Original Research Article

Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Terms of the Green's Function, in Nonstandard Analysis

Type of Article

Abstract

Discussions are presented by Morita and Sato on the problem of obtaining the particular solution of an inhomogeneous differential equation with polynomial coefficients in terms of the Green's function. In a recent paper, a simple recipe based on nonstandard analysis, which is closely related with distribution theory, is presented, is applied to Kummer's differential equation. In the present paper, it is applied to the hypergeometric, the Hermite, a simple ordinary and a fractional differential equation.

Keywords: Green's function; differential equations with polynomial coefficients; nonstandard analysis; fractional differential equation; Hermite differential equation

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1 Introduction

In the present paper, we treat the problem of obtaining the particular solutions of a differential equation with polynomial coefficients in terms of the Green's function.

In a preceding paper [(7)], this problem is studied in the framework of distribution theory, where the method is applied to Kummer's and the hypergeometric differential equation. In another paper [(9)], this problem is studied in the framework of nonstandard analysis, where a recipe of solution of the present problem is presented, and it is applied to a simple fractional and a first-order ordinary differential equation. In a recent paper [(11)], we presented a compact recipe based on nonstandard analysis, which is obtained by revising the one given in [(9)]. It is applied to Kummer's differential equation.

In the present paper, we apply the recipe presented in [(11)] to the differential equations treated in [(9)] and the Hermite differential equation.

The presentation in this paper follows those in [(7; 9; 11)], in Introduction and in many descriptions in the following sections.

We consider a fractional differential equation, which takes the form:

$$p_n(t, {}_RD_t)u(t) := \sum_{l=0}^{n} a_l(t){}_RD_t^{\rho_l}u(t) = f(t), \tag{1.1}$$

where $n \in \mathbb{Z}_{>-1}$, $t \in \mathbb{R}$, $a_l(t)$ for $l \in \mathbb{Z}_{[0,n]}$ are polynomials of t, $\rho_l \in \mathbb{C}$ for $l \in \mathbb{Z}_{[0,n]}$ satisfy $\mathrm{Re}\,\rho_0 > \mathrm{Re}\,\rho_1 \geq \cdots \geq \mathrm{Re}\,\rho_n$ and $\mathrm{Re}\,\rho_0 > 0$. We use Heaviside's step function $H(t)$, which is equal to 1 if $t > 0$ and, to 0 if $t \leq 0$. Here ${}_RD_t^{\rho_l}$ are the Riemann-Liouville fractional integrals and derivatives defined by the following definition; see [(6; 12)].

Definition 1.1. Let $t \in \mathbb{R}$, $\tau \in \mathbb{R}$, $u_0(t)$ be locally integrable on $\mathbb{R}_{>\tau}$, $u(t) = u_0(t)H(t-\tau)$, $\lambda \in \mathbb{C}_+$, $n \in \mathbb{Z}_{>-1}$ and $\rho = n - \lambda$. Then ${}_RD_t^{-\lambda}u(t)$ is the Riemann-Liouville fractional integral defined by

$$\begin{aligned} {}_RD_t^{-\lambda}u(t) &= \frac{1}{\Gamma(\lambda)}\int_{-\infty}^{t}(t-x)^{\lambda-1}u_0(x)H(x-\tau)dx \\ &= \frac{1}{\Gamma(\lambda)}\int_{\tau}^{t}(t-x)^{\lambda-1}u_0(x)dx \cdot H(t-\tau), \end{aligned} \tag{1.2}$$

and ${}_RD_t^{-\lambda}u(t) = 0$ for $t \leq \tau$, where $\Gamma(\lambda)$ is the gamma function, ${}_RD_t^{\rho}u(t) = {}_RD_t^{n-\lambda}u(t)$ is the Riemann-Liouville fractional derivative defined by

$${}_RD_t^{\rho}u(t) = {}_RD_t^{n-\lambda}u(t) = \frac{d^n}{dt^n}[{}_RD_t^{-\lambda}u_0(t)] \cdot H(t-\tau), \tag{1.3}$$

when $n \geq \mathrm{Re}\,\lambda$, and ${}_RD_t^{n}u(t) = \frac{d^n}{dt^n}u_0(t) \cdot H(t-\tau)$ when $\rho = n \in \mathbb{Z}_{>-1}$.

Here $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all integers, all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$, $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$ and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{R}_{>a} = \{x \in \mathbb{R} \mid x > a\}$ for $a \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}\,z > 0\}$.

In accordance with Definition 1.1, when $u_0(t) = \frac{1}{\Gamma(\nu)}(t-\tau)^{\nu-1}$, we adopt

$${}_RD_t^{\rho}\frac{(t-\tau)^{\nu-1}}{\Gamma(\nu)}H(t-\tau) = \begin{cases} \frac{(t-\tau)^{\nu-\rho-1}}{\Gamma(\nu-\rho)}H(t-\tau), & \nu-\rho \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \\ 0, & \nu-\rho \in \mathbb{Z}_{<1}, \end{cases} \tag{1.4}$$

for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $\tau \in \mathbb{R}$. Here ${}_RD_t$ is used in place of usually used notation ${}_\tau D_R$, in order to show that the variable is t.

Remark 1.1. Let $g_\nu(t) := \frac{1}{\Gamma(\nu)}t^{\nu-1}H(t)$ for $\nu \in \mathbb{C}$. Then $g_\nu(t) = 0$ if $\nu \in \mathbb{Z}_{<1}$, and Equation (1.4) shows that if $\nu \notin \mathbb{Z}_{<1}$, ${}_RD_t^{\rho}g_\nu(t) = g_{\nu-\rho}(t)$. As a consequence, we have ${}_RD_t^{\nu+n}g_\nu(t) = g_{-n}(t) = 0$ for $n \in \mathbb{Z}_{>-1}$.

Remark 1.2. Let $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$, $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $g_\nu(t) := \frac{1}{\Gamma(\nu)}t^{\nu-1}H(t)$. Then the index law: ${}_RD_t^{\rho_1}{}_RD_t^{\rho_2}g_\nu(t) = {}_RD_t^{\rho_2}{}_RD_t^{\rho_1}g_\nu(t)$ does not always hold. An example is given in the book [(5, p. 108)]; see also [(10, p. 48)].

In distribution theory [(7; 13; 2; 14)], we use distribution $\tilde{H}(t)$, which corresponds to function $H(t)$, differential operator D and distribution $\delta(t) = D\tilde{H}(t)$, which is called Dirac's delta function.

1.1 Preliminaries on Nonstandard Analysis

In the present paper, we use nonstandard analysis [(1)], where infinitesimal numbers are used. We denote the set of all infinitesimal real numbers by $\mathbb{R}^0$. We also use $\mathbb{R}^0_{>0} = \{\epsilon \in \mathbb{R}^0 \mid \epsilon > 0\}$, which is such that if $\epsilon \in \mathbb{R}^0_{>0}$ and $N \in \mathbb{Z}_{>0}$, then $\epsilon < \frac{1}{N}$. We use $\mathbb{R}^{ns}$, which has subsets $\mathbb{R}$ and $\mathbb{R}^0$. If $x \in \mathbb{R}^{ns}$ and $x \notin \mathbb{R}$, x is expressed as $x_1 + \epsilon$ by $x_1 \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^0$, where x_1 may be $0 \in \mathbb{R}$. Equation $x \simeq y$

for $x \in \mathbb{R}^{ns}$ and $y \in \mathbb{R}^{ns}$, is used, when $x - y \in \mathbb{R}^0$. We denote the set of all infinitesimal complex numbers by $\mathbb{C}^0$, which is the set of complex numbers z which satisfy $|\mathrm{Re}\, z| + |\mathrm{Im}\, z| \in \mathbb{R}^0$. We use $\mathbb{C}^{ns}$, which has subsets $\mathbb{C}$ and $\mathbb{C}^0$. If $z \in \mathbb{C}^{ns}$ and $z \notin \mathbb{C}$, z is expressed as $z_1 + \epsilon$ by $z_1 \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^0$, where z_1 may be $0 \in \mathbb{C}$.

Remark 1.3. In nonstandard analysis [(1)], in addition to infinitesimal numbers, we use unlimited numbers, which are often called infinite numbers. In the present paper, we do not use them, but if we use them, we have to consider sets $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$ such that if $\omega \in \mathbb{R}^\infty$, there exists $\epsilon \in \mathbb{R}^0$ satisfying $\omega = \frac{1}{\epsilon}$, and if $\omega \in \mathbb{C}^\infty$, there exists $\epsilon \in \mathbb{C}^0$ satisfying $\omega = \frac{1}{\epsilon}$, and then $\mathbb{R}^{ns} = \mathbb{R} \cup \mathbb{R}^0 \cup \mathbb{R}^\infty$ and $\mathbb{C}^{ns} = \mathbb{C} \cup \mathbb{C}^0 \cup \mathbb{C}^\infty$.

In place of (1.4), we now use

$$ {}_R D_t^\rho \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu-1+\epsilon} H(t) = \frac{1}{\Gamma(\nu-\rho+\epsilon)} t^{\nu-\rho-1+\epsilon} H(t), \tag{1.5} $$

for all $\rho \in \mathbb{C}$ and $\nu \in \mathbb{C}$, where $\epsilon \in \mathbb{R}^0_{>0}$.

Lemma 1.1. *Let $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$, $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{R}^0_{>0}$ and $g_{\nu+\epsilon}(t) := \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu+\epsilon-1} H(t)$. Then the index law:*

$$ {}_R D_t^{\rho_1} \, {}_R D_t^{\rho_2} g_{\nu+\epsilon}(t) = {}_R D_t^{\rho_1+\rho_2} g_{\nu+\epsilon}(t) = g_{\nu-\rho_1-\rho_2+\epsilon}(t), \tag{1.6} $$

always holds.

