-(\varepsilon\mu_1+|c||z(\theta_t\omega)|\mu_3)\|u\|_2^2-\varepsilon\mu_2-\mu_4|c||z(\theta_t\omega)|.\end{aligned} \tag{4.14}$$

Whereas $\tilde{G}(u)=\int_U G(u)dx$. Collecting (4.5)-(4.14) and (4.4), showing that

$$(Q(\varphi,\omega,t),\varphi)\leq -\frac{d}{dt}\int_U G(u)dx-\varepsilon kG(u)+(\varepsilon\mu_1+|c||z(\theta_t\omega)|\mu_3)\|u\|_2^2+\varepsilon\mu_2+\mu_4|c||z(\theta_t\omega)$$
$$+\frac{c^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_0}}(\|u\|_2^2+\|w\|^2)+\frac{4\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_0}}\|u\|_2^2+\frac{\alpha}{4}\|w\|^2+\frac{2}{\alpha}\|f\|^2. \quad (4.15)$$

Then substituting all together into (4.2) yield

$$\frac{1}{2}\frac{d}{dt}(\|\varphi\|_E^2+2G(u))+\frac{\varepsilon}{2}(\|u\|_2^2+\|w\|^2)+\frac{\delta}{4}\|\eta\|_{\mu,2}^2+\frac{\alpha}{2}\|w\|^2+\varepsilon kG(u)$$
$$\leq \frac{c^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_0}}(\|u\|_2^2+\|w\|^2)+(\frac{4\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_0}}+|c||z(\theta_t\omega)|\mu_3+\varepsilon\mu_1)\|u\|_2^2$$
$$+\frac{|c||z(\theta_t\omega)|}{2}\|u\|_2^2+\frac{2}{\alpha}\|f\|^2+\varepsilon\mu_2+\mu_4|c|\ |z(\theta_t\omega)|. \quad (4.16)$$

Since $\sigma=\min\{\varepsilon,\varepsilon k,\frac{\delta}{2}\}$ and $\|\varphi\|^2=(\|u\|_2^2+\|w\|^2)$, then there is the following equivalent system

$$\frac{1}{2}\frac{d}{dt}(\|\varphi\|_E^2+2G(u))+\rho(t,\theta_t\omega)(\|\varphi\|_E^2+2G(u))\leq\frac{2}{\alpha}\|f\|^2+\varepsilon\mu_2+\mu_4|c||z(\theta_t\omega)|. \quad (4.17)$$

Whereas

$$\rho(t,\theta_t\omega)=\sigma-\mu_3|c||z(\theta_t\omega)|-(\frac{c^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_0}}+\frac{4\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_0}}+\varepsilon\mu_1+\frac{|c||z(\theta_t\omega)|}{2}). \quad (4.18)$$

By applying Gronwall's inequality to (4.17) over [0,t], it has been that

$$\|\varphi(t,\omega,\varphi_0)\|_E^2+2G(u)\leq e^{-2\int_0^t\rho(s,\theta_s\omega)ds}[\|\varphi_0\|_E^2+2G(u_0)]$$
$$+(\frac{2}{\alpha}\|f\|^2+\varepsilon\mu_2)\int_0^t e^{-\int_s^t\rho(\tau,\theta_\tau\omega)d\tau}ds$$
$$+\mu_4|c|\int_0^t e^{-\int_s^t\rho(\tau,\theta_\tau\omega)d\tau}|z(\theta_t\omega)|ds. \quad (4.19)$$

Substitiuting ω by $\theta_{-t}\omega$, from (4.19) as a result

$$\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2+2G(u)\leq e^{-2\int_0^t\rho(s-t,\theta_{s-t}\omega)ds}[\|\varphi_0(\theta_{-t}\omega)\|_E^2+2G(u_0)]$$
$$+(\frac{2}{\alpha}\|f\|^2+\varepsilon\mu_2+)\int_0^t e^{-\int_s^t\rho(\tau-t,\theta_{\tau-t}\omega)d\tau}ds$$
$$+\mu_4|c|\int_0^t e^{-\int_s^t\rho(\tau-t,\theta_{\tau-t}\omega)d\tau}|z(\theta_{s-t}\omega)|ds.$$
$$\leq e^{\int_{-t}^0\rho(s,\theta_s\omega)ds}\|\varphi_0\|_E^2$$
$$+(\frac{2}{\alpha}\|f\|^2+\varepsilon\mu_2)\int_{-t}^0 e^{-\int_s^0\rho(\tau,\theta_\tau\omega)d\tau}ds$$
$$+\mu_4|c|\int_{-t}^0 e^{-\int_s^0\rho(\tau,\theta_\tau\omega)d\tau}|z(\theta_s\omega)|ds. \quad (4.20)$$

From (4.18) it is known that

$$|c|(\mu_3+\frac{1}{2})\frac{1}{\alpha}+c^2(\frac{1}{2\sqrt{\lambda_0}}\frac{1}{\sqrt{2}\alpha}+\frac{4\varepsilon^2}{\alpha\sqrt{\lambda_0}}\frac{1}{\sqrt{2}\alpha})<\sigma. \tag{4.21}$$

Note that by (4.12) and (4.13), it is arrived that

$$kG(u)\leq(g(u),u)+\mu_1\|u\|_2^2+\mu_2\leq(\mu_1+\mu_3)\|u\|_2^2+\mu_2+\mu_4. \tag{4.22}$$

Then it follows from Lemma 4.1 and $\varphi_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$, and the fact that $B(\omega)$ is tempered that

$$\lim_{t\to+\infty}e^{2\int_{-t}^{0}-\rho(\tau,\theta_\tau\omega)d\tau}[\|\varphi_0(\theta_{-t}\omega)\|_\tau^2+2G(u_0)]=0. \tag{4.23}$$

Since $|z(\theta_s\omega)|$ is tempered, it is seen that the following integral is convergent

$$\rho^2(\omega)=(\frac{2}{\alpha}\|f\|^2+\varepsilon\mu_2+c)\int_{-\infty}^{0}e^{-\int_s^0\rho(\tau,\theta_\tau\omega)d\tau}(1+|z(\theta_s\omega)|)ds. \tag{4.24}$$

Then by Lemma 3.1 and $g\in L^2(U)$, it is yield

$$\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2\leq\rho^2(\omega).$$

Then by (4.22)-(4.23), and Lemma 4.1 there exists $B_0(\omega)=\{\varphi\in E:\|\varphi_0(\theta_{-t}\omega)\|_E\leq\rho^2(\omega)\}$ is closed measurable absorbing ball in $\mathcal{D}(E)$ and $T=T(0,B,\omega)>0$ such that $\varphi(t,\theta_{-t}\omega,\varphi_0)=\varphi_0\in B_0(\omega)$ satisfy the following result p-a.s $\omega\in\Omega$

$$\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2\leq\rho^2(\omega),$$

it is completed the proof. □

5. Decomposition of solutions

In order to obtain regularity estimates later, as in [33] the equations were decomposed (3.3) by decomposing the nonlinear term. At first, the following decomposition given on nonlinearity $g(u)=g_1(u)+g_2(u)$ whereas $g_1,g_2\in\mathbb{C}^1$ satisfies the following conditions for some proper constant: there is a constant $C>0$ such that

$$\begin{cases}|g_1(s)|\leq C(|s|+|s|^5),\ \forall\ s\in\ \mathbb{R},\\ sg_1(s)\geq0\ ,\\ \exists\ \rho_2,\ \vartheta_1\geq0\ \textit{such that}\ \forall\ \vartheta\in(0,\vartheta_1],\\ \exists\ c_\vartheta\in\mathbb{R},\ \rho_2G_1(s)+\vartheta s^2-c_\vartheta\leq sg_1(s),\ \forall\ s\in\mathbb{R},\end{cases} \tag{5.1}$$

and

$$\begin{cases}|g_2'(s)|\leq C(1+|s|^p),\ \forall\ s\in\ \mathbb{R},0<p<5,\\ 3G_2(s)-C\leq sg_2(s),\\ -\frac{\lambda}{8}s^2-C\leq G_2(s),\ \forall\ s\in\mathbb{R},\end{cases} \tag{5.2}$$

whereas
$G_i(s) = \int_0^s g_i(r)dr, i = 1, 2.$
The solution decomposed $\varphi = (u, w)^T$ of the system (3.15) into the two parts

$$\varphi = \varphi_L + \varphi_N$$

whereas $\varphi_L = (u_L, w_L)$, $\varphi_N = (u_N, w_N)$ solves the following equations respectively

$$\begin{cases} \varphi_L' + H(\varphi_L) + Q_1(\varphi_L) = 0, \\ \varphi_L(0,\omega) = (u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_t\omega))^T, \ t \geq 0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \varphi_N' + H(\varphi_N) + Q_2(\varphi, \varphi_L) = \tilde{Q}_2(\omega), \\ \varphi_N(0,\omega) = (0, \varepsilon z(\theta_t\omega), 0)^T, \ t \geq 0, \end{cases} \tag{5.4}$$

whereas

$$\begin{array}{l} Q_1(\varphi_L) = \begin{pmatrix} 0 \\ g_1(u_L) \\ 0 \end{pmatrix}, \ Q_2(\varphi, \varphi_L) = \begin{pmatrix} 0 \\ g(u) - g_1(u_L) \\ 0 \end{pmatrix}, \\ \tilde{Q}_2(\omega) = \begin{pmatrix} cu_N z(\theta_t\omega) \\ -cz(\theta_t\omega)(v_N - 2\varepsilon u_N + cu_N z(\theta_t\omega)) - g(u) + f(x) \\ cu_N z(\theta_t\omega) \end{pmatrix}. \end{array} \tag{5.5}$$

To prove the existence of a compact random attractor for the RDS Φ, it is obtained that the solutions of systems (5.3) and (5.4) similar to solution of system (4.2), which one decays exponentially and another is bounded in higher regular space. In order to get the regularity estimate, it will be proved some priori estimate for the solutions of systems (5.3) on $U \times [0, \infty]$ as follows.

Lemma 5.1 Let B be a bounded non-random subset of E, for any $\varphi_L(0,\omega) = (u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_t\omega))^T \in B$, there holds

$$\|\varphi_L(0,\omega; \varphi_L(0,\omega))\|_E^2 \leq r_3^2(\omega), \tag{5.6}$$

whereas $\varphi_L = (u_L, v_L)^T$ satisfies (5.3)
Proof. Taking the inner product $(\cdot,\cdot)_E$ of (5.3) in $L^2(U)$ with $\varphi_L = (u_L, v_L)^T$, in which $v_L = u_{Lt} + \varepsilon u_L$, whose initial values are $(u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_t\omega))^T$, it is given that

$$\frac{1}{2}\frac{d}{dt}\|\varphi_L\|_E^2 + (H(\varphi_L), \varphi_L)_E + (Q_1(\varphi_L), \varphi_L) = 0, \tag{5.7}$$

by simple computation there holds

$$(H(\varphi_L), \varphi_L)_E \geq \frac{\varepsilon}{2}\left(\|u_L\|_2^2 + \|v_L\|^2\right) + \frac{\alpha}{2}\|v_L\|^2, \tag{5.8}$$

whereas ε satisfy (4.4). Now it is estimated that the Third term of (5.7) such as

$$\begin{array}{l} (Q(\varphi_L), \varphi_L) = \begin{pmatrix} 0 \\ g_1(u_L) \end{pmatrix} \begin{pmatrix} u_L \\ v_L \end{pmatrix} \\ = (g_1(u_L), u_{Lt} + \varepsilon u_L) \\ = \frac{d}{dt}G_1(u_L) + \varepsilon \int_U g_1(u_L)u_L dx. \end{array} \tag{5.9}$$

According $(5.1)_2$ and $(5.1)_3$, it is resulted

$$\begin{array}{l} G_1(u_L) \geq 0\ ,\ g_1(u_L)u_L \geq 0, \\ \frac{d}{dt}G_1(u_L) + \varepsilon \int_U g_2(u_L)u_L dx \geq \frac{d}{dt}G_1(u_L) + k_0\varepsilon G_1(u_L) + \varepsilon\vartheta\|u_L\|^2 - \varepsilon c_\vartheta. \end{array} \tag{5.10}$$

Thus combining with (5.7)-(5.10) and (5.3) it follows that

$$\frac{d}{dt}\left(\|\varphi_L\|_E^2 + 2\tilde{G}_1(u_L)\right) + 2\sigma_L\left(\|\varphi_L\|_E^2 + 2\tilde{G}_1(u_L)\right) \leq \rho, \tag{5.11}$$

whereas $\rho = \varepsilon c_\vartheta$ and $\sigma_L = \min(\frac{\varepsilon}{2}, \frac{\alpha}{2}, \frac{\varepsilon}{4}, k_0\varepsilon)$

$$\|\varphi_L\|_E^2 + 2\tilde{G}_1(u_L) \geq \|\varphi_L\|_E^2 \geq 0, \tag{5.12}$$

hence

$$\begin{array}{l} \varphi_{L(0,\omega)} = (\varphi_0(\theta_{-t}\omega) + cuz(\theta_{-t}\omega))^\top \\ \leq (r_2(\omega) + cuz(\theta_t\omega)) = \rho_2(\omega) \in B_0(\omega). \end{array} \tag{5.13}$$

By putting $(5.1)_1$, (5.11) and (5.13) together. Then taking Gronwall's inequality to result over $[0,t]$ such that by definition of $B_0(\omega)$ and Lemma 4.2, it is proved

$$\|\varphi_L(0,\omega,\varphi_{L(\tau,\omega)})\|_E \leq r_3^2(\omega). \tag{5.14}$$

The proof is completed.

Lemma 5.2 Let B be a bounded non-random subset of E, there exists positive constant $\sigma_1 \geq 0$ such that for any $\varphi_L(0,\omega) = (u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_t\omega))^T \in B$, we have

$$\|\varphi_L(0,\omega;\varphi_L(0,\omega))\|_E^2 \leq r_4^2(\omega)e^{2\sigma_1(\omega)t}, t \geq 0, \tag{5.15}$$

whereas $\varphi_L = (u_L, v_L)^T$ satisfies (5.3),

Proof Similar to Lemma 5.1, consider (5.7). By (5.1), $(g_1(u_L),(u_L)) \geq 0,\ g_1(0) = 0 and.$ By Sobolev embedding theorem $H^1 \subset L^6 \subset L^4 \subset L^2$ and (5.6), it is concluded

$$\begin{array}{l} 0 \leq \tilde{G}_1(u_L) \leq \displaystyle\int_U G_1(u^1)dx \\ \leq C(\|u_L\|^2 + \|u_L\|_{L^6}^6) \\ \leq \rho_2(\omega)\|u_L\|_1^2, \\ \sigma_1\|u_L\|_1^2 \geq \frac{\sigma_1}{\rho_2(\omega)}\tilde{G}_1(u_L),\ \forall\ u_L \in \mathbb{R}, \end{array} \tag{5.16}$$

due to (5.7) and (5.16) it could be obtained the following result,

$$\frac{d}{dt}(\|\varphi_L\|_E^2 + 2\tilde{G}_1(u_L)) + 2\sigma_1\|\varphi_L\|_E^2 + \frac{\sigma_1}{2\rho_2(\omega)}\tilde{G}_1(u_L) \leq \rho. \tag{5.17}$$

Since $\sigma_1(\omega) = \min\ [\sigma_1, \frac{\sigma_1}{2\rho_2(\omega)}]$.

By applying Gronwall's inequality to (5.17) it yield

$$\begin{array}{l} \|\varphi_L(0,\omega,\varphi_L(0,\omega)\|_E^2 \leq \left(\|\varphi_L(0,\omega)\|_E^2 + \tilde{G}_1(u_L(0))\right)e^{2\sigma_1(\omega)t} + \rho\displaystyle\int_0^t e^{-2\sigma_1(s,\omega)}ds \\ \leq \left(\rho_1^2(\omega) + \tilde{G}_1(u_L(0))\right)e^{2\sigma_1(\omega)t} + \rho\int_0^t e^{-2\sigma_1(s,\omega)}ds, \\ \leq r_4^2(\omega) \end{array} \tag{5.18}$$

by $(5.1)_1$ the following estimate will be got

$$\tilde{G}_1(u_L)=\int_U G_1(u^1)dx\le C(\|u_L\|^2+\|u_L\|_{L^6}^6)\le C_g\|u_L\|_{H^1}^6\le C_p{\rho_1}^6(\omega),\ \forall\ u_L\in\mathbb{R}. \tag{5.19}$$

Thus collecting all (5.13) and (5.18)-(5.19) together, to arrive to (5.15), where

$$r_4^2(\omega)\le\left(\rho_1^2(\omega)+C_p{\rho_1}^6(\omega)\right)e^{2\sigma_1(\omega)t}+\rho\int_0^t e^{-2\sigma_1(s,\omega)}ds.$$

The proof is completed.

Lemma 5.3 Assume that $(h_1)-(h_3)$, and (5.1)-(5.2) holds. there exists a random radius $r_5(\omega)$, such that for P-a.e.$\omega\in\Omega$,

$$\left\|A^{\frac{1+\nu}{2}}u_N\right\|^2+\left\|A^{\frac{\nu}{2}}u_{Nt}\right\|^2\le r_5(\omega), \tag{5.20}$$

whereas

$$\nu=min\{\frac{1}{4},\frac{4-p}{4}\},\ \forall\ 0\le p\le 4. \tag{5.21}$$

Proof By (5.6),(4.1) and $\varphi_N=\varphi-\varphi_L$, there exists a random variable $r(\omega)>0$ such that

$$\max\{\|\varphi(0,\omega,\varphi(0,\omega))\|_E,\|\varphi_N((0,\omega,\varphi_N(0,\omega)))\|_E\}\le r(\omega). \tag{5.22}$$

Taking the inner product of $(\cdot,\cdot)_E$ of (5.4) with $(A^\nu\varphi_N,A^\nu w_N)^T$ to find that

$$(\varphi_N',A^\nu\varphi_N)+(H(\varphi_N),A^\nu\varphi_N)=\left(\tilde{Q}_2(\varphi_N,\omega,t),A^\nu\varphi_N\right) \tag{5.23}$$

According to (5.21) and Lemma 4.1, it is found

$$(H(\varphi_N),A^\nu\varphi_N)_E\ge\frac{\varepsilon}{2}\left(\left\|A^{\frac{1+\nu}{2}}u_N\right\|_2^2+\left\|A^{\frac{\nu}{2}}w_N\right\|^2\right)+\frac{\alpha}{2}\left\|A^{\frac{\nu}{2}}w_N\right\|^2, \tag{5.24}$$

next, it will be estimated the right hand side of (5.23), yield

$$\begin{aligned}&\left(\tilde{Q}_2(\varphi_N,\omega,t),A^\nu\varphi_N\right)=\\&((cu_Nz(\theta_t\omega),A^\nu u_N))-(cw_Nz(\theta_t\omega),A^\nu w_N)\\&+(2c\varepsilon u_Nz(\theta_t\omega),A^\nu w_N)-(c^2u_Nz^2(\theta_t\omega),A^\nu w_N)\\&-(g(u)-g_1(u_L),A^\nu w_N)+(f(x),A^\nu w_N).\end{aligned} \tag{5.25}$$

Now, it is dealt with the right term in (5.25), by using (4.5)-(4.10) and (5.21), it leads to

$$((cu_Nz(\theta_t\omega),A^\nu u_N))\le|c||z(\theta_t\omega)|\|A^{\frac{1+\nu}{2}}u_N\|^2, \tag{5.26}$$

$$(cw_Nz(\theta_t\omega),A^\nu w_N)\le|c||z(\theta_t\omega)|\|A^{\frac{\nu}{2}}w_N\|^2, \tag{5.27}$$

$$(2c\varepsilon u_Nz(\theta_t\omega),A^\nu w_N)\le\frac{\varepsilon|c||z(\theta_t\omega)|}{2\sqrt{\lambda_0}}\left(\|A^{\frac{1+\nu}{2}}u_N\|^2+\|A^{\frac{\nu}{2}}w_N\|^2\right), \tag{5.28}$$

$$(c^2u_Nz^2(\theta_t\omega),A^\nu w_N)\le\frac{|c|^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_0}}(\|A^{\frac{1+\nu}{2}}u_N\|^2+\|A^{\frac{\nu}{2}}w_N\|^2), \tag{5.29}$$

$$(f(x),A^\nu w_N)\le\frac{1}{\alpha}\|A^{\frac{\nu}{2}}f(x)\|^2+\frac{\alpha}{4}\|A^{\frac{\nu}{2}}w_N\|^2, \tag{5.30}$$

for the nonlinear term, It is easy to show that

$$\begin{aligned}&(g(u)-g_1(u_L),A^\nu w_N)=(g(u)-g_1(u_N),A^\nu(u_N+\varepsilon u_N-cu_Nz(\theta_t\omega)))\\&\le \tfrac{d}{dt}\int_U(g(u)-g_1(u_L))A^\nu u_Ndx+\int_U(g(u)-g_1(u_L))A^\nu u_Ndx\\&-\int_U(g'(u)u_t-g_1'(u_L)u_{Lt})A^\nu u_Ndx-C\int_U(g(u)-g_1(u_L))A^\nu u_Nz(\theta_t\omega)dx,\end{aligned}$$

Next, due to (1.2),(5.1)-(5.2), the Cauchy-Schwartz inequality and the Young inequality, it arrives to

$$\int_U(g'(u)u_t-g_1'(u_L)u_{Lt})A^\nu u_Ndx=\int_U((g_1'(u)-g_1'(u_L))u_t+g_1'(u_{_L})u_{Nt}+g_2'(u)u_t)A^\nu u_Ndx,\tag{5.31}$$

then, the following inequalities

$$\begin{aligned}&\int_U(g_1'(u)-g_1'(u_L))u_tA^\nu u_Ndx\le C\int_U g_1''(u+\theta(u-u_L))|u-u_L||u_t||A^\nu u_N|dx\\&\le C\int_U\left(1+|u|^3+|u_L|^3\right)|u_N||A^\nu u_N||u_t|dx\\&\le C\left(1+\|u\|^3_{L^{10}}+\|u_L\|^3_{L^{10}}\right)\|u_N\|_{L^{\frac{10}{1-4\nu}}}\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}\|u_t\|_{L^{10}}\\&\le k_1(\omega)\|A^{\frac{1+\nu}{2}}u_N\|\\&\le 4\varepsilon k_1^2(\omega)+\tfrac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}u_N\|^2,\end{aligned}\tag{5.32}$$

and note that $\nu\le\frac{4-p}{4}$

$$\begin{aligned}&\int_U g_2'(u)u_tA^\nu u_Ndx\le C\int_U(1+|u|^p)|u_t||A^\nu u_N|dx\\&\le C(1+\|u\|^p_{L^{\frac{10}{4-4\nu}}})\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}\|u_t\|_{L^2}\\&\le C(1+\|\nabla u\|^p_2)\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}\|u_t\|_{L^6}\\&\le 4\varepsilon k_2^2(\omega)+\tfrac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}u_N\|^2,\end{aligned}\tag{5.33}$$

$$\begin{aligned}&\int_U g_1'(u_L)u_{Nt}A^\nu u_Ndx\le C(1+\|u_L\|^4_{L^{10}})\|A^{\frac{1+\nu}{2}}u_N\|_{L^{\frac{10}{1+4\nu}}}\|A^\nu u_{Nt}\|_{L^{\frac{10}{1+4\nu}}}\\&\le C(1+\|u_L\|^4_{L^{10}})\|A^{\frac{1+\nu}{2}}u_N\|_{L^{\frac{10}{1+4\nu}}}\|A^\nu u_{Nt}\|_{L^{\frac{10}{1+4\nu}}}\\&\le 4\varepsilon k_3(\omega)(\|A^{\frac{\nu}{2}}u_N\|^2+|\varepsilon|^2)+\tfrac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}u_N\|^2_{L^{\frac{10}{1+4\nu}}}\end{aligned}\tag{5.34}$$

and

$$\begin{aligned}&\int_U\left(g(u)-g_1(u_L)\right)|A^\nu u_N||z(\theta_t\omega)|dx\\&\le C\int_U g'\left(u+\theta(u-u_l)\right)|u-uL||A^\nu u_N||z(\theta_t\omega)|dx\\&\le C\int_U\left(1+|u|^4+|u_L|^4\right)|u_N||A^\nu u_N||z(\theta_t\omega)|dx\\&\le C\left(1+\|u\|^4_{L^{10}}+\|u_L\|^4_{L^{10}}\right)\|u_N\|_{L^{\frac{10}{1-4\nu}}}\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}|z(\theta_t\omega)|\\&\le 4\varepsilon\left(k_4^2(\omega)+|z(\theta_t\omega)|^2\right)+\tfrac{\varepsilon}{16}\left\|A^{\frac{1+\nu}{2}}u_N\right\|^2.\end{aligned}\tag{5.35}$$

Thus, by putting (5.24)-(5.35) into (5.23), can show that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|A^{\frac{\nu}{2}}\varphi_2\|^2_E+2\left(g(u)-g_1(u_L)\right)\right)+\frac{\varepsilon}{4}\left\|A^{\frac{\nu}{2}}\varphi_2\right\|^2_E+\frac{k\varepsilon}{2}(g(u)-g_1(u_L))\\&\le\mu_2|c||z(\theta_t\omega)|\|A^{\frac{\nu}{2}}\varphi_2\|^2_E+C(\omega)[1+k_1^2(\omega)+k_2^2(\omega)\\&+k_3^2(\omega)+k_4^2(\omega)+|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^4+\|A^{\frac{\nu}{2}}f(x)\|^2].\end{aligned}\tag{5.36}$$

By Gronwall's inequality to (5.36), is satisfied

$$\begin{aligned}
&\left\|A^{\frac{\nu}{2}}\varphi_2(t,\omega,\varphi(0,\omega))\right\|_E^2 \le \left(\|A^{\frac{\nu}{2}}\varphi_2(0,\omega,\varphi(0,\omega))\|_E^2 + 2(g(u(0,\omega,\varphi(0,\omega))) - g_1(u_L\,(0,\omega,\varphi(0,\omega)))\right)\\
&\le \left(\|A^{\frac{\nu}{2}}\varphi_2\|_E^2 + (g(u) - g_1(u_L))\right) e^{2\int_0^t(\sigma-\mu_2|c||z(\theta_s\omega)|))(s,\omega)ds}\\
&+\int_{-t}^0 \rho_1(\omega)e^{2\int_0^s(\sigma-\mu_2|c||z(\theta_\varsigma\omega)|)(\varsigma,\omega)d\varsigma}ds.
\end{aligned} \tag{5.37}$$

put

$$\begin{aligned}
\rho_1(\omega) &= C(\omega)[1+k_1^2(\omega)+k_2^2(\omega)+k_3^2(\omega)+k_4^2(\omega)\\
&+|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^4+\|A^{\frac{\nu}{2}}f(x)\|^2]
\end{aligned} \tag{5.38}$$

similar to above equation

$$\begin{aligned}
&\int_U (g(u)-g_1(u_L))\,A^\nu u_N dx \le C\int_U (g'(u+\theta(u-u_L))|u-u_L||A^\nu u_N|dx\\
&\le C\int_U \left(1+|u|^4+|u_L|^4\right)|u_N||A^\nu u_N|dx\\
&\le C(1+\|u\|_{L^{10}}^4+\|u_L\|_{L^{10}}^4)\|u_N\|_{L^{\frac{10}{1-4\nu}}}\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}\\
&\le k_5\,(\omega)\,\|A^{\frac{1+\nu}{2}}u_N\|\|A^\nu u_N\|\\
&\le \varepsilon k_5^2\,(\omega)\,\|A^{\frac{\nu}{2}}u_N\|^2+\frac{\varepsilon}{4}\|A^{\frac{1+\nu}{2}}u_N\|^2,
\end{aligned} \tag{5.39}$$

by (5.38) and (5.39), to get

$$\|A^\nu\varphi_2(t,\omega,\varphi(0,\omega))\|_E^2 \le r_5(\omega),$$

this complete the proof.

6. Random attractors

In this Section, it established the existence of a $\mathcal{D}$-random attractor for the random dynamical system Φ associated with system (3.15) on $\mathbb{R}^5$ that is, by Lemma 4.1, Φ has a closed random absorbing set in $\mathcal{D}$, which along with the $\mathcal{D}$-pullback asymptotic compactness and then imply the existence of a unique $\mathcal{D}$-random attractor. Next due to decomposition of solutions shall prove the $\mathcal{D}$-pullback asymptotic compactness of Φ (see[9, 32]).
Since $\omega\in\Omega, t\ge 0$, reads as

Lemma 6.1. assume that $(h_1)-(h_3)$ hold, then for any $t\ge 0,\omega\in\Omega$, the RDS Φ associated with (3.5) possesses a uniformly attracting set $\Lambda(0,\omega)\subset E$, and possesses a random attractor $\mathcal{A}(0,\omega)\subseteq\Lambda(0,\omega)\cap B_0(\omega)$.

Proof For any $t\ge 0,\omega\in\Omega$, as in Lemma 5.3 let $B_\nu(0,\omega)$ be the closed ball in $H_{2+2\nu}\times H_{2\nu}$, with radius $r_5(\omega)$. Set

$$\Lambda(0,\omega)=B_\nu(0,\omega), \tag{6.1}$$

then $\Lambda(0,\omega)\in\mathfrak{D}(E)$. Since $H_{2+2\nu}\times H_{2\nu}\hookrightarrow H_0^2(U)\times L^2(U)$. Now to show the following attraction property of $\Lambda(0,\omega)$: for every $B(0,\omega)\in\mathfrak{D}(E)$,

$$\lim_{t\to\infty} d_H(\Phi(t,\theta_{-t}\omega,B(0,\theta_{-t}\omega)),\Lambda(0,\omega))=0. \tag{6.2}$$

From Lemma 5.2, implies that

$$\varphi_N(0,\omega,\varphi(0,\omega)) = \varphi(0,\omega,\varphi(0,\omega)) - \varphi_L(0,\omega,\varphi(0,\omega)) \in \Lambda(0,\omega). \tag{6.3}$$

Thus, by Lemma 5.2, yields

$$\inf_{\psi\in\Lambda(0,\omega)} \|\varphi(0,\omega,\varphi(0,\omega)) - \psi\|_E^2 \le \|\varphi_L(0,\omega,\varphi_L(0,\omega))\|_E^2 \le r_4^2(\omega)e^{2\sigma_1(\omega)t},\ t \ge 0. \tag{6.4}$$

However, for all $t > 0$

$$dist(\Phi(t,\theta_{-t}\omega, B(0,\theta_{-t}\omega)),\Lambda(0,\omega)) \le r_4^2(\omega)e^{-2\sigma_1(\omega)t}. \tag{6.5}$$

Finally, from the relation between Φ and Ψ one can easily obtain that for any non-random bounded $B \subset E$ P-a.s.

$$dist(\Psi(t,\theta_{-t}\omega, B(0,\theta_{-t}\omega)),\Lambda(0,\omega)) \to 0, t \to +\infty. \tag{6.6}$$

Therefore, the RDS Φ associated with (3.5) possesses a random attractor $\mathcal{A}(0,\omega) \subseteq \Lambda(0,\omega) \cap B(\omega)$, $\mathcal{A} = \{\mathcal{A}(0,\omega) : t \ge 0, \omega \in \Omega\} \in U$ in $\mathbb{R}^5$.
Then the proof is completed. □

Theorem 6.2. we assume that$(h_1) - (h_3)$ hold. Then the continuous cocycle Φ associated with problem (3.8)or random dynamical system Φ has a unique D-pullback attractor$A = \{A(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ in $\mathbb{R}^5$.

Proof. Notice that the continuous cocycle Φ has a closed random absorbing set $\{A(\omega)\}_{\omega\in\Omega}$ in D by Lemma 4.2. On the other hand, by(3.17) and Lemma 6.1 the continuous cocycle Φ is D-pullback asymptotically compact in $\mathbb{R}^5$. Hence the existence of a unique D- random attractor for Φ follows from Lemma 2.6 immediately. □

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