Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Nonstandard Analysis

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Abstract

Discussions are presented by Morita and Sato on the problem of obtaining the particular solution of an inhomogeneous differential equation with polynomial coefficients in terms of the Green's function. In the present paper, solution is given without using the Green's function, on the basis of nonstandard analysis. It is applied to the hypergeometric, the Hermite, a simple ordinary and a fractional differential equation.

Keywords: differential equations with polynomial coefficients; nonstandard analysis; hypergeometric differential equation; fractional differential equation; Hermite differential equation

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1 Introduction

In the present paper, the problem of obtaining the particular solutions of a differential equation with polynomial coefficients, without using the Green's function, is studied.

In a preceding paper [(6)], this problem is studied in the framework of distribution theory, where the method is applied to Kummer's and the hypergeometric differential equation. In another paper [(8)], this problem is studied in the framework of nonstandard analysis, where a recipe of solution of the present problem is presented, and it is applied to a simple fractional and a first-order ordinary differential equation. In a recent paper [(9)], a compact recipe based on nonstandard analysis, which is obtained by revising the one given in [(8)], is presented, and is applied to Kummer's differential equation.

In the present paper, we adopt a recipe without the Green' functuion, and is applied to the hypergeometric differential equation, the differential equations treated in [(8)] and the Hermite differential equation.

The presentation in this paper follows those in [(6; 8; 9)], in Introduction and in many descriptions in the following sections.

We consider a fractional differential equation, which takes the form:

$$p_n(t, {}_RD_t)u(t) = \sum_{l=0}^{n} a_l(t)\,{}_RD_t^{\rho_l}u(t) = f(t), \tag{1.1}$$

where (i) $n \in \mathbb{Z}_{>-1}$, $t \in \mathbb{R}$, (ii) $a_l(t)$ for $l \in \mathbb{Z}_{[0,n]}$ are polynomials of t, (iii) $\rho_l \in \mathbb{C}$ for $l \in \mathbb{Z}_{[0,n]}$ satisfy $\mathrm{Re}\,\rho_0 > \mathrm{Re}\,\rho_1 \geq \cdots \geq \mathrm{Re}\,\rho_n$ and $\mathrm{Re}\,\rho_0 > 0$.

Here $\mathbb{Z}$ is the set of all integers, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$, $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$ and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{R}_{>a} = \{x \in \mathbb{R} \mid x > a\}$ for $a \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}\,z > 0\}$.

We use Heaviside's step function $H(t)$, which is equal to 1 if $t > 0$ and, to 0 if $t \leq 0$. Here ${}_RD_t^{\rho_l}$ are the Riemann-Liouville fractional integrals and derivatives defined by the following definition; see [(5; 10)].

Definition 1.1. Let $t \in \mathbb{R}$, $\tau \in \mathbb{R}$, $u_0(t)$ be locally integrable on $\mathbb{R}_{>\tau}$, $u(t) = u_0(t)H(t-\tau)$, $\lambda \in \mathbb{C}_+$, $n \in \mathbb{Z}_{>-1}$ and $\rho = n - \lambda$. Then ${}_RD_t^{-\lambda}u(t)$ is the Riemann-Liouville fractional integral defined by

$$\begin{aligned} {}_RD_t^{-\lambda}u(t) &= \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{t} (t-x)^{\lambda-1} u_0(x) H(x-\tau)dx \\ &= \frac{1}{\Gamma(\lambda)} \int_{\tau}^{t} (t-x)^{\lambda-1} u_0(x)dx \cdot H(t-\tau), \end{aligned} \tag{1.2}$$

and ${}_RD_t^{-\lambda}u(t) = 0$ for $t \leq \tau$, where $\Gamma(\lambda)$ is the gamma function, ${}_RD_t^{\rho}u(t) = {}_RD_t^{n-\lambda}u(t)$ is the Riemann-Liouville fractional derivative defined by

$${}_RD_t^{\rho}u(t) = {}_RD_t^{n-\lambda}u(t) = \frac{d^n}{dt^n}[{}_RD_t^{-\lambda}u_0(t)] \cdot H(t-\tau), \tag{1.3}$$

when $n \geq \mathrm{Re}\,\lambda$, and ${}_RD_t^{n}u(t) = \frac{d^n}{dt^n}u_0(t) \cdot H(t-\tau)$ when $\rho = n \in \mathbb{Z}_{>-1}$.

In accordance with Definition 1.1, when $u_0(t) = \frac{1}{\Gamma(\nu)}(t-\tau)^{\nu-1}$, we adopt

$${}_RD_t^{\rho}\frac{(t-\tau)^{\nu-1}}{\Gamma(\nu)}H(t-\tau) = \begin{cases} \frac{(t-\tau)^{\nu-\rho-1}}{\Gamma(\nu-\rho)}H(t-\tau), & \nu-\rho \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \\ 0, & \nu-\rho \in \mathbb{Z}_{<1}, \end{cases} \tag{1.4}$$

for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $\tau \in \mathbb{R}$. Here ${}_RD_t$ is used in place of usually used notation ${}_\tau D_R$, in order to show that the variable is t.

Remark 1.1. Let $g_\nu(t) = \frac{1}{\Gamma(\nu)}t^{\nu-1}H(t)$ for $\nu \in \mathbb{C}$. Then $g_\nu(t) = 0$ if $\nu \in \mathbb{Z}_{<1}$, and Equation (1.4) shows that if $\nu \notin \mathbb{Z}_{<1}$, ${}_RD_t^{\rho}g_\nu(t) = g_{\nu-\rho}(t)$. As a consequence, we have ${}_RD_t^{\nu+n}g_\nu(t) = g_{-n}(t) = 0$ for $n \in \mathbb{Z}_{>-1}$.

In distribution theory [(6; 11; 2; 12)], we use distribution $\tilde{H}(t)$, which corresponds to function $H(t)$, differential operator D and distribution $\delta(t) = D\tilde{H}(t)$, which is called Dirac's delta function.

1.1 Preliminaries on Nonstandard Analysis

In nonstandard analysis [(1)], where infinitesimal numbers appear. We denote the set of all infinitesimal real numbers by $\mathbb{R}^0$. We also use $\mathbb{R}^0_{>0} = \{\epsilon \in \mathbb{R}^0 \mid \epsilon > 0\}$, which is such that if $\epsilon \in \mathbb{R}^0_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ satisfying $\epsilon < \frac{1}{N}$. We use $\mathbb{R}^{ns}$, which has subsets $\mathbb{R}$ and $\mathbb{R}^0$. If $x \in \mathbb{R}^{ns}$ and $x \notin \mathbb{R}$, x is expressed as $x_1 + \epsilon$ by $x_1 \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^0$, where x_1 may be $0 \in \mathbb{R}$. Equation $x \simeq y$ for $x \in \mathbb{R}^{ns}$ and $y \in \mathbb{R}^{ns}$, is used, when $x - y \in \mathbb{R}^0$. We denote the set of all infinitesimal complex numbers by $\mathbb{C}^0$, which is the set of complex numbers z which satisfy $|\mathrm{Re}\,z| + |\mathrm{Im}\,z| \in \mathbb{R}^0$. We use $\mathbb{C}^{ns}$, which has subsets $\mathbb{C}$ and $\mathbb{C}^0$. If $z \in \mathbb{C}^{ns}$ and $z \notin \mathbb{C}$, z is expressed as $z_1 + \epsilon$ by $z_1 \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^0$, where z_1 may be $0 \in \mathbb{C}$.

In place of (1.4), we now use

$$ {}_RD_t^\rho g_{\nu+\epsilon}(t) = {}_RD_t^\rho \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu-1+\epsilon} H(t) = g_{\nu-\rho+\epsilon}(t) = \frac{1}{\Gamma(\nu-\rho+\epsilon)} t^{\nu-\rho-1+\epsilon} H(t), \tag{1.5} $$

for all $\rho \in \mathbb{C}$ and $\nu \in \mathbb{C}$, where $\epsilon \in \mathbb{R}^0_{>0}$.

Lemma 1.1. *Let $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$, $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{R}^0_{>0}$ and $g_{\nu+\epsilon}(t)=\frac{1}{\Gamma(\nu+\epsilon)} t^{\nu+\epsilon-1} H(t)$. Then the index law:*

$$ {}_RD_t^{\rho_1} \, {}_RD_t^{\rho_2} g_{\nu+\epsilon}(t) = {}_RD_t^{\rho_1+\rho_2} g_{\nu+\epsilon}(t) = g_{\nu-\rho_1-\rho_2+\epsilon}(t), \tag{1.6} $$

always holds.

In the present study in nonstandard analysis, in place of $\tilde{H}(t)$ and $\delta(t)$ in distribution theory, $H_\epsilon(t)$ and $\delta_\epsilon(t)$ are used, which are given by

$$ H_\epsilon(t) = {}_RD_t^{-\epsilon} H(t) = g_{1+\epsilon}(t) = \frac{1}{\Gamma(\epsilon+1)} t^\epsilon H(t), \tag{1.7} $$

$$ \delta_\epsilon(t) = \frac{d}{dt} H_\epsilon(t) \tag{1.8} $$

for $\epsilon \in \mathbb{R}^0_{>0}$. We note that they tend to $H(t)$ and 0, respectively, in the limit of $\epsilon \to 0$.

Lemma 1.2. *In the notation in Remark 1.1, $H_\epsilon(t) = g_{1+\epsilon}(t)$, $\delta_\epsilon(t) = g_\epsilon(t)$, and*

$$ {}_RD_t^\epsilon H_\epsilon(t) = {}_RD_t^\epsilon g_{1+\epsilon}(t) = g_1(t) = H(t), \quad {}_RD_t^\epsilon \delta_\epsilon(t) = {}_RD_t^\epsilon g_\epsilon(t) = g_0(t) = 0. \tag{1.9} $$

1.2 Summary of the Following Sections

In solving Equation (1.1) for $u(t)$, in nonstandard analysis, we consider the solution of the following transformed differential equation for $\tilde{u}(t) = {}_RD_t^{-\epsilon} u(t)$:

$$ \tilde{p}_{n,\epsilon}(t, {}_RD_t) \tilde{u}(t) = \tilde{f}(t), \tag{1.10} $$

for $\epsilon \in \mathbb{R}^0_{>0}$, where

$$ \tilde{p}_{n,\epsilon}(t, {}_RD_t) = {}_RD_t^{-\epsilon} p_n(t, {}_RD_t) {}_RD_t^\epsilon. \tag{1.11} $$

In the present study in nonstandard analysis, we adopt the following assumption.

Assumption 1.1. Solutions of Equations (1.1) and (1.10), $u(t)$ and $\tilde{u}(t)$, are linear combinations of $g_\nu(t)$ and $g_{\nu+\epsilon}(t)$, respectively, for ν satisfying $\mathrm{Re}\,\nu > \nu_0 \in \mathbb{R}$. As a consequence, the following condition is satisfied.

Condition 1.1. $\tilde{f}(t)$ and $f(t)$ are expressed as follows:

$$ \tilde{f}(t) = \sum_{l=1}^\infty c_l \cdot g_{\nu_l+\epsilon}(t) = \sum_{l=1}^\infty c_l \cdot \frac{t^{\nu_l+\epsilon-1}}{\Gamma(\nu_l+\epsilon)} H(t), \quad f(t) = \sum_{l=1}^\infty d_l \cdot g_{\nu_l+\epsilon}(t), \tag{1.12} $$

respectively, where $c_l \in \mathbb{C}$ are constants, $\nu_l \in \mathbb{C}$ satisfy $\mathrm{Re}\,\nu_l \geq \mathrm{Re}\,\nu_1$, for all $l \in \mathbb{Z}_{>0}$, and $d_l = c_l$ if $\nu_l \notin \mathbb{Z}_{<1}$, and $d_l = 0$ if $\nu_l \in \mathbb{Z}_{<1}$.

In order to obtain the solutions when Condition 1.1 is satisfied, we solve the equations which satisfy the following condition.

Condition 1.2. $\tilde{f}(t) = g_{\nu+\epsilon}(t) = {}_RD_t^{-\nu} g_\epsilon(t) = {}_RD_t^{\beta}\delta_\epsilon(t)$ for $\nu = -\beta \in \mathbb{C}$ and $\epsilon \in \mathbb{R}^0_{>0}$. When $\nu = -\beta \in \mathbb{Z}_{<1}$, $f(t) = 0$, and when $\nu = -\beta \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, $f(t) = g_\nu(t)$.

The solutions for the cases in which Condition 1.1 is satisfied, are obtained from those for the cases in which Condition 1.2 is satisfied, with the aid of the following lemma.

Lemma 1.3. *Let the solution of Equations* (1.10) *and* (1.1) *satisfying Condition 1.2, be expressed by* $\tilde{u}_\nu(t)$ *and* $u_\nu(t)$*, respecively. Then the solution of Equations* (1.10) *and* (1.1) *satisfying Condition 1.1, are given by* $\tilde{u}(t) = \sum_{l=1}^\infty c_l \cdot \tilde{u}_{\nu_l}(t)$ *and* $u(t) = \sum_{l=1}^\infty d_l \cdot u_{\nu_l}(t)$.

In the following sections, only the solutions satisfying Condition 1.2 are obtained. In Sections 2 and 4, solutions of the hypergeometric the Hermite differential equation are given. In Sections 3 and 5, solutions of a simple ordinary and a fractional differential equation, which are studied in [(8)], are given. The fractional differential equation is the one presented in [(3)].

In the preceding paper [(9)], the inhomogeneous terms $f(t)$ and $\tilde{f}(t)$ are assumed to satisfy one of four conditions, which include the above two conditions.

In [(9)], full expressions of the Green's functions and the solutions, are derived along the recipe given there, for Kummer's differential equation.

Section 6 is for Conclusion.

2 Solution of the Hypergeometric Differential Equation

The hypergeometric differential equation is described by

$$p_H(t, {}_RD_t)u(t) = [t(1-t)\frac{d^2}{dt^2} + (c - (a+b+1)t)\frac{d}{dt} - ab]u(t) = f(t), \tag{2.1}$$

where a, b and c are constants satifying $a \neq 0$ and $b \neq 0$.

Lemma 2.1. *Let* $c \notin \mathbb{Z}_{<1}$*. Then there exist two complementary solutions of Equation* (2.1)*, which are given by*

$$H_1(t) = {}_2F_1(a, b; c; t) = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{k!(c)_k} t^k, \quad t > 0, \tag{2.2}$$

$$H_2(t) = \frac{1}{\Gamma(2-c)} t^{1-c} \cdot {}_2F_1(1+a-c, 1+b-c; 2-c; t), \quad t > 0. \tag{2.3}$$

In the present paper, this is proved in Lemma 2.3 given below.

We construct the transformed differential equation of Equation (2.1), which corresponds to Equation (1.10). For this purpose, we use the following lemma.

Lemma 2.2. *Let* $\lambda \in \mathbb{C}_+$*,* $m \in \mathbb{Z}_{>-1}$ *and* $\rho = m - \lambda$*. Then*

$$ {}_RD_t^\rho[tu(t)] = t \cdot {}_RD_t^\rho u(t) + \rho \cdot {}_RD_t^{\rho-1} u(t), \tag{2.4}$$

$$ {}_RD_t^\rho[t^2 u(t)] = t^2 \cdot {}_RD_t^\rho u(t) + 2\rho t \cdot {}_RD_t^{\rho-1} u(t) + \rho(\rho-1) \cdot {}_RD_t^{\rho-2} u(t). \tag{2.5}$$

Proof. When $\rho = -\lambda$, by using (1.2), we have

$$t \cdot {}_RD_t^{-\lambda} u(t) - {}_RD_t^{-\lambda}[tu(t)] = \frac{1}{\Gamma(\lambda)} \int_{-\infty}^t (t-x)^\lambda u(x)dx = \lambda \cdot {}_RD_t^{-\lambda-1} u(t), \tag{2.6}$$

which gives (2.4) for $\rho = -\lambda$. We then prove this equation for $\rho \in \mathbb{Z}_{>0}$ by mathematical induction. By replacing $u(t)$ by $tu(t)$ in Equation (2.4), and then by using Equation (2.4) in the obtained equation, we obtain

$$\begin{aligned} {}_RD_t^\rho[t^2 u(t)] &= t \cdot {}_RD_t^\rho[tu(t)] + \rho \cdot {}_RD_t^{\rho-1}[tu(t)] \\ &= t[t \cdot {}_RD_t^\rho u(t) + \rho \cdot {}_RD_t^{\rho-1} u(t)] + \rho[t \cdot {}_RD_t^{\rho-1} u(t) + (\rho-1) \cdot {}_RD_t^{\rho-2} u(t)], \end{aligned} \tag{2.7}$$

which gives (2.5). □

With the aid of formulas (2.4) and (2.5) for $\rho = -\epsilon$, we obtain the following transformed differential equation of Equation (2.1), which is satisfied by $\tilde{u}(t) = {}_R D_t^{-\epsilon} u(t)$ and $\tilde{f}(t) = {}_R D_t^{-\epsilon} f(t)$:

$$\begin{aligned}\tilde{p}_{H,\epsilon}(t, {}_R D_t)\tilde{u}(t) &= {}_R D_t^{-\epsilon} p_H(t, {}_R D_t)\, {}_R D_t^{\epsilon}\tilde{u}(t)\\ &= {}_R D_t^{-\epsilon}[t(1-t)\frac{d^2}{dt^2} + (c-(a+b+1)t)\frac{d}{dt} - ab]\, {}_R D_t^{\epsilon}\tilde{u}(t)\\ &= [t(1-t)\frac{d^2}{dt^2} + (c-\epsilon-(a+b+1-2\epsilon)t)\frac{d}{dt} - (a-\epsilon)(b-\epsilon)\tilde{u}(t) = \tilde{f}(t). \quad (2.8)\end{aligned}$$

Lemma 2.3. *Let $H_1(t)$ and $H_2(t)$ be given by Equations* (2.2) *and* (2.3)*, respectively. Then*

(i) *when $\nu \in \mathbb{C}$ and $\tilde{f}(t) = g_{\nu+\epsilon}(t)$, $\tilde{u}_\nu(t)$, given by*

$$\tilde{u}_\nu(t) = \frac{1}{\nu - 1 + c}\sum_{k=0}^{\infty}\frac{(a+\nu)_k(b+\nu)_k}{(c+\nu)_k\Gamma(k+\nu+\epsilon+1)}t^{k+\nu+\epsilon}H(t), \quad (2.9)$$

is a particular solution of Equation (2.8)*,*

(ii) *when $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $f(t) = g_\nu(t)$, $u_\nu(t) = {}_R D_t^{\epsilon}\tilde{u}_\nu(t)$, given by*

$$u_\nu(t) = \frac{1}{\nu - 1 + c}\sum_{k=0}^{\infty}\frac{(a+\nu)_k(b+\nu)_k}{(c+\nu)_k\Gamma(k+\nu+1)}t^{k+\nu}H(t), \quad \nu \notin \mathbb{Z}_{<1}, \quad (2.10)$$

is a particular solution of Equation (2.1)*,*

(iii) *when $\nu = -\beta \in \mathbb{Z}_{<1}$ and $C_\beta = \frac{(a-\beta)_\beta(b-\beta)_\beta}{(-1+c-\beta)_\beta}$, $u_{-\beta}(t) = u_\nu(t) = {}_R D_t^{\epsilon}\tilde{u}_\nu(t)$, given by*

$$u_{-\beta}(t) = \frac{1}{-\beta - 1 + c}\sum_{k=\beta}^{\infty}\frac{(a-\beta)_k(b-\beta)_k}{(c-\beta)_k\Gamma(k-\beta+1)}t^{k-\beta}H(t) = C_\beta\frac{1}{-1+c}H_1(t)H(t), \quad (2.11)$$

is a complementary solution of Equation (2.1)*, and*

(iv) *$\tilde{u}_c(t)$ and $u_c(t)$, given by*

$$\tilde{u}_c(t) = \sum_{k=0}^{\infty}\frac{(a-c+1)_k(b-c+1)_k}{k!\Gamma(2-c+k+\epsilon)}t^{1-c+\epsilon+k}H(t), \quad (2.12)$$

$$u_c(t) = {}_R D_t^{\epsilon}\tilde{u}_c(t) = H_2(t)H(t), \quad (2.13)$$

are complementary solutions of Equations (2.8) *and* (2.1)*, respectively.*

When $\nu = -\beta \in \mathbb{Z}_{<1}$, with the aid of Equation (2.11)*, $\tilde{u}_\nu(t)$ given by Equation* (2.9) *is expressed as*

$$\tilde{u}_{-\beta}(t) = {}_R D_t^{-\epsilon}u_{-\beta}(t) + \epsilon\sum_{k=0}^{\beta-1}\frac{(a-\beta)_k(b-\beta)_k}{(-1+c-\beta)_{k+1}}(\beta-k-1)!t^{k-\beta+\epsilon}H(t). \quad (2.14)$$

Proof. In Frobenius' method, solution $\tilde{u}(t)$ is assumed to be expressed by

$$\tilde{u}(t) = \sum_{k=0}^{\infty}p_k\frac{1}{\Gamma(\alpha+k+1)}t^{\alpha+k}H(t), \quad (2.15)$$

where α and p_k are constants, and $p_0 \neq 0$. By using this in Equation (2.8), we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}p_k[(\alpha+k-1+c-\epsilon)\frac{t^{\alpha+k-1}}{\Gamma(\alpha+k)} - [(\alpha+k-1)(\alpha+k)\\ &\quad+(a-\epsilon+b-\epsilon+1)(\alpha+k) + (a-\epsilon)(b-\epsilon)]\frac{t^{\alpha+k}}{\Gamma(\alpha+k+1)}]H(t)\\ &= p_0(\alpha-1+c-\epsilon)\frac{t^{\alpha-1}}{\Gamma(\alpha)}H(t) + \sum_{k=1}^{\infty}[p_k(\alpha+k-1+c-\epsilon)\\ &\quad-p_{k-1}(\alpha+k-1+a-\epsilon)(\alpha+k-1+b-\epsilon)]\frac{t^{\alpha+k-1}}{\Gamma(\alpha+k)}H(t) = \tilde{f}(t). \quad (2.16)\end{aligned}$$

When $\tilde{f}(t) = g_{\nu+\epsilon}(t) = \frac{t^{\nu+\epsilon-1}}{\Gamma(\nu+\epsilon)} H(t)$, Equation (2.16) is satisfied, if $\alpha = \nu + \epsilon$, $p_0 = \frac{1}{-1+c+\nu}$ and

$$p_k = p_{k-1} \frac{(k-1+a+\nu)(k-1+b+\nu)}{k-1+c+\nu} = p_0 \frac{(a+\nu)_k (b+\nu)_k}{(c+\nu)_k}, \quad k \in \mathbb{Z}_{>0}. \tag{2.17}$$

Using these α, p_0 and p_k in Equation (2.15), and putting $\tilde{u}(t) = \tilde{u}_\nu(t)$, we obtain Equation (2.9).

When $\tilde{f}(t) = 0$, Equation (2.16) is satisfied, if $\alpha = 1 - c + \epsilon$, $p_0 \neq 0$ and

$$p_k = p_{k-1} \frac{(k+a-c)(k+b-c)}{k} = p_0 \frac{(a-c+1)_k (b-c+1)_k}{k!}, \quad k \in \mathbb{Z}_{>0}. \tag{2.18}$$

Using these α, p_0 and p_k in Equation (2.15), and putting $\tilde{u}(t) = p_0 \tilde{u}_c(t)$, we obtain Equation (2.12).

Remark 2.1. We note here that the above derivation of Equation (2.9) becomes that of Equation (2.10), when we replace $\tilde{u}$ by u, ϵ by 0, $\tilde{f}$ by f, (2.8) by (2.1), and (2.9) by (2.10).

Remark 2.2. We note here that if $\nu = -\beta = 0$, the above derivation of Equation (2.9) becomes that of Equation (2.11), when we replace $\tilde{u}$ by u, (2.8) by (2.1), ϵ by 0, $\tilde{f}$ by f, ν by 0, (2.9) by (2.11).

Remark 2.3. We note here that the above derivation of Equation (2.12) becomes that of Equation (2.13), when we replace $\tilde{u}$ by u, (2.8) by (2.1), ϵ by 0, $\tilde{f}$ by f, (2.12) by (2.13).

In obtaining Equation (2.14) from (2.9), the following formulas are used:

$$\frac{1}{\Gamma(z)} = \frac{\sin(\pi z)\Gamma(1-z)}{\pi}; \quad \frac{1}{\Gamma(-m+\epsilon)} \simeq (-1)^m \epsilon \cdot m!, \quad m \in \mathbb{Z}_{>-1}, \tag{2.19}$$

which are given by (65) in [(9)]. □

Remark 2.4. For $\tau \in \mathbb{R}$, the Green's function $G_{H,\epsilon}(t,\tau)$ is so defined that it satisfies

$$\tilde{p}_{H,\epsilon}(t, {}_R D_t) G_{H,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau). \tag{2.20}$$

Comparing Equation (2.20) with the solutions $\tilde{u}_\nu(t)$ in Lemma 2.3, we see that $G_{H,\epsilon}(t,0) = \tilde{u}_0(t)$, and $G_{H,0}(t,0) = {}_R D_t^{-\epsilon} G_{H,\epsilon}(t,0) = u_0(t)$.

Remark 2.5. In [(9)], studies are made on Kummer's differential equation, where we have equations which correspond to the equations given above for the hypergeometric differential equation. The equations which correspond to Equations (2.1), (2.2), (2.3), (2.8), (2.9), (2.11), (2.12) and (2.13) given above, are (24), (41), (42), (39), (62), (63), (45) and (46) in [(9)], in this order. Following Equation (2.11), Equation (63) in [(9)] is expressed as

$$u_f(t) = \frac{(a-n)_n b^n}{(-1+c-n)_{n+1}} \sum_{l=0}^{\infty} \frac{(a)_l b^l}{(c)_l \Gamma(l+1)} t^l H(t) = \frac{(a-n)_n b^n}{(-1+c-n)_n} \frac{1}{-1+c} K_1(t) H(t). \tag{2.21}$$

3 Solution of a Simple Ordinary and a Fractional Differential Equation

In a preceding paper [(8)], solutions of a simple ordinary and a fractional differential equation are discussed in the framework of nonstandard analysis, by using the recipe of solution given there.

The simple fractional differential equation is

$$p_F(t, {}_R D_t) v(t) = [{}_R D_t^{3/2} - at \cdot {}_R D_t^{1/2}] v(t) = f(t), \tag{3.1}$$

which is presented in [(3)], and the ordinary differential equation is

$$p_L(t, \frac{d}{dt}) y(t) = [\frac{d}{dt} - at] y(t) = f(t), \tag{3.2}$$

where a is a constant. A brief discussion of this problem is given by using the revised recipe adopted in the preceding section.

Putting $y(t) = {}_RD_t^\epsilon \tilde{y}(t)$ and $f(t) = {}_RD_t^\epsilon \tilde{f}(t)$ in Equation (3.2), and using Equation (2.4), we obtain the following transformed differential equation of Equation (3.2):

$$\tilde{p}_{L,\epsilon}(t, {}_RD_t)\tilde{y}(t) = {}_RD_t^{-\epsilon} p_L(t, \frac{d}{dt}) {}_RD_t^\epsilon \tilde{y}(t) = {}_RD_t^{-\epsilon}[\frac{d}{dt} - at] {}_RD_t^\epsilon \tilde{y}(t)$$
$$= [\frac{d}{dt} - at + a\epsilon \cdot {}_RD_t^{-1}]\tilde{y}(t) = \tilde{f}(t). \tag{3.3}$$

In the case of Equation (3.1), putting $v(t) = {}_RD_t^\epsilon \tilde{v}(t)$ and $f(t) = {}_RD_t^\epsilon \tilde{f}(t)$, we obtain

$$\tilde{p}_{F,\epsilon}(t, {}_RD_t)\tilde{v}(t) = {}_RD_t^{-\epsilon} p_F(t, {}_RD_t) {}_RD_t^\epsilon \tilde{v}(t) = {}_RD_t^{-\epsilon} p_L(t, {}_RD_t) {}_RD_t^{1/2} {}_RD_t^\epsilon \tilde{v}(t)$$
$$= \tilde{p}_{L,\epsilon}(t, {}_RD_t) {}_RD_t^{1/2} \tilde{v}(t) = \tilde{f}(t). \tag{3.4}$$

Remark 3.1. For $\tau \in \mathbb{R}$, the Green's function $G_{L,\epsilon}(t, \tau)$ for Equation (3.2), is so defined that it satisfies

$$\tilde{p}_{L,\epsilon}(t, {}_RD_t) G_{L,\epsilon}(t, \tau) = [\frac{d}{dt} - at + a\epsilon \cdot {}_RD_t^{-1}] G_{L,\epsilon}(t, \tau) = \delta_\epsilon(t - \tau). \tag{3.5}$$

Lemma 3.1. *Let $y_0(t) = e^{at^2/2}H(t)$. Then by using the formula $(2k)! = 4^k k! (\frac{1}{2})_k$, we can easily confirm that $G_{L,0}(t, 0)$ and $G_{L,\epsilon}(t, 0)$, given by*

$$y_0(t) = G_{L,0}(t, 0) = e^{at^2/2}H(t) = \sum_{k=0}^{\infty} \frac{a^k}{2^k k!} t^{2k} H(t) = \sum_{k=0}^{\infty} \frac{2^k a^k (\frac{1}{2})_k}{(2k)!} t^{2k} H(t), \tag{3.6}$$

$$\tilde{y}_0(t) = G_{L,\epsilon}(t, 0) = {}_RD_t^{-\epsilon} G_{L,0}(t, 0) = \sum_{k=0}^{\infty} \frac{2^k a^k (\frac{1}{2})_k}{\Gamma(2k + 1 + \epsilon)} t^{2k+\epsilon} H(t), \tag{3.7}$$

are a complementary solution of Equation (3.2) *and a particular solution of* (3.3) *for $\tilde{f}(t) = \delta_\epsilon(t) = \frac{1}{\Gamma(\epsilon)} t^{\epsilon-1} H(t)$, respectively.*

Remark 3.2. For $\tau \in \mathbb{R}$, the Green's function $G_{F,\epsilon}(t, \tau)$ for Equation (3.1), is so defined that it satisfies

$$\tilde{p}_{F,\epsilon}(t, {}_RD_t) G_{F,\epsilon}(t, \tau) = [{}_RD_t^{3/2} - at \cdot {}_RD_t^{1/2} + a\epsilon \cdot {}_RD_t^{-1/2}] G_{F,\epsilon}(t, \tau) = \delta_\epsilon(t - \tau). \tag{3.8}$$

Lemma 3.2. *Let $G_{L,0}(t, 0)$ and $G_{L,\epsilon}(t, 0)$ be given by Equations* (3.6) *and* (3.7)*. Then comparing Equation* (3.2) *with* (3.1)*, and Equation* (3.5) *with* (3.8)*, we confirm that*

$$v_0(t) = G_{F,0}(t, 0) = {}_RD_t^{-1/2} y_0(t) = {}_RD_t^{-1/2} G_{L,0}(t, 0) = \sum_{k=0}^{\infty} \frac{2^k a^k (\frac{1}{2})_k}{\Gamma(2k + \frac{3}{2})} t^{2k+1/2} H(t), \tag{3.9}$$

$$\tilde{v}_0(t) = G_{F,\epsilon}(t, 0) = {}_RD_t^{-1/2} \tilde{y}_0(t) = {}_RD_t^{-1/2} G_{L,\epsilon}(t, 0) = \sum_{k=0}^{\infty} \frac{2^k a^k (\frac{1}{2})_k}{\Gamma(2k + \frac{3}{2} + \epsilon)} t^{2k+1/2+\epsilon} H(t), \tag{3.10}$$

are a complementary solution of Equation (3.1) *and a particular solution of* (3.4) *for $\tilde{f}(t) = \delta_\epsilon(t) = \frac{1}{\Gamma(\epsilon)} t^{\epsilon-1} H(t)$, respectively.*

4 Solutions of the Hermite Differential Equation

The Hermite differential equation is described by

$$p_{He,b}(t, \frac{d}{dt}) u(t) = [\frac{d^2}{dt^2} - at\frac{d}{dt} + ab] u(t) = f(t), \quad t > 0, \tag{4.1}$$

where $a \in \mathbb{C} \backslash \{0\}$ and $b \in \mathbb{Z}_{>-1}$ are constants.

Lemma 4.1. *We have two complementary solutions of Equation* (4.1)*, which are*

$$he_{b,0}(t) = \sum_{k=0}^{\infty} \frac{(-\frac{b}{2})_k (2a)^k}{\Gamma(2k+1)} t^{2k} = {}_1F_1(-\frac{b}{2}; \frac{1}{2}; \frac{1}{2}at^2), \quad t > 0, \tag{4.2}$$

$$he_{b,1}(t) = \sum_{k=0}^{\infty} \frac{(\frac{1-b}{2})_k (2a)^k}{\Gamma(2k+2)} t^{2k+1} = t \cdot {}_1F_1(\frac{1-b}{2}; \frac{3}{2}; \frac{1}{2}at^2), \;\; t > 0. \tag{4.3}$$

These are related by $he_{b,0}(t) = {}_RD_t he_{b+1,1}(t)$ *and* $he_{b,1}(t) = {}_RD_t^{-1} he_{b-1,0}(t)$.

In Lemma 4.2, derivation of (4.2) and (4.3) by using Frobenius' method is presented.

Remark 4.1. Let $n \in \mathbb{Z}_{>-1}$. Then if $b = 2n+1$, $(-\frac{b}{2})_k = (-\frac{1}{2} - n)_k$ and $(\frac{1-b}{2})_k = (-n)_k$, and if $b = 2n$, $(-\frac{b}{2})_k = (-n)_k$ and $(\frac{1-b}{2})_k = (\frac{1}{2} - n)_k$. Since

$$(-n)_k = \begin{cases} (-1)^k \frac{n!}{(n-k)!}, & k \in \mathbb{Z}_{[0,n]}, \\ 0, & k \in \mathbb{Z}_{>n}, \end{cases} \tag{4.4}$$

$he_{2n,0}(t)$ and $he_{2n+1,1}(t)$ are

$$he_{2n,0}(t) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(2a)^k}{(2k)!} t^{2k}, \quad he_{2n+1,1}(t) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(2a)^k}{(2k+1)!} t^{2k+1}, \tag{4.5}$$

which are polynomials of degree $2n$ and $2n+1$, respectively, and $he_{2n+1,0}(t)$ and $he_{2n,1}(t)$ are infinite series.

Remark 4.2. In the notation of [(4, Chaper V, Section 2)], the Hermite polynomials $He_{2n}(x)$ and $He_{2n+1}(x)$ represent

$$He_{2n}(x) = \frac{(-1)^n}{2^n} \frac{(2n)!}{n!} \cdot he_{2n,0}(\frac{x}{\sqrt{a}}), \quad He_{2n+1}(x) = \frac{(-1)^n}{2^n} \frac{(2n+1)!}{n!} \sqrt{a} \cdot he_{2n+1,1}(\frac{x}{\sqrt{a}}), \tag{4.6}$$

and the Hermite functions of the second kind $he_{2n}(x)$ and $he_{2n+1}(x)$ represent

$$he_{2n}(x) = (-1)^n 2^n n! \sqrt{a} \cdot he_{2n,1}(\frac{x}{\sqrt{a}}), \quad he_{2n+1}(x) = (-1)^{n+1} 2^n n! \cdot he_{2n+1,0}(\frac{x}{\sqrt{a}}). \tag{4.7}$$

The transformed differential equation of Equation (4.1) for $\tilde{u}(t) = {}_RD_t^{-\epsilon} u(t)$ and $\tilde{f}(t) = {}_RD_t^{-\epsilon} f(t)$, is

$$\begin{aligned} \tilde{p}_{He,b+\epsilon}(t, {}_RD_t)\tilde{u}(t) &= {}_RD_t^{-\epsilon} p_{He,b}(t, \frac{d}{dt})_R D_t^{\epsilon} \tilde{u}(t) \\ &= [\frac{d^2}{dt^2} - at\frac{d}{dt} + a(b+\epsilon)]\tilde{u}(t) = \tilde{f}(t). \end{aligned} \tag{4.8}$$

Remark 4.3. For $\tau \in \mathbb{R}$, the Green's function $G_{He,b,\epsilon}(t,\tau)$ for Equation (4.1), is so defined that it satisfies

$$\tilde{p}_{He,b+\epsilon}(t, \frac{d}{dt}) G_{He,b,\epsilon}(t,\tau) = [\frac{d^2}{dt^2} - at\frac{d}{dt} + a(b+\epsilon)] G_{He,b,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau). \tag{4.9}$$

Lemma 4.2. *Let* $he_{b,0}(t)$ *and* $he_{b,1}(t)$ *be given by Equations* (4.2) *and* (4.3)*. Then*

(i) *when* $\nu \in \mathbb{C}$ *and* $\tilde{f}(t) = g_{\nu+\epsilon}(t) = {}_RD_t^{-\nu}\delta_\epsilon(t)$, $\tilde{u}_\nu(t)$ *given by*

$$\tilde{u}_\nu(t) = {}_RD_t^{-\nu-\epsilon} he_{b-\nu,1}(t) H(t) = \sum_{k=0}^{\infty} \frac{(\frac{\nu+1-b}{2})_k (2a)^k}{\Gamma(2k+2+\nu+\epsilon)} t^{2k+1+\nu+\epsilon} H(t), \tag{4.10}$$

is a particular solution of Equation (4.8)*. We denote the equation which is obtaind from Equation* (4.10) *by replacing* $\tilde{u}$ *by* u*, and* ϵ *by* 0*, by Equation* (4.10-A)*, and then*

(ii) *when* $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ *and* $f(t) = g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1} H(t)$*,* $u_\nu(t) = {}_R D_t^\epsilon \tilde{u}_\nu(t)$*, satisfying Equation* (4.10-A)*, is a particular solution of Equation* (4.1)*.*

(iii) *When* $\beta \in \mathbb{Z}_{>-1}$*, for* $\nu = -2\beta$ *and* $\nu = -2\beta - 1$*, we express* $u_\nu(t) = {}_R D_t^\epsilon \tilde{u}_\nu(t)$ *as*

$$u_{-2\beta}(t) = {}_R D_t^{2\beta} he_{b+2\beta,1}(t) H(t) = \sum_{k=\beta}^{\infty} \frac{(-\beta + \frac{1-b}{2})_k (2a)^k}{\Gamma(2k+2-2\beta)} t^{2k+1-2\beta} H(t)$$

$$= (-\beta + \frac{1-b}{2})_\beta \cdot he_{b,1}(t) H(t), \tag{4.11}$$

$$u_{-2\beta-1}(t) = {}_R D_t^{2\beta} he_{b+2\beta,0}(t) H(t) = \sum_{k=\beta}^{\infty} \frac{(-\beta - \frac{b}{2})_k (2a)^k}{\Gamma(2k+1-2\beta)} t^{2k-2\beta} H(t)$$

$$= (-\beta - \frac{b}{2})_\beta \cdot he_{b,0}(t) H(t), \tag{4.12}$$

which are complementary solutions of Equation (4.1)*.*

Equation (4.5) *shows that when* $n \in \mathbb{Z}_{>-1}$*,* $he_{2n,0}(t)$ *and* $he_{2n+1,1}(t)$ *are polynomials of degree* $2n$ *and* $2n+1$*, respectively.*

Proof. We now give a derivation of Equation (4.10), by Frobenius' Method.

In Frobenius' method, solution $\tilde{u}(t)$ is assumed to be expressed by

$$\tilde{u}(t) = \sum_{k=0}^{\infty} p_k \frac{1}{\Gamma(\alpha+2k+1)} t^{\alpha+2k} H(t), \tag{4.13}$$

where α and p_k are constants, and $p_0 \neq 0$. Using these in Equation (4.8), we obtain

$$\sum_{k=0}^{\infty} p_k \frac{t^{\alpha+2k-2}}{\Gamma(\alpha+2k-1)} - a(\alpha+2k-b-\epsilon) \frac{t^{\alpha+2k}}{\Gamma(\alpha+2k+1)}] H(t)$$

$$= p_0 \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} H(t) + \sum_{k=1}^{\infty} [p_k - a p_{k-1}(\alpha+2k-2-b-\epsilon)] \frac{t^{\alpha+2k-2}}{\Gamma(\alpha+2k-1)} H(t) = \tilde{f}(t). \tag{4.14}$$

When $\tilde{f}(t) = {}_R D_t^{-\nu} \delta_\epsilon(t) = \frac{t^{\epsilon-1+\nu}}{\Gamma(\epsilon+\nu)} H(t)$, Equation (4.14) is satisfied, if $\alpha = 1 + \epsilon + \nu$, $p_0 = 1$ and

$$p_k = 2a p_{k-1}(k - 1 + \frac{\nu+1-b}{2}) = p_0 (2a)^k (\frac{\nu+1-b}{2})_k, \quad k \in \mathbb{Z}_{>0}. \tag{4.15}$$

Using these in Equation (4.13), and putting $\tilde{u}(t) = \tilde{u}_\nu(t)$, we obtain Equation (4.10).

Remark 4.4. We note here that the above derivation of Equation (4.10) becomes that of Equation (4.10-A), when we replace $\tilde{u}$ by u, (4.8) by (4.1), ϵ by 0, $\tilde{f}$ by f, and (4.10) by (4.10-A).

Remark 4.5. We note here that if $\nu = -2\beta = 0$, the above derivation of Equation (4.10) becomes that of Equation (4.11), when we replace $\tilde{u}$ by u, (4.8) by (4.1), ϵ by 0, $\tilde{f}$ by f, ν by 0, $\alpha = 1 + \epsilon + \nu$ by $\alpha = 1$, and (4.10) by (4.11).

Remark 4.6. We note here that if $\nu = -2\beta - 1 = -1$, the above derivation of Equation (4.10) becomes that of Equation (4.12), when we replace $\tilde{u}$ by u, (4.8) by (4.1), ϵ by 0, $\tilde{f}$ by f, ν by -1, $\alpha = 1 + \epsilon + \nu$ by $\alpha = 0$, and (4.10) by (4.12).

□

Remark 4.7. When $b = 0$, Equations (4.3) and (3.6), and Equation (4.5) show that

$$he'_{0,1}(t) H(t) = e^{at^2/2} H(t) = y_0(t), \quad he_{0,0}(t) H(t) = H(t), \tag{4.16}$$

respectively.

5 Solutions of Equations (3.3) and (3.4)

Lemma 5.1. *Let $\tilde{u}_\nu(t)$ be a particular solution of Equation* (4.8) *for $b=0$, that is given by Equation* (4.10) *for $b=0$. Then $\tilde{y}_\nu(t)$ and $\tilde{v}_\nu(t)$, given by $\tilde{y}_\nu(t)=\frac{d}{dt}\tilde{u}_\nu(t)$ and $\tilde{v}_\nu(t)={}_RD_t^{-1/2}\tilde{y}_\nu(t)$, are particular solutions of Equations* (3.3) *and* (3.4)*, respectively, for $\tilde{f}(t)=g_{\nu+\epsilon}(t)={}_RD_t^{-\nu}\delta_\epsilon(t)$.*

By using Lemmas 5.1 and 4.2 and Remark 4.7, we obtain the following lemmas.

Lemma 5.2. *When $\beta\in\mathbb{Z}_{>-1}$, $y_{-2\beta}(t)$ and $y_{-2\beta-1}(t)$ are given by*

$$y_{-2\beta}(t)=u'_{-2\beta}(t)=(-\beta-\frac{1}{2})_\beta\cdot y_0(t),\quad y_{-2\beta-1}(t)=u'_{-2\beta-1}(t)=0, \tag{5.1}$$

and $y_{-2\beta}(t)$ is a complementary solution of Equation (3.2)*.*

Lemma 5.3. (i) *For $\nu\in\mathbb{C}$ and $\tilde{f}(t)=g_{\nu+\epsilon}(t)={}_RD_t^{-\nu}\delta_\epsilon(t)$, $\tilde{v}_\nu(t)$, given by*

$$\tilde{v}_\nu(t)={}_RD_t^{1/2}\tilde{u}_\nu(t)=\sum_{k=0}^{\infty}\frac{(\frac{\nu+1}{2})_k(2a)^k}{\Gamma(2k+\frac{3}{2}+\nu+\epsilon)}t^{2k+1/2+\nu+\epsilon}H(t), \tag{5.2}$$

is a particular solution of Equation (3.4)*. We denote the equation which is obtaind from Equation* (5.2) *by replacing $\tilde{u}$ by u, and ϵ by 0, by Equation* (5.2-A)*, and then*

(ii) *when $\nu\in\mathbb{C}\backslash\mathbb{Z}_{<1}$ and $f(t)=g_\nu(t)=\frac{1}{\Gamma(\nu)}t^{\nu-1}H(t)$, $v_\nu(t)={}_RD_t^\epsilon\tilde{v}_\nu(t)$, satisfying Equation* (5.2-A)*, is a particular solution of Equation* (3.1)*.*

(iii) *When $\beta\in\mathbb{Z}_{>-1}$, for $\nu=-2\beta$ and $\nu=-2\beta-1$, we express $v_\nu(t)={}_RD_t^\epsilon\tilde{v}_\nu(t)$ as*

$$v_{-2\beta}(t)=\sum_{k=0}^{\infty}\frac{(-\beta+\frac{1}{2})_k(2a)^k}{\Gamma(2k+\frac{3}{2}-2\beta)}t^{2k+1/2-2\beta}H(t),\quad \beta\in\mathbb{Z}_{>-1}, \tag{5.3}$$

$$v_{-2\beta}(t)=(-\beta+\frac{1}{2})_\beta[v_0(t)+\sum_{l=1}^{\beta}\frac{(-1)^l}{(\frac{1}{2})_l\Gamma(\frac{3}{2}-2l)}t^{1/2-2l}H(t)],\quad \beta\in\mathbb{Z}_{>0}, \tag{5.4}$$

$$\begin{aligned}v_{-2\beta-1}(t)&=\sum_{k=0}^{\beta}\frac{(-\beta)_k(2a)^k}{\Gamma(2k+\frac{1}{2}-2\beta)}t^{2k-1/2-2\beta}H(t)\\ &=(-1)^\beta\beta!\sum_{l=0}^{\beta}\frac{(-1)^l}{l!\Gamma(\frac{1}{2}-2l)}t^{-1/2-2l}H(t),\quad \beta\in\mathbb{Z}_{>-1},\end{aligned} \tag{5.5}$$

which are complementary solutions of Equation (4.1)*. Here $v_0(t)$ is given by Equation* (3.9)*, which is obtained by putting $\beta=0$ in Equation* (5.3)*.*

(iv) *When $\beta\in\mathbb{Z}_{>-1}$, for $\nu=-2\beta+\frac{1}{2}$ and $\nu=-2\beta-\frac{1}{2}$, we express $v_\nu(t)={}_RD_t^\epsilon\tilde{v}_\nu(t)$ as*

$$v_{-2\beta+1/2}(t)=\sum_{k=\beta}^{\infty}\frac{(-\beta+\frac{3}{4})_k(2a)^k}{\Gamma(2k+2-2\beta)}t^{2k+1-2\beta}H(t)=(-\beta+\frac{3}{4})_\beta(2a)^\beta\cdot v_{1/2}(t), \tag{5.6}$$

$$v_{-2\beta-1/2}(t)=\sum_{k=\beta}^{\infty}\frac{(-\beta+\frac{1}{4})_k(2a)^k}{\Gamma(2k+1-2\beta)}t^{2k-2\beta}H(t)=(-\beta+\frac{1}{4})_\beta(2a)^\beta\cdot v_{-1/2}(t), \tag{5.7}$$

which are particular solutions of Equation (4.1)*, for $f(t)={}_RD_t^{2\beta-1/2}H(t)$ and $f(t)={}_RD_t^{2\beta+1/2}H(t)$, respectively. Here expressions of $v_{1/2}(t)$ and $v_{-1/2}(t)$ are obtained by putting $\beta=0$ in Equations* (5.6) *and* (5.7)*, respectively.*

6 Conclusion

In [(6)], the problem of obtaining the particular solution of an inhomogeneous ordinary differential equation with polynomial coefficients is discussed in terms of the Green's function, in the framework of distribution theory. It is applied to Kummer's and the hypergeometric differential equation.

In [(8)], a compact recipe is presented, which is applicable to the case of an inhomogeneous fractional differential equation, which is expressed by Equation (1.1). In the recipe, the particular solution is given, assuming that the inhomogeneous part satisfies one of three conditions, in the framework of nonstandard analysis. It is applied to a simple fractional and an ordinary differential equation.

In [(9)], a compact revised recipe in nonstandard analysis is presented, which is more closely related with distribution theory. In this case, the particular solution is given, assuming that the inhomogeneous part satisfies one of four conditions. In that paper, it is applied to Kummer's differential equation.

In the present paper, in solving Equation (1.1) in the framework of nonstandard analysis, we solve Equations (1.1) and (1.10) assuming Assumption 1.1. As a consequence, we adopt Condition 1.1. Only the solutions satisfying Condition 1.2 are obtained. In Sections 2 and 4, solutions of the hypergeometric and the Hermite differential equation are given. In Sections 3 and 5, solutions of a simple ordinary and a fractional differential equation, which are studied in [(8)], are given. The solutions for the cases in which Condition 1.1 is satisfied, are obtained from those for the cases in which Condition 1.2 is satisfied, with the aid of Lemma 1.3.

In [(7)], an ordinary differential equation is expressed in terms of blocks of classified terms. When the equation is expressed by two blocks of classified terms, the complementary solutions are obtained by using Frobenius' method. In Sections 2 and 4, particular and complementary solutions are derived by using Frobenius' method.

References

[1] Diener, F.; Diener, M. (1995). Tutorial, in Nonstandard Analysis in Practice. Springer, Berlin, pp. 1-21.

[2] Gelfand, I. M.; Silov, G. E. (1964). Generalized Functions. Vol. 1. Academic Press Inc., New York.

[3] Kim, M.-H.; O, H.-C. (2014). Explicit Representation of Green's Function for Linear Fractional Differential Operator with Variable Coefficients. J. Fractional Calculus and Applications, 5(1), 26–36.

[4] Magnus, M.; Oberhettinger, F. (1949). Formulas and Theorems for the Functions of Mathematical Physics. Chelsea Publ. Co., New York.

[5] Morita, T.; Sato, K. (2013). Liouville and Riemann-Liouville Fractional Derivatives via Contour Integrals. Frac. Calc. Appl. Anal. 16, 630–653.

[6] Morita, T.; Sato, K. (2017). Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Terms of the Green's Function. Mathematics. 5, 62, 1–24. https://doi.org/10.3390/math5040062

[7] Morita, T.; Sato, K. (2018). A Study on the Solution of Linear Differential Equations with Polynomial Coefficients. J. Adv. Math. Comput. Sci. 28 (3), 1–15. https://doi.org/10.9734/JAMCS/2018/43000

[8] Morita, T.; Sato, K. (2021). Solution of Inhomogeneous Fractional Differential Equations with Polynomial Coefficients in Terms of the Green's Function, in Nonstandard Analysis. Mathematics. 9, 1944, 1–24. https://doi.org/10.3390/math9161944

[9] Morita, T. (2022). Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Terms of the Green's Function, in Nonstandard Analysis. AppliedMath. 2, 379–392. https://doi.org/10.3390/appliedmath2030022

[10] Podlubny, I. (1999). Fractional Differential Equations. Academic Press, San Diego, Section 2.3.2.

[11] Schwartz, L. (1966). Théorie des Distributions. Hermann, Paris.

[12] Zemanian, A. H. (1965). Distribution Theory and Transform Analysis. Dover Publ. Inc., New York.