Remark 1.4. When $\epsilon \in \mathbb{R}^0$ or $\epsilon \in \mathbb{C}^0$, we often ignore terms of $O(\epsilon)$ compared with a term of $O(\epsilon^0)$. For instance, when $\nu \in \mathbb{R}_{>0}$ and $\nu - \rho \in \mathbb{R}_{>0}$, we adopt $\frac{1}{\Gamma(\nu+\epsilon)} t^{\nu-1+\epsilon} H(t) \simeq \frac{1}{\Gamma(\nu)} t^{\nu-1+\epsilon} H(t)$, and also

$$ {}_R D_t^\rho \frac{1}{\Gamma(\nu)} t^{\nu-1+\epsilon} H(t) \simeq \frac{1}{\Gamma(\nu-\rho)} t^{\nu-\rho-1+\epsilon} H(t), \tag{1.7} $$

in place of (1.5). In the following, we often use "$=$" in place of "$\simeq$".

In the present study in nonstandard analysis, $\epsilon \in \mathbb{R}^0_{>0}$ is used, and $H(t)$ and $\delta(t) = D\tilde{H}(t)$, respectively, are replaced by

$$ H_\epsilon(t) := {}_R D_t^{-\epsilon} H(t) = \frac{1}{\Gamma(\epsilon+1)} t^\epsilon H(t) \simeq t^\epsilon H(t), \tag{1.8} $$

which tends to $H(t)$ in the limit $\epsilon \to 0$, and by

$$ \delta_\epsilon(t) := \frac{d}{dt} H_\epsilon(t) = \frac{d}{dt} \frac{1}{\Gamma(\epsilon+1)} t^\epsilon H(t) = \frac{1}{\Gamma(\epsilon)} t^{\epsilon-1} H(t) \simeq \epsilon t^{\epsilon-1} H(t). \tag{1.9} $$

Lemma 1.2. *In the notation in Remark 1.1, $H_\epsilon(t) = g_{1+\epsilon}(t)$, $\delta_\epsilon(t) = g_\epsilon(t)$, and we have*

$$ {}_R D_t^\epsilon H_\epsilon(t) = {}_R D_t^\epsilon g_{1+\epsilon}(t) = g_1(t) = H(t), \quad {}_R D_t^\epsilon \delta_\epsilon(t) = {}_R D_t^\epsilon g_\epsilon(t) = g_0(t) = 0. \tag{1.10} $$

Lemma 1.3. *Let $\epsilon \in \mathbb{R}^0_{>0}$, $\tau \in \mathbb{R}$, and $f(t)$ be locally integrable on $\mathbb{R}_{>\tau}$. Then*

$$ \int_{-\infty}^{\infty} \delta_\epsilon(t-x) f(x) H(x-\tau) dx = {}_R D_t^{-\epsilon} [f(t) H(t-\tau)]. \tag{1.11} $$

Proof. Since $\delta_\epsilon(t-x) = {}_R D_t^{-\epsilon+1} H(t-x)$, we have

$$ \int_{-\infty}^{\infty} \delta_\epsilon(t-x) f(x) H(x-\tau) dx = {}_R D_t^{-\epsilon+1} \int_{-\infty}^{\infty} H(t-x) f(x) H(x-\tau) dx $$
$$ = {}_R D_t^{-\epsilon+1} \int_\tau^t f(x) H(x-\tau) dx = {}_R D_t^{-\epsilon} [f(t) H(t-\tau)]. \tag{1.12} $$

□

1.2 Summary of the Following Sections

In Section 2, a recipe of solution of Equation (1.1), in nonstandard analysis, is presented. We there consider the solution of the following equation for $\tilde{u}(t)$:

$$\tilde{p}_{n,\epsilon}(t, {}_RD_t)\tilde{u}(t) = \tilde{f}(t), \tag{1.13}$$

where $\epsilon \in \mathbb{R}^0_{>0}$ and

$$\tilde{p}_{n,\epsilon}(t, {}_RD_t) := {}_RD_t^{-\epsilon} p_n(t, {}_RD_t) {}_RD_t^{\epsilon}. \tag{1.14}$$

Here the inhomogeneous terms $f(t)$ and $\tilde{f}(t)$ are assumed to satisfy one of the following four conditions.

Condition 1.1. Let $\epsilon \in \mathbb{R}^0_{>0}$ and $\beta \in \mathbb{C}$.

(i) $f(t) = f_{0,0}(t)H(t)$ and $\tilde{f}(t) = {}_RD_t^{-\epsilon} f(t) + c_\epsilon \delta_\epsilon(t)$, where $f_{0,0}(t)$ is locally integrable on $\mathbb{R}_{>0}$ and c_ϵ is a constant.

(ii) $f(t) = {}_RD_t^{\beta} f_\beta(t)$ and $\tilde{f}(t) = {}_RD_t^{\beta} \tilde{f}_\beta(t)$, where

$$\tilde{f}_\beta(t) = {}_RD_t^{-\epsilon} f_\beta(t) + c_{\beta,\epsilon}\delta_\epsilon(t), \quad f_\beta(t) = f_{\beta,0}(t)H(t), \tag{1.15}$$

$f_{\beta,0}(t)$ is locally integrable on $\mathbb{R}_{>0}$, and $c_{\beta,\epsilon}$ is a constant.

(iii) $\tilde{f}(t) = {}_RD_t^{\beta} \tilde{f}_\beta(t)$, where $\tilde{f}_\beta(t) = {}_RD_t H_\epsilon(t) = \delta_\epsilon(t)$. When $\beta \in \mathbb{Z}_{>-1}$, $f(t) = 0$, and when $\beta \notin \mathbb{Z}_{>-1}$, $f(t) = {}_RD_t^{\beta+1} H(t)$.

(iv) $\tilde{f}(t)$ and $f(t)$ are expressed as follows:

$$\tilde{f}(t) = \sum_{l=1}^{\infty} c_l \cdot {}_RD_t^{\beta_l} \delta_\epsilon(t) = \sum_{l=1}^{\infty} c_l \cdot \frac{t^{\epsilon-1-\beta_l}}{\Gamma(\epsilon-\beta_l)} H(t), \quad f(t) = \sum_{l=1}^{\infty} d_l \cdot {}_RD_t^{\beta_l+1} H(t), \tag{1.16}$$

respectively, where $c_l \in \mathbb{C}$ are constants, $\beta_l \in \mathbb{C}$ satisfy $-\mathrm{Re}\,\beta_l \geq -\mathrm{Re}\,\beta_1 \in \mathbb{R}$, for all $l \in \mathbb{Z}_{>0}$, and $d_l = c_l$ if $\beta_l \notin \mathbb{Z}_{>-1}$, and $d_l = 0$ if $\beta_l \in \mathbb{Z}_{>-1}$.

Remark 1.5. When Condition 1.1(ii) is satisfied for $\beta = 0$, Condition 1.1(i), in which $f_0(t) = f(t)$, $\tilde{f}_0(t) = \tilde{f}(t)$ and $c_\epsilon = c_{0,\epsilon}$, is satisfied, and vice versa.

Remark 1.6. Lemma 1.2 shows that when Condition 1.1(i) is satisfied ${}_RD_t^{\epsilon}\tilde{f}(t) = f(t)$, and $\tilde{f}(t) = {}_RD_t^{-\epsilon} f(t)$ does not always hold, and when Condition 1.1(iii) is satisfied, ${}_RD_t^{\epsilon}\tilde{f}_\beta(t) = 0$.

In [(11)], full expressions of the Green's functions and the solutions, are derived along the recipe given in Section 2, for Kummer's differential equation. In Section 3, they are applied to the hypergeometric differential equation:

$$p_H(t, {}_RD_t)u(t) := [t(1-t)\frac{d^2}{dt^2} + (c-(a+b+1)t)\frac{d}{dt} - ab]u(t) = f(t), \tag{1.17}$$

where a, b and c are constants satifying $a \neq 0$ and $b \neq 0$.

In Sections 4, 5.1.1 and 5.2.1, solutions of a simple ordinary and a fractional differential equation, which are studied in [(9)], are given for the case, when Conditions 1.1(i)∼(iii) are satisfied. The fractional differential equation is the one presented in [(3)]. In Section 5 excluding Sections 5.1.1 and 5.2.1, solutions of the Hermite differential equation are given for the cases when Conditions 1.1(i)∼(iii) are satisfied.

The solutions for the cases when Condition 1.1(iv) are easily obtained from the results for the cases for Condition 1.1(iii), and hence are not given.

Section 6 is for Conclusion.

2 Recipe of Solution of Differential Equation, in Nonstandard Analysis

2.1 Solution of Equation (1.1) when Condition 1.1(i) is satisfied

Definition 2.1. Let $\tilde{p}_{n,\epsilon}(t, {}_RD_t)$ be given by Equation (1.14). Then for $\epsilon \in \mathbb{R}^0_{>0}$ and $\tau \in \mathbb{R}$, the Green's function $G_\epsilon(t,\tau)$ for Equation (1.1) satisfies

$$\tilde{p}_{n,\epsilon}(t, {}_RD_t)G_\epsilon(t,\tau) = \delta_\epsilon(t-\tau). \tag{2.1}$$

Lemma 2.1. *Let $G_\epsilon(t,\tau)$ be defined as in Definition 2.1, and $G_0(t,\tau) := {}_RD_t^\epsilon G_\epsilon(t,\tau)$. Then $G_0(t,\tau)$ is a complementary solution of Equation* (1.1) *on $\mathbb{R}_{>\tau}$, and ${}_RD_t^{-1}p_n(t, {}_RD_t)G_0(t,\tau) = 1$ at any value of t satisfying $t > \tau$.*

Proof. These are confirmed by applying ${}_RD_t^\epsilon$ and ${}_RD_t^{-1+\epsilon}$ to Equation (2.1), by noting Lemma 1.2. □

Lemma 2.2. *Let $\tilde{u}_c(t)$ be a complementary solution of Equation* (1.13) *on $\mathbb{R}_{>0}$, and $u_c(t) := {}_RD_t^\epsilon \tilde{u}_c(t)$. Then $u_c(t)$ is a complementary solution of Equation* (1.1) *on $\mathbb{R}_{>0}$.*

Proof. This is confirmed by replacing $\tilde{u}(t)$ and $\tilde{f}(t)$ by $\tilde{u}_c(t)$ and 0 in Equation (1.13), and then applying ${}_RD_t^\epsilon$ to the equation. □

Theorem 2.1. Let Condition 1.1(i) be satisfied, $G_\epsilon(t,\tau)$ and $G_0(t,\tau)$ be given as in Lemma 2.1. Then $\tilde{u}_f(t)$ given by

$$\tilde{u}_f(t) = \int_{-\infty}^{\infty} G_\epsilon(t,\tau)f(\tau)d\tau + c_\epsilon G_\epsilon(t,0), \tag{2.2}$$

is the particular solution of Equation (1.13) for the term $\tilde{f}(t)$, and $u_f(t)$ given by

$$u_f(t) = {}_RD_t^\epsilon \tilde{u}_f(t) = \int_{-\infty}^{\infty} G_0(t,\tau)f(\tau)d\tau + c_\epsilon G_0(t,0), \tag{2.3}$$

consists of the particular solution for the term $f(t)$ and a complementary solution of Equation (1.1).

Proof. By using Equations (2.3), (2.1) and (1.11), we obtain

$$\begin{aligned}\tilde{p}_{n,\epsilon}(t, {}_RD_t)\tilde{u}_f(t) &= \tilde{p}_{n,\epsilon}(t, {}_RD_t)[\int_{-\infty}^{\infty} G_\epsilon(t,\tau)f(\tau)d\tau + c_\epsilon G_\epsilon(t,0)] \\ &= \int_{-\infty}^{\infty} \delta_\epsilon(t-\tau)f(\tau)d\tau + c_\epsilon\delta_\epsilon(t) = {}_RD_t^{-\epsilon}f(t) + c_\epsilon\delta_\epsilon(t) = \tilde{f}(t),\end{aligned} \tag{2.4}$$

which is a proof for $\tilde{u}_f(t)$. □

2.2 Solution of Equation (1.13) when Condition 1.1(ii) or (iii) or (iv) is satisfied

When Condition 1.1(ii) is satisfied, we introduce the transformed differential equations for $w(t) = {}_RD_t^{-\beta}u(t)$ and $\tilde{w}(t) = {}_RD_t^{-\epsilon}w(t)$ from Equations (1.1) and (1.13), respectively, by

$$\tilde{p}_{n,\beta}(t, {}_RD_t)w(t) = f_\beta(t), \tag{2.5}$$

$$\tilde{p}_{n,\beta+\epsilon}(t, {}_RD_t)\tilde{w}(t) = \tilde{f}_\beta(t), \tag{2.6}$$

where

$$\tilde{p}_{n,\beta}(t, {}_RD_t) := {}_RD_t^{-\beta} p_n(t, {}_RD_t) {}_RD_t^{\beta}, \tag{2.7}$$

$$\tilde{p}_{n,\beta+\epsilon}(t, {}_RD_t) := {}_RD_t^{-\beta-\epsilon} p_n(t, {}_RD_t) {}_RD_t^{\beta+\epsilon}. \tag{2.8}$$

Lemma 2.3. *Let Equation* (2.6) *and* $\tilde{f}(t) = {}_RD_t^{\beta} \tilde{f}_\beta(t)$ *hold. Then by using* (2.8)*, we confirm that Equation* (1.13) *for* $\tilde{u}(t) = {}_RD_t^{\beta} \tilde{w}(t)$ *holds.*

Remark 2.1*.* Let $\tilde{u}_c(t)$ and $\tilde{w}_c(t)$ be complementary solutions of Equation (1.13) and (2.6), respectively, on $\mathbb{R}_{>0}$. Then by using (2.8), we confirm that they are related by $\tilde{u}_c(t) = {}_RD_t^{\beta} \tilde{w}_c(t)$.

Definition 2.2. For $\epsilon \in \mathbb{R}_{>0}^0$ and $\tau \in \mathbb{R}$, the Green's function $G_{\beta,\epsilon}(t,\tau)$ for Equation (2.5) satisfies

$$\tilde{p}_{n,\beta+\epsilon}(t, {}_RD_t) G_{\beta,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau). \tag{2.9}$$

Lemma 2.4. *Let* $G_{\beta,\epsilon}(t,\tau)$ *be defined as in Definition 2.2, and* $G_{\beta,0}(t,\tau) := {}_RD_t^{\epsilon} G_{\beta,\epsilon}(t,\tau)$*. Then* $G_{\beta,0}(t,\tau)$ *is a complementary solution of Equation* (2.5) *on* $\mathbb{R}_{>\tau}$*.*

Proof. A proof of this lemma is obtained from that of Lemma 2.1, by replacing (2.1) by (2.9), $\tilde{p}_{n,\epsilon}$ by $\tilde{p}_{n,\beta+\epsilon}$, G_ϵ by $G_{\beta,\epsilon}$, p_n by $\tilde{p}_{n,\beta}$, G_0 by $G_{\beta,0}$, and (1.1) by (2.5). □

Theorem 2.2*.* Let Condition 1.1(ii) be satisfied, and $G_{\beta,\epsilon}(t,\tau)$ satisfy Equation (2.9). Then $\tilde{w}_f(t)$ and $\tilde{u}_f(t)$ given by

$$\tilde{w}_f(t) := \int_{-\infty}^{\infty} G_{\beta,\epsilon}(t,\tau) f_\beta(\tau) d\tau + c_{\beta.\epsilon} G_{\beta,\epsilon}(t,0), \quad \tilde{u}_f(t) := {}_RD_t^{\beta} \tilde{w}_f(t), \tag{2.10}$$

are particular solutions of Equations (2.6) and (1.13), respectively.

Proof. Theorem 2.1 states that when $\tilde{f}(t)$ satisfies Condition 1.1(i) and $G_\epsilon(t,\tau)$ satisfies (2.6), the solution $\tilde{u}_f(t)$ of (1.13) is expressed as (2.2). This shows that when $\tilde{f}_\beta(t)$ satisfies Condition 1.1(ii) and $G_{\beta,\epsilon}(t,\tau)$ satisfies (2.9), the solution $\tilde{w}_f(t)$ of (2.6) is given by the first equation in (2.10). The second equation in it is due to Lemma 2.3. □

When Condition 1.1(iii) is satisfied, Equation (1.13) is expressed as

$$\tilde{p}_{n,\epsilon}(t, {}_RD_t) \tilde{u}(t) = {}_RD_t^{\beta} \delta_\epsilon(t) = \frac{1}{\Gamma(\epsilon-\beta)} t^{\epsilon-\beta-1} H(t). \tag{2.11}$$

Since Condition 1.1(iii) is a special case of Condition 1.1(ii) in which $f_\beta(t) = 0$ and $c_{\beta,\epsilon} = 1$, we obtain the following theorem from Theorem 2.2.

Theorem 2.3*.* Let Condition 1.1(iii) be satisfied, and $G_{\beta,\epsilon}(t,0)$ satisfy Equation (2.9) for $\tau = 0$. Then $\tilde{w}_f(t)$ and $\tilde{u}_f(t)$ given by

$$\tilde{w}_f(t) = G_{\beta,\epsilon}(t,0), \quad \tilde{u}_f(t) = {}_RD_t^{\beta} G_{\beta,\epsilon}(t,0), \tag{2.12}$$

are particular solutions of Equations (2.6) and (1.13), respectively.

Corollary 2.5. *When* $\beta = m \in \mathbb{Z}_{>0}$*,* $\tilde{u}_f(t) = {}_RD_t^{m} G_{m,\epsilon}(t,0)$ *is a particular solution of* (1.13)*, and then* $u_f(t) = {}_RD_t^{-\epsilon} \tilde{u}_f(t) = {}_RD_t^{m} G_{m,\epsilon}(t,0)$ *is a complementary solution of Equation* (1.1)*.*

Proof. In this case, $u_f(t) = {}_RD_t^{m} G_{m,0}(t,0)$ is a solution of (1.1) for $f(t) = D^{m-\epsilon}\delta_\epsilon(t)$, which is 0 by Lemma 1.2. □

Theorem 2.3 shows that if $\tilde{f}(t) = {}_RD_t^{\beta} \delta_\epsilon(t)$, the particular solution of (1.13) is given by $\tilde{u}_f(t) = {}_RD_t^{\beta} G_{\beta,\epsilon}(t,0)$. As a consequence, we have

Theorem 2.4*.* Let $\tilde{f}(t)$ satisfy Condition 1.1(iv), so that it is given by Equation (1.16). Then the particular solution of Equation (1.13) is given by

$$\tilde{u}_f(t) = \sum_{l=1}^{\infty} c_l \cdot {}_RD_t^{\beta_l} G_{\beta_l,\epsilon}(t,0). \tag{2.13}$$

3 Solution of the Hypergeometric Differential Equation

We construct the transformed differential equation of Equation (1.17), which corresponds to Equation (1.13). For this purpose, we use the following lemma.

Lemma 3.1. *Let* $\lambda \in \mathbb{C}_+$*,* $m \in \mathbb{Z}_{>-1}$ *and* $\rho = m - \lambda$*. Then*

$$ {}_RD_t^\rho[tu(t)] = t \cdot {}_RD_t^\rho u(t) + \rho \cdot {}_RD_t^{\rho-1} u(t), \tag{3.1}$$
$$ {}_RD_t^\rho[t^2 u(t)] = t^2 \cdot {}_RD_t^\rho u(t) + 2\rho t \cdot {}_RD_t^{\rho-1} u(t) + \rho(\rho-1) \cdot {}_RD_t^{\rho-2} u(t). \tag{3.2}$$

Proof. When $\rho = -\lambda$, by using (1.2), we have

$$ t \cdot {}_RD_t^{-\lambda} u(t) - {}_RD_t^{-\lambda}[tu(t)] = \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{t} (t-x)^\lambda u(x) dx = \lambda \cdot {}_RD_t^{-\lambda-1} u(t), \tag{3.3}$$

which gives (3.1) for $\rho = -\lambda$. We then prove this equation for $\rho \in \mathbb{Z}_{>0}$ by mathematical induction. By using Equation (3.1), we confirm Equation (3.2) as follows:

$$ \begin{aligned} &{}_RD_t^\rho[t^2 u(t)] = t \cdot {}_RD_t^\rho[tu(t)] + \rho \cdot {}_RD_t^{\rho-1}[tu(t)] \\ &= t[t \cdot {}_RD_t^\rho u(t) + \rho \cdot {}_RD_t^{\rho-1} u(t)] + \rho[t \cdot {}_RD_t^{\rho-1} u(t) + (\rho-1) \cdot {}_RD_t^{\rho-2} u(t)], \end{aligned} \tag{3.4}$$

which gives (3.2). □

With the aid of formulas (3.1) and (3.2) for $\rho = -\epsilon$, we construct the following transformation of Equation (1.17) for $\tilde{u}(t) = {}_RD_t^{-\epsilon} u(t)$, which corresponds to Equation (1.13):

$$ \begin{aligned} &\tilde{p}_{H,\epsilon}(t, {}_RD_t)\tilde{u}(t) := {}_RD_t^{-\epsilon} p_H(t, {}_RD_t) {}_RD_t^\epsilon \tilde{u}(t) \\ &= {}_RD_t^{-\epsilon}[t(1-t)\frac{d^2}{dt^2} + (c - (a+b+1)t)\frac{d}{dt} - ab] {}_RD_t^\epsilon \tilde{u}(t) \\ &= [t(1-t)\frac{d^2}{dt^2} + (c - \epsilon - (a+b+1-2\epsilon)t)\frac{d}{dt} - (a-\epsilon)(b-\epsilon)\tilde{u}(t) = \tilde{f}(t). \end{aligned} \tag{3.5}$$

In accordance with Definition 2.1, we define the Green's function $G_{H,\epsilon}(t,\tau)$, which satisfies

$$ \tilde{p}_{H,\epsilon}(t, {}_RD_t) G_{H,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau), \tag{3.6}$$

for $\tau \in \mathbb{R}$. The solutions of Equations (3.5) and (1.17) are then given with the aid of Theorem 2.1 and the following lemma.

Lemma 3.2. *Let* $c \notin \mathbb{Z}_{<1}$*. Then there exist two complementary solutions of Equation* (1.17)*, which are given by*

$$ H_1(t) = {}_2F_1(a,b;c;t) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} t^k, \quad t > 0, \tag{3.7}$$

$$ H_2(t) = \frac{1}{\Gamma(2-c)} t^{1-c} \cdot {}_2F_1(1+a-c, 1+b-c; 2-c; t), \quad t > 0. \tag{3.8}$$

In the present paper, this is proved in Lemmas 3.3 and 3.4 given below.

Lemma 3.3. *Let* $H_1(t)$ *be given by Equation* (3.7)*. Then* $G_{H,\epsilon}(t,0)$ *and* $G_{H,0}(t,0)$*, given by*

$$ G_{H,\epsilon}(t,0) = \frac{1}{-1+c} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \Gamma(k+\epsilon+1)} t^{k+\epsilon} H(t), \tag{3.9}$$

$$ G_{H,0}(t,0) = {}_RD_t^\epsilon G_{H,\epsilon}(t,0) = \frac{1}{-1+c} H_1(t) H(t), \tag{3.10}$$

are a particular solution of Equation (3.6) *for* $\tau = 0$*, and a complementary solution of Equation* (1.17)*, respectively.*

A proof of this lemma is given in Sections 3.0.1 and 3.0.2, and also the statement for $G_{H,0}(t,0)$ is due to Lemma 2.1.

Lemma 3.4. *Let $H_2(t)$ be given by Equation* (3.8)*. Then $\tilde{u}_c(t)$ and $u_c(t)$, given by*

$$\tilde{u}_c(t) = \sum_{k=0}^{\infty} \frac{(a-c+1)_k (b-c+1)_k}{k!\Gamma(2-c+k+\epsilon)} t^{1-c+\epsilon+k} H(t), \tag{3.11}$$

$$u_c(t) = {}_R D_t^{\epsilon} \tilde{u}_c(t) = H_2(t) H(t), \tag{3.12}$$

are complementary solutions of Equations (3.5) *and* (1.17)*, respectively.*

A proof of this lemma is given in Sections 3.0.1 and 3.0.2, and also Equation (3.12) is due to Lemma 2.2.

3.0.1 A Particular and a Complementary Solution of Equation (3.5), by Frobenius' Method, I

Lemmas 3.3 and 3.4 state that $\tilde{u}(t) = G_{H,\epsilon}(t,0)$ given by Equation (3.9), and $\tilde{u}(t) = \tilde{u}_c(t)$ given by Equation (3.11), are a particular and a complementary solution, respectively, of Equation (3.5). We derive them by using Frobenius' method.

We assume that $\tilde{u}(t)$ is expressed by

$$\tilde{u}(t) = \sum_{k=0}^{\infty} p_k \frac{1}{\Gamma(\alpha+k+1)} t^{\alpha+k} H(t), \tag{3.13}$$

where α and p_k are constants, and $p_0 \neq 0$. By using this in Equation (3.5), we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} p_k [(\alpha+k-1+c-\epsilon) \frac{t^{\alpha+k-1}}{\Gamma(\alpha+k)} - [(\alpha+k-1)(\alpha+k) \\
&\quad +(a-\epsilon+b-\epsilon+1)(\alpha+k) + (a-\epsilon)(b-\epsilon)] \frac{t^{\alpha+k}}{\Gamma(\alpha+k+1)}] H(t) \\
&= p_0(\alpha-1+c-\epsilon) \frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t) + \sum_{k=1}^{\infty} [p_k(\alpha+k-1+c-\epsilon) \\
&\quad -p_{k-1}(\alpha+k-1+a-\epsilon)(\alpha+k-1+b-\epsilon)] \frac{t^{\alpha+k-1}}{\Gamma(\alpha+k)} H(t) = \tilde{f}(t).
\end{aligned} \tag{3.14}$$

When $\tilde{f}(t) = \delta_\epsilon(t) = \frac{t^{\epsilon-1}}{\Gamma(\epsilon)} H(t)$, Equation (3.14) is satisfied, if $\alpha = \epsilon$, $p_0 = \frac{1}{-1+c}$ and

$$p_k = p_{k-1} \frac{(k-1+a)(k-1+b)}{k-1+c} = p_0 \frac{(a)_k (b)_k}{(c)_k}, \quad k \in \mathbb{Z}_{>0}. \tag{3.15}$$

By using these α, p_0 and p_k in Equation (3.13), and putting $\tilde{u}(t) = G_{H,\epsilon}(t,0)$, we obtain Equation (3.9).

When $\tilde{f}(t) = 0$, Equation (3.14) is satisfied, if $\alpha = 1-c+\epsilon$, $p_0 \neq 0$ and

$$p_k = p_{k-1} \frac{(k+a-c)(k+b-c)}{k} = p_0 \frac{(a-c+1)_k (b-c+1)_k}{k!}, \quad k \in \mathbb{Z}_{>0}. \tag{3.16}$$

By using these α, p_0 and p_k in Equation (3.13), and putting $\tilde{u}(t) = p_0 \tilde{u}_c(t)$, we obtain Equation (3.11).

3.0.2 Complementary Solutions of Equation (1.17), by Frobenius' Method

We now give the derivations of Equations (3.10) and (3.12), by Frobenius' Method.

We note that the statements in Section 3.0.1 hold, even when we replace "particular" by "complementary", ϵ by 0, $\tilde{u}(t)$ by $u(t)$, $\tilde{u}_c(t)$ by $u_c(t)$, $\tilde{f}(t)$ by $f(t)$, (3.5) by (1.17), (3.9) by (3.10), and (3.11) by (3.12).

3.1 Solutions of Equations (3.5) and (1.17) Satisfying Condition 1.1(iii)

We construct the transformed differential equations of Equation (1.17), which appear in Theorems 2.2, 2.3 and 2.4. With the aid of formulas (3.1) and (3.2), we have the following equations for $w(t) = {}_RD_t^{-\beta}u(t)$ and $\tilde{w}(t) = {}_RD_t^{-\epsilon}w(t)$ from Equation (1.17) satisfying Condition 1.1(ii), as follows:

$$\tilde{p}_{H,\beta}(t, {}_RD_t)w(t) := {}_RD_t^{-\beta}p_H(t, {}_RD_t){}_RD_t^{\beta}w(t) = [t(1-t)\frac{d^2}{dt^2}$$
$$+(c-\beta-(a+b+1-2\beta)t)\frac{d}{dt} - (a-\beta)(b-\beta)]w(t) = f_\beta(t), \tag{3.17}$$
$$\tilde{p}_{H,\beta+\epsilon}(t, {}_RD_t)\tilde{w}(t) := {}_RD_t^{-\beta-\epsilon}p_H(t, {}_RD_t){}_RD_t^{\beta+\epsilon}\tilde{w}(t) = [t(1-t)\frac{d^2}{dt^2}$$
$$+(c-\beta-\epsilon-(a+b+1-2\beta-2\epsilon)t)\frac{d}{dt} - (a-\beta-\epsilon)(b-\beta-\epsilon)]\tilde{w}(t) = \tilde{f}_\beta(t). \tag{3.18}$$

Remark 3.1. In this section, we consider Equations (3.17) and (3.18) in place of Equations (1.17) and (3.5), respectively, and hence the equations in this section are obtained from the corresponding equations in Section 3, by replacing a by $a-\beta$, b by $b-\beta$, c by $c-\beta$, f by f_β, $\tilde{f}$ by $\tilde{f}_\beta$, u by w, and $\tilde{u}$ by $\tilde{w}$. They will be given without derivation.

Lemma 3.5. *Let $c-\beta \notin \mathbb{Z}_{<1}$. Then Lemma 3.2 and Remark 3.1 show that there exist two complementary solutions of Equation* (3.17)*, which are given by*

$$H_{\beta,1}(t) = {}_2F_1(a-\beta, b-\beta; c-\beta; t) = \sum_{k=0}^{\infty}\frac{(a-\beta)_k(b-\beta)_k}{k!(c-\beta)_k}t^k, \quad t>0, \tag{3.19}$$
$$H_{\beta,2}(t) = \frac{1}{\Gamma(2-c+\beta)}t^{1-c+\beta}\cdot {}_2F_1(1+a-c, 1+b-c; 2-c+\beta; t)$$
$$= {}_RD_t^{-\beta}H_2(t)H(t), \quad t>0. \tag{3.20}$$

In accordance with Definition 2.2, we define the Green's function $G_{H,\beta,\epsilon}(t,\tau)$, which satisfies

$$\tilde{p}_{H,\beta+\epsilon}(t, {}_RD_t)G_{H,\beta,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau), \tag{3.21}$$

for $\tau \in \mathbb{R}$. The solutions of Equations (3.18), (3.17), (3.5) and (1.17) are then given with the aid of Theorems 2.2, 2.3 and 2.4 and Lemma 3.5.

Remark 3.2. Equation (3.21) is obtained from Equation (3.6), by replacing c by $c-\beta$, a by $a-\beta$, b by $b-\beta$, and $G_{H,\epsilon}$ by $G_{H,\beta,\epsilon}$.

In Section 3.1, formulas are derived with the aid of two complementary solutions given by Equations (3.19) and (3.20), and hence they hold when $c-\beta \notin \mathbb{Z}_{<1}$.

Lemma 3.6. *Let $H_{\beta,1}(t)$ be given by Equation* (3.19)*. Then Lemma 3.3, Remark 3.1 and Lemmas 3.5 and 2.1 show that $G_{H,\beta,\epsilon}(t,0)$ and $G_{H,\beta,0}(t,0)$, given by*

$$G_{H,\beta,\epsilon}(t,0) = {}_RD_t^{-\epsilon}G_{H,\beta,0}(t,0), \quad G_{H,\beta,0}(t,0) = \frac{1}{-1+c-\beta}H_{\beta,1}(t)H(t), \tag{3.22}$$

are a particular solution of Equation (3.21) *for $\tau = 0$, and a complementary solution of Equation* (3.17)*, respectively.*

Theorem 3.1. Let Condition 1.1(iii) be satisfied, and $G_{H,\beta,\epsilon}(t,0)$ be given by Equation (3.22). Then Theorem 2.3 shows that $\tilde{w}_f(t)$ and $\tilde{u}_f(t)$, given by $\tilde{w}_f(t) = G_{H,\beta,\epsilon}(t,0)$ and by

$$\tilde{u}_f(t) = {}_R D_t^{\beta} \tilde{w}_f(t) = \frac{1}{-1+c-\beta} \sum_{k=0}^{\infty} \frac{(a-\beta)_k (b-\beta)_k}{(c-\beta)_k \Gamma(k-\beta+1+\epsilon)} t^{k-\beta+\epsilon} H(t), \tag{3.23}$$

are the particular solutions of Equations (3.18) and (3.5), respectively.

Corollary 3.7. *Let $\beta = n \in \mathbb{Z}_{>-1}$, $G_{H,\epsilon}(t,0)$ and $G_{H,0}(t,0)$ be given by* (3.9) *and* (3.10)*, respectively, $\tilde{u}_f(t)$ be the solution of Equation* (3.5)*, given by Equation* (3.23)*, and $C_1 = \frac{(a-n)_n (b-n)_n}{(-1+c-n)_n}$. Then $u_f(t) = {}_R D_t^{\epsilon} \tilde{u}_f(t)$ and $\tilde{u}_f(t)$ are expressed by*

$$\tilde{u}_f(t) \simeq C_1 \cdot G_{H,\epsilon}(t,0) + \epsilon C_1 \sum_{l=1}^{n} \frac{(-1+c-l)_l (-1)^{l-1}}{(a-l)_l (b-l)_l} (l-1)! \cdot t^{-l+\epsilon} H(t), \tag{3.24}$$

$$u_f(t) = C_1 \cdot G_{H,0}(t,0), \tag{3.25}$$

where $u_f(t)$ is a complementary solution of Equation (1.17)*, for $n \in \mathbb{Z}_{>-1}$, as shown in Corollary 2.5.*

Proof. When $\beta = n \in \mathbb{Z}_{>0}$, the terms of $k < n$ on the righthand side of (3.23) are $O(\epsilon)$. We confirm that the terms of $k \geq n$ gives the first term on the righthand side of (3.24). The second term is obtained by using the following formulas for $z \in \mathbb{C}$ and $m \in \mathbb{Z}_{>-1}$:

$$\frac{1}{\Gamma(z)} = \frac{\sin(\pi z)\Gamma(1-z)}{\pi}; \quad \frac{1}{\Gamma(-m+\epsilon)} \simeq (-1)^m \epsilon \cdot m!. \tag{3.26}$$

□

Remark 3.3. In the preceding paper [(11)], we have Corollary 1 for Kummer's differential equation. We note here that the first terms on the righthand sides of Equations (63) and (64) in that paper can be written as $C_1 \cdot G_{K,\epsilon}(t,0)$ and $C_1 \cdot G_{K,0}(t,0)$, where $C_1 = \frac{(a-n)_n \cdot b^n}{(-1+c-n)_n}$.

Lemma 3.8. *Lemma 3.5, Remark 3.1 and Lemma 2.2 show that $\tilde{w}_c(t)$ and $w_c(t)$, given by*

$$\tilde{w}_c(t) = {}_R D_t^{-\epsilon} w_c(t), \quad w_c(t) := H_{\beta,2}(t) H(t) = {}_R D_t^{-\beta} H_2(t) H(t), \tag{3.27}$$

are complementary solutions of Equations (3.18) *and* (3.17)*, respectively, and then Remark 2.1 shows that $\tilde{u}_c(t)$ and $u_c(t)$, given by $\tilde{u}_c(t) = {}_R D_t^{\beta} \tilde{w}_c(t)$ and $u_c(t) = {}_R D_t^{\epsilon} \tilde{u}_c(t)$, respectively, are the complementary solutions of Equations* (3.5) *and* (1.17) *which are given in Lemma 3.5.*

3.2 Solutions of Equations (1.17) and (3.5) Satisfying Condition 1.1(i) or (ii)

The differential equation, which is satisfied by the Green's function $G_{H,\epsilon}(t,\tau)$ for Equation (1.17), is given by Equation (3.6).

Lemma 3.9. *Let $0 < \tau < t$, $H_1(t)$ and $H_2(t)$ be those in Lemma 3.2, and $G_{H,0}(t,\tau)$ be given by*

$$G_{H,0}(t,\tau) = \frac{1}{\tau_H \psi'_\tau(\tau)} \psi_\tau(t) H(t-\tau) = \frac{1}{\tau_H \psi'_\tau(\tau)} \sum_{k=1}^{\infty} \frac{1}{k!} \psi_\tau^{(k)}(\tau)(t-\tau)^k H(t-\tau), \tag{3.28}$$

where $\tau_H = \tau(1-\tau)$ and $\psi_\tau(t) = H_1(\tau) H_2(t) - H_2(\tau) H_1(t)$.

Then $G_{H,\epsilon}(t,\tau)$, given by $G_{H,\epsilon}(t,\tau) = {}_R D_t^{-\epsilon} G_{H,0}(t,\tau)$, satisfies Equation (3.6).

Proof. Taking account of Lemma 2.1, we choose the complementary solution of Equation (1.17) on $\mathbb{R}_{>\tau}$, given by $\tilde{G}_{H,0}(t,\tau) = C_1 \cdot G_{H,0}(t,\tau)$, where C_1 is a constant, and then confirm that $\tilde{G}_{H,\epsilon}(t,\tau) = C_1 \cdot {}_RD_t^{-\epsilon} G_{H,0}(t,\tau)$ satisfies (3.6), when $C_1 = 1$, as follows.

We put $x = t - \tau$, and we express $\tilde{G}_{H,\epsilon}(t,\tau)$ by

$$\tilde{v}(x) := \tilde{G}_{H,\epsilon}(\tau + x, \tau) = \sum_{k=1}^{\infty} a_k \frac{x^{k+\epsilon}}{\Gamma(k+\epsilon+1)} H(x), \tag{3.29}$$

where a_k are constants, and $a_1 \neq 0$. Then (3.6) is expressed as

$$\tilde{p}_{H,\epsilon}(\tau + x, {}_RD_x)\tilde{v}(x) = [\tau_H a_1 \frac{x^{\epsilon-1}}{\Gamma(\epsilon)} + O(x^\epsilon)]H(x) = \frac{x^{\epsilon-1}}{\Gamma(\epsilon)} H(x). \tag{3.30}$$

This is satisfied when $a_1 = \frac{1}{\tau_H}$. □

Theorem 3.2. Let $\tilde{f}(t)$ satisfy Condition 1.1(i), $G_{H,\epsilon}(t,\tau)$ satisfy Equation (3.6), $G_{H,\epsilon}(t,\tau)$ and $G_{H,0}(t,\tau)$ for $\tau > 0$ be given in Lemma 3.9, and $G_{H,\epsilon}(t,0)$ and $G_{H,0}(t,0)$ be given in Lemma 3.3. and be given in Lemma 3.9, Then Theorem 2.1 shows that we have the solutions $\tilde{u}_f(t)$ and $u_f(t)$ of Equations (3.5) and (1.17), respectively, which are given by

$$\tilde{u}_f(t) = \int_{-\infty}^{\infty} G_{H,\epsilon}(t,\tau) f(\tau) d\tau + c_\epsilon G_{H,\epsilon}(t,0), \tag{3.31}$$

$$u_f(t) = {}_RD_t^\epsilon \tilde{u}_f(t) = \int_{-\infty}^{\infty} G_{H,0}(t,\tau) f(\tau) d\tau + c_\epsilon G_{H,0}(t,0). \tag{3.32}$$

See Lemma 3.4 for the complementary solutions $\tilde{u}_c(t)$ and $u_c(t)$.

This lemma is derived by using the complementary solutions given by Equations (3.7) and (3.8), and hence by assuming $c \notin \mathbb{Z}_{<1}$.

With the aid of Remark 3.2, we have the following lemma for $G_{H,\beta,\epsilon}(t,\tau)$ for $\tau > 0$.

Lemma 3.10. *The lemma, which is obtained from Lemma 3.9 by replacing H_1 by $H_{\beta,1}$, H_2 by $H_{\beta,2}$, Lemma 3.2 by Lemma 3.5, $G_{H,\epsilon}$ by $G_{H,\beta,\epsilon}$, and $G_{H,0}$ by $G_{H,\beta,0}$, holds.*

Theorem 3.3. Let $f_\beta(t)$ satisfy Condition 1.1(ii), $G_{H,\beta,\epsilon}(t,\tau)$ for $\tau > 0$, satisfy Equation (3.21), and be determined by Lemma 3.10, and $G_{H,\beta,\epsilon}(t,0)$ be given in Lemma 3.6. Then Theorem 2.2 shows that the particular solutions of Equations (3.5) and (3.18), respectively, are given by

$$\tilde{u}_f(t) = {}_RD_t^\beta \int_{-\infty}^{\infty} G_{H,\beta,\epsilon}(t,\tau) f_\beta(\tau) d\tau + c_{\beta,\epsilon} \cdot {}_RD_t^\beta G_{H,\beta,\epsilon}(t,0), \tag{3.33}$$

and $\tilde{w}_f(t) = {}_RD_t^{-\beta} \tilde{u}_f(t)$. Their complementary solutions $\tilde{u}_c(t)$ and $\tilde{w}_c(t)$ are given in Lemma 3.8.

Corollary 3.11. *Let $\beta = n \in \mathbb{Z}_{>0}$. Then the expression of the second term on the righthand side of Equation* (3.33) *is given by $\tilde{u}_f(t)$ in Equation* (3.24) *in Corollary 3.7, multiplied by $c_{\beta,\epsilon}$.*

4 Solution of a Simple Ordinary and a Fractional Differential Equation, I

In a preceding paper [(9)], solutions of a simple ordinary and a fractional differential equations are discussed in the framework of nonstandard analysis, by using the recipe of solution given in [(9)]. The simple fractional differential equation is

$$p_F(t, {}_RD_t)v(t) := [{}_RD_t^{3/2} - at \cdot {}_RD_t^{1/2}]v(t) = f(t), \tag{4.1}$$

which is presented in [(3)], and the ordinary differential equation is

$$p_L(t, \frac{d}{dt})y(t) := [\frac{d}{dt} - at]y(t) = f(t), \tag{4.2}$$

where a is a constant. We give here a brief discussion of this problem by using the recipe given in Section 2.

We put $\tilde{y}(t) = {}_RD_t^{-\epsilon}y(t)$, in Equation (4.2), by using Equation (3.1), we have the following equation for $\tilde{y}(t)$, which is a transformation of Equation (4.2):

$$\tilde{p}_{L,\epsilon}(t, {}_RD_t)\tilde{y}(t) := {}_RD_t^{-\epsilon}p_L(t, \frac{d}{dt}){}_RD_t^{\epsilon}\tilde{y}(t) = {}_RD_t^{-\epsilon}[\frac{d}{dt} - at]{}_RD_t^{\epsilon}\tilde{y}(t)$$
$$= [\frac{d}{dt} - at + a\epsilon \cdot {}_RD_t^{-1}]\tilde{y}(t) = \tilde{f}(t). \tag{4.3}$$

In the case of Equation (4.1), we put $\tilde{v}(t) = {}_RD_t^{-\epsilon}v(t)$, and then we have

$$\tilde{p}_{F,\epsilon}(t, {}_RD_t)\tilde{v}(t) := {}_RD_t^{-\epsilon}p_F(t, {}_RD_t){}_RD_t^{\epsilon}\tilde{v}(t) = {}_RD_t^{-\epsilon}p_L(t, {}_RD_t){}_RD_t^{1/2}{}_RD_t^{\epsilon}\tilde{v}(t)$$
$$= \tilde{p}_{L,\epsilon}(t, {}_RD_t){}_RD_t^{1/2}\tilde{v}(t) = \tilde{f}(t). \tag{4.4}$$

Corresponding to Equation (2.9), the differential equation satisfied by the Green's function $G_{L,\epsilon}(t,\tau)$ for Equation (4.2), is given by

$$\tilde{p}_{L,\epsilon}(t, {}_RD_t)G_{L,\epsilon}(t,\tau) = [\frac{d}{dt} - at + a\epsilon \cdot {}_RD_t^{-1}]G_{L,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau). \tag{4.5}$$

Lemma 4.1. *Let $y_1(t) := e^{at^2/2}$. Then by using the formula $(2k)! = 4^k k!(\frac{1}{2})_k$, we can easily confirm that $G_{L,0}(t,0)$ and $G_{L,\epsilon}(t,0)$, given by*

$$G_{L,0}(t,0) = y_1(t)H(t) = \sum_{k=0}^{\infty} \frac{a^k}{2^k k!}t^{2k}H(t) = \sum_{k=0}^{\infty} \frac{2^k a^k (\frac{1}{2})_k}{(2k)!}t^{2k}H(t), \tag{4.6}$$

$$G_{L,\epsilon}(t,0) = {}_RD_t^{-\epsilon}G_{L,0}(t,0) = \sum_{k=0}^{\infty} \frac{2^k a^k (\frac{1}{2})_k}{\Gamma(2k+1+\epsilon)}t^{2k+\epsilon}H(t), \tag{4.7}$$

are a complementary solution of Equation (4.2) *and a particular solution of* (4.5)*, respectively, for $\tau = 0$, and $G_{L,0}(t,\tau)$ and $G_{L,\epsilon}(t,\tau)$, given by*

$$G_{L,0}(t,\tau) = \sum_{k=0}^{\infty} A_k(\tau)\frac{(t-\tau)^k}{k!}H(t-\tau), \tag{4.8}$$

$$G_{L,\epsilon}(t,\tau) = {}_RD_t^{-\epsilon}G_{L,0}(t,\tau) = \sum_{k=0}^{\infty} A_k(\tau)\frac{(t-\tau)^{k+\epsilon}}{\Gamma(k+\epsilon+1)}H(t-\tau), \tag{4.9}$$

are those for $\tau > 0$, where

$$A_{2k+r}(\tau) := \frac{y_1^{(2k+r)}(\tau)}{y_1(\tau)} = \sum_{l=0}^{k} \frac{(2k+r)!a^{k+l+r}\tau^{2l+r}}{(k-l)!(2l+r)!2^{k-l}}, \quad k \in \mathbb{Z}_{>-1}, \;\; r \in \mathbb{Z}_{[0,1]}, \tag{4.10}$$

so that $A_0(\tau) = 1$, $A_1(\tau) = a\tau$, $A_2(\tau) = a(1+a\tau^2)$, $\;A_3(\tau) = a^2\tau(3+a\tau^2)$, $\;A_4(\tau) = a^2(3+6a\tau^2+a^2\tau^2)$, and so on.

Theorem 4.1. Let Condition 1.1(i) be satisfied. Then Theorem 2.1 shows that the solutions of Equations (4.3) and (4.2), respectively, are given by

$$\tilde{y}_f(t) = \int_{-\infty}^{\infty} G_{L,\epsilon}(t,\tau)f(\tau)d\tau + c_\epsilon G_{L,\epsilon}(t,0), \tag{4.11}$$

$$y_f(t) = \int_{-\infty}^{\infty} G_{L,0}(t,\tau)f(\tau)d\tau + c_\epsilon G_{L,0}(t,0). \tag{4.12}$$

The differential equation satisfied by the Green's function $G_{F,\epsilon}(t,\tau)$ for Equation (4.1), is

$$\tilde{p}_{F,\epsilon}(t, {}_RD_t)G_{F,\epsilon}(t,\tau) = [{}_RD_t^{3/2} - at \cdot {}_RD_t^{1/2} + a\epsilon \cdot {}_RD_t^{-1/2}]G_{F,\epsilon}(t,\tau)$$
$$= \delta_\epsilon(t-\tau). \tag{4.13}$$

Lemma 4.2. *Let* $G_{L,0}(t,0)$*,* $G_{L,\epsilon}(t,0)$*,* $G_{L,0}(t,\tau)$ *and* $G_{L,\epsilon}(t,\tau)$ *be given by Equations* (4.6)∼(4.9)*. Then comparing Equation* (4.2) *with* (4.1)*, and Equation* (4.5) *with* (4.13)*, we confirm that*

$$G_{F,0}(t,0) = {}_RD_t^{-1/2}G_{L,0}(t,0), \quad G_{F,\epsilon}(t,0) = {}_RD_t^{-1/2}G_{L,\epsilon}(t,0), \tag{4.14}$$
$$G_{F,0}(t,\tau) = {}_RD_t^{-1/2}G_{L,0}(t,\tau), \quad G_{F,\epsilon}(t,\tau) = {}_RD_t^{-1/2}G_{L,\epsilon}(t,\tau), \quad \tau > 0. \tag{4.15}$$

Corollary 4.3. *Comparing Equation* (4.3) *with* (4.4)*, and Equation* (4.2) *with* (4.1)*, we see that the solutions of Equations* (4.4) *and* (4.1) *are obtained from Equations* (4.11) *and* (4.12)*, by replacing* $\tilde{y}_f$ *by* $\tilde{v}_f$*,* y_f *by* v_f*,* $G_{L,\epsilon}$ *by* ${}_RD_t^{-1/2}G_{L,\epsilon}$*, and* $G_{L,0}$ *by* ${}_RD_t^{-1/2}G_{L,0}$*.*

Solutions of Equations (4.3) and (4.4) satisfying Conditions 1.1(iii) and (ii), respectively, are given in Sections 5.1.1 and 5.2.1, by using the results for the Hermite differential equation.

5 Solutions of the Hermite Differential Equation

We consider the Hermite differential equation:

$$p_{He,b}(t, \frac{d}{dt})u(t) := [\frac{d^2}{dt^2} - at\frac{d}{dt} + ab]u(t) = f(t), \quad t > 0, \tag{5.1}$$

where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{Z}_{>-1}$ are constants.

Lemma 5.1. *We have two complementary solutions of Equation* (5.1)*, which are*

$$he_{b,0}(t) := \sum_{k=0}^{\infty} \frac{(-\frac{b}{2})_k (2a)^k}{\Gamma(2k+1)} t^{2k} = {}_1F_1(-\frac{b}{2}; \frac{1}{2}; \frac{1}{2}at^2), \quad t > 0, \tag{5.2}$$
$$he_{b,1}(t) := \sum_{k=0}^{\infty} \frac{(\frac{1-b}{2})_k (2a)^k}{\Gamma(2k+2)} t^{2k+1} = t \cdot {}_1F_1(\frac{1-b}{2}; \frac{3}{2}; \frac{1}{2}at^2), \quad t > 0. \tag{5.3}$$

Proof. In Section 5.0.1, derivation of (5.2) and (5.3) by using Frobenius' method is presented. □

Remark 5.1. Let $n \in \mathbb{Z}_{>-1}$. Then if $b = 2n+1$, $(-\frac{b}{2})_k = (-\frac{1}{2} - n)_k$ and $(\frac{1-b}{2})_k = (-n)_k$, and if $b = 2n$, $(-\frac{b}{2})_k = (-n)_k$ and $(\frac{1-b}{2})_k = (\frac{1}{2} - n)_k$. Since

$$(-n)_k = \begin{cases} (-1)^k \frac{n!}{(n-k)!}, & k \in \mathbb{Z}_{[0,n]}, \\ 0, & k \in \mathbb{Z}_{>n}, \end{cases} \tag{5.4}$$

$he_{2n,0}(t)$ and $he_{2n+1,1}(t)$ are

$$he_{2n,0}(t) := \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(2a)^k}{(2k)!} t^{2k}, \quad he_{2n+1,1}(t) := \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(2a)^k}{(2k+1)!} t^{2k+1}, \tag{5.5}$$

which are polynomials of degree $2n$ and $2n+1$, respectively, and $he_{2n+1,0}(t)$ and $he_{2n,1}(t)$ are infinite series.

Remark 5.2. In the notation of [(4, Chaper V, Section 2)], the Hermite polynomials $He_{2n}(x)$ and $He_{2n+1}(x)$ represent

$$He_{2n}(x) = \frac{(-1)^n}{2^n}\frac{(2n)!}{n!}\cdot he_{2n,0}(\frac{x}{\sqrt{a}}), \quad He_{2n+1}(x) = \frac{(-1)^n}{2^n}\frac{(2n+1)!}{n!}\sqrt{a}\cdot he_{2n+1,1}(\frac{x}{\sqrt{a}}), \tag{5.6}$$

and the Hermite functions of the second kind $he_{2n}(x)$ and $he_{2n+1}(x)$ represent

$$he_{2n}(x) = (-1)^n 2^n n!\sqrt{a}\cdot he_{2n,1}(\frac{x}{\sqrt{a}}), \quad he_{2n+1}(x) = (-1)^{n+1}2^n n!\cdot he_{2n+1,0}(\frac{x}{\sqrt{a}}). \tag{5.7}$$

When Condition 1.1(i) is satisfied, the equation for $\tilde{u}(t) := {}_RD_t^{-\epsilon}u(t)$ is

$$\begin{aligned}\tilde{p}_{He,b+\epsilon}(t, {}_RD_t)\tilde{u}(t) &:= {}_RD_t^{-\epsilon}p_{He}(t, \frac{d}{dt}){}_RD_t^{\epsilon}\tilde{u}(t)\\ &= [\frac{d^2}{dt^2} - at\frac{d}{dt} + a(b+\epsilon)]\tilde{u}(t) = \tilde{f}(t).\end{aligned} \tag{5.8}$$

Corresponding to Equation (2.1), the differential equation satisfied by the Green's function $G_{He,b,\epsilon}(t,\tau)$ for Equation (5.1), is given by

$$\tilde{p}_{He,b+\epsilon}(t, \frac{d}{dt})G_{He,b,\epsilon}(t,\tau) = [\frac{d^2}{dt^2} - at\frac{d}{dt} + a(b+\epsilon)]G_{He,b,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau). \tag{5.9}$$

Lemma 5.2. *Let $he_{b,0}(t)$ and $he_{b,1}(t)$ be given by Equations* (5.2) *and* (5.3)*. Then $G_{He,b,\epsilon}(t,0)$ and $G'_{He,b+1,\epsilon}(t,0)$ given by*

$$G_{He,b,\epsilon}(t,0) = {}_RD_t^{-\epsilon}he_{b,1}(t)H(t) = \sum_{k=0}^{\infty}\frac{(\frac{1-b}{2})_k(2a)^k}{\Gamma(2k+2+\epsilon)}t^{2k+1+\epsilon}H(t), \tag{5.10}$$

$$G'_{He,b+1,\epsilon}(t,0) = {}_RD_t^{-\epsilon}he_{b,0}(t)H(t) = \sum_{k=0}^{\infty}\frac{(-\frac{b}{2})_k(2a)^k}{\Gamma(2k+1+\epsilon)}t^{2k+\epsilon}H(t), \tag{5.11}$$

are particular solutions of Equation (5.8) *for $\tilde{f}(t) = \delta_\epsilon(t)$ and $\frac{d}{dt}\delta_\epsilon(t)$, respectively. As a consequence, Corollary 2.5 shows that $G_{He,b,0}(t,0)$ and $G'_{He,b+1,0}(t,0)$ given by*

$$G_{He,b,0}(t,0) = {}_RD_t^{\epsilon}G_{He,b,\epsilon}(t,0) = he_{b,1}(t)H(t), \tag{5.12}$$

$$G'_{He,b+1,0}(t,0) = {}_RD_t^{\epsilon+1}G_{He,b+1,\epsilon}(t,0) = he_{b,0}(t)H(t), \tag{5.13}$$

are complementary solutions of Equation (5.1)*. Equation* (5.5) *shows that when $n \in \mathbb{Z}_{>-1}$, $he_{2n,0}(t)$ and $he_{2n+1,1}(t)$ are polynomials of degree $2n$ and $2n+1$, respectively.*

Proof. When we know the statement for $G_{He,b,0}(t,0)$, we confirm the statement for $G_{He,b,\epsilon}(t,0)$ by using Lemma 2.1, since ${}_RD_t^{-1}p_{He}(t, \frac{d}{dt})he_{b,1}(t)H(t) = 1 + O(t)$ at $t \geq 0$. In Sections 5.0.1 and 5.0.2, Equations (5.10), (5.11), (5.12) and (5.13) are derived by using Frobenius' method. ☐

5.0.1 Particular Solutions of Equation (5.8) by Frobenius' Method

We now give the derivations of Equations (5.10) and (5.11), by Frobenius' Method.

In Frobenius' method, the solution $\tilde{u}(t)$ is assumed to be expressed by

$$\tilde{u}(t) = \sum_{k=0}^{\infty}p_k\frac{1}{\Gamma(\alpha+k+1)}t^{\alpha+k}H(t), \tag{5.14}$$

where α and p_k are constants, and $p_0 \neq 0$. By using this in Equation (5.8), we obtain

$$\sum_{k=0}^{\infty} p_k[\frac{t^{\alpha+k-2}}{\Gamma(\alpha+k-1)} - a(\alpha+k-b-\epsilon)\frac{t^{\alpha+k}}{\Gamma(\alpha+k+1)}]H(t) = p_0\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}H(t)$$
$$+p_1\frac{t^{\alpha-1}}{\Gamma(\alpha)}H(t) + \sum_{k=2}^{\infty}[p_k - ap_{k-2}(\alpha+k-2-b-\epsilon)]\frac{t^{\alpha+k-2}}{\Gamma(\alpha+k-1)}H(t) = \tilde{f}(t). \quad (5.15)$$

When $\tilde{f}(t) = \delta_\epsilon(t) = \frac{t^{\epsilon-1}}{\Gamma(\epsilon)}H(t)$, Equation (5.15) is satisfied, if $\alpha = 1+\epsilon$, $p_0 = 1$, $p_1 = 0$ and

$$p_{2k} = 2ap_{2k-2}(k-1+\frac{1-b}{2}) = p_0(2a)^k(\frac{1-b}{2})_k, \quad p_{2k+1} = 0; \quad k \in \mathbb{Z}_{>0}. \quad (5.16)$$

By using these in Equation (5.14), and putting $\tilde{u}(t) = G_{He,b,\epsilon}(t,0)$, we obtain Equation (5.10).

When $\tilde{f}(t) = \frac{d}{dt}\delta_\epsilon(t) = \frac{t^{\epsilon-2}}{\Gamma(\epsilon-1)}H(t)$, Equation (5.15) is satisfied, if $\alpha = \epsilon$, $p_0 = 1$, $p_1 = 0$, and

$$p_{2k} = 2ap_{k-1}(k-1-\frac{b}{2}) = p_0(2a)^k(-\frac{b}{2})_k, \quad p_{2k+1} = 0; \quad k \in \mathbb{Z}_{>0}. \quad (5.17)$$

By using these in Equation (5.14), and putting $\tilde{u}(t) = G'_{He+1,b,\epsilon}(t,0)$, we obtain Equation (5.11).

5.0.2 Complementary Solutions of Equation (5.1) by Frobenius' Method

We now give the derivations of Equations (5.12) and (5.13), by using Frobenius' method.

We note that the statements in Section 5.0.1 hold, even when we replace ϵ by 0, $\tilde{u}(t)$ by $u(t)$, $\tilde{f}(t)$ by $f(t)$, (5.8) by (5.1), (5.10) by (5.12), and (5.11) by (5.13).

5.1 Solutions of Equations (5.8) and (5.1) Satisfying Condition 1.1(iii)

When Condition 1.1(ii) or (iii) is satisfied, the equations for $w(t) := {}_RD_t^{-\beta}u(t)$ and $\tilde{w}(t) := {}_RD_t^{-\beta}\tilde{u}(t)$, respectively, are

$$\tilde{p}_{He,b+\beta}(t,\frac{d}{dt})w(t) := {}_RD_t^{-\beta}p_{He}(t,\frac{d}{dt}){}_RD_t^{\beta}w(t)$$
$$= [\frac{d^2}{dt^2} - at\frac{d}{dt} + a(b+\beta)]w(t) = f_\beta(t), \quad t > 0, \quad (5.18)$$
$$\tilde{p}_{He,b+\beta+\epsilon}(t,\frac{d}{dt})\tilde{w}(t) := {}_RD_t^{-\beta}\tilde{p}_{He,b+\epsilon}(t,\frac{d}{dt}){}_RD_t^{\beta}\tilde{w}(t)$$
$$= [\frac{d^2}{dt^2} - at\frac{d}{dt} + a(b+\beta+\epsilon)]\tilde{w}(t) = \tilde{f}_\beta(t). \quad (5.19)$$

Lemma 5.3. *We have two complementary solutions of Equation* (5.18)*. They are expressed by* $he_{b+\beta,0}(t)$ *and* $he_{b+\beta,1}(t)$*, which are obtained from Equations* (5.2) *and* (5.3)*, respectively, by replacing* b *by* $b+\beta$*.*

Corresponding to Equation (2.9), the differential equation satisfied by the Green's function $G_{He,b+\beta,\epsilon}(t,\tau)$ for Equation (5.18), is given by

$$\tilde{p}_{He,b+\beta+\epsilon}(t,\frac{d}{dt})G_{He,b+\beta,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau). \quad (5.20)$$

Remark 5.3. Equation (5.20) is obtained from Equation (5.9), by replacing b by $b+\beta$.

Lemma 5.4. *Let* $he_{b+\beta,1}(t)$ *be given by Equation* (5.3)*, in which* b *is replaced by* $b+\beta$*. Then Remark 5.3 shows that the lemma, which is obtained from Lemma 5.2, by replacing* b *by* $b+\beta$*,* (5.9) *by* (5.20)*, and* (5.1) *by* (5.18)*, holds.*

Theorem 5.1. Let Condition 1.1(iii) be satisfied, and $G_{He,b+\beta,\epsilon}(t,0)$ be given by Equation (5.10), in which b is replaced by $b+\beta$, as stated in Lemma 5.4. Then Theorem 2.3 shows that $\tilde{w}_f(t) = G_{He,b+\beta,\epsilon}(t,0)$ and $\tilde{u}_f(t) = {}_RD_t^{\beta}\tilde{w}_f(t)$, given by

$$\tilde{u}_f(t) = {}_RD_t^{\beta}G_{He,b+\beta,\epsilon}(t,0) = \sum_{k=0}^{\infty}\frac{(\frac{1-b-\beta}{2})_k(2a)^k}{\Gamma(2k+2-\beta+\epsilon)}t^{2k+1-\beta+\epsilon}H(t), \tag{5.21}$$

are particular solutions of Equations (5.19) and (5.8), respectively.

Corollary 5.5. *Let $m \in \mathbb{Z}_{>-1}$, $r \in \mathbb{Z}_{[0,1]}$ and $\beta = 2m+r$. Then by using the second formula in* (3.26)*, $\tilde{u}_f(t)$ given by* (5.21) *is expressed as*

$$\begin{aligned}\tilde{u}_f(t) &= {}_RD_t^{2m+r}G_{He,b+2m+r,\epsilon}(t,0) = \sum_{k=0}^{\infty}\frac{(\frac{1-b-r}{2}-m)_k(2a)^k}{\Gamma(2k+2-2m-r+\epsilon)}t^{2k+1-2m-r+\epsilon}H(t)\\ &= (\frac{1-b-r}{2}-m)_m(2a)^m\sum_{l=0}^{\infty}\frac{(\frac{1-b-r}{2})_l(2a)^l}{\Gamma(2l+2-r+\epsilon)}t^{2l+1-r+\epsilon}H(t)+O(\epsilon).\end{aligned} \tag{5.22}$$

Corollary 5.6. *Let $m \in \mathbb{Z}_{>-1}$ and $\tilde{u}_f(t)$ be given by* (5.21)*. Then comparing Equation* (5.22) *with* (5.10)*, we see that if $\beta = 2m$,*

$$\tilde{u}_f(t) = {}_RD_t^{2m}G_{He,b+2m,\epsilon}(t,0) = (\frac{1-b}{2}-m)_m(2a)^mG_{He,b,\epsilon}(t,0)+O(\epsilon), \tag{5.23}$$

and if $\beta = 2m+1$,

$$\tilde{u}_f(t) = {}_RD_t^{2m+1}G_{He,b+2m+1,\epsilon}(t,0) = (-\frac{b}{2}-m)_m(2a)^mG'_{He,b+1,\epsilon}(t,0)+O(\epsilon). \tag{5.24}$$

Remark 5.4. Let $\tilde{u}_f(t)$ be given by (5.21). Then Corollary 5.6 shows that $u_f(t) = {}_RD_t^{-\epsilon}\tilde{u}_f(t)$ for $\beta \in \mathbb{Z}_{>0}$ is a complementary solution of Equation (5.1), as shown in Corollary 2.5.

5.1.1 Solutions of Equations (4.3) and (4.4) Satisfying Condition 1.1(iii)

Lemma 5.7. *By comparing Equation* (4.5) *and* (4.13) *with* (5.9)*, we see that*

$$G_{L,\epsilon}(t,\tau) = \frac{d}{dt}G_{He,0,\epsilon}(t,\tau), \quad G_{F,\epsilon}(t,\tau) = {}_RD_t^{-1/2}G_{L,\epsilon}(t,\tau). \tag{5.25}$$

Lemma 5.8. *Let $\tilde{u}_f(t)$ be a particular solution of Equation* (5.8) *for $b=0$. Then $\tilde{y}_f(t)$ and $\tilde{u}_f(t)$, given by $\tilde{y}_f(t) = \frac{d}{dt}\tilde{u}_f(t)$ and $\tilde{v}_f(t) = {}_RD_t^{-1/2}\tilde{y}_f(t)$, are particular solutions of Equations* (4.3) *and* (4.4)*, respectively.*

Theorem 5.2. Let Condition 1.1(iii) be satisfied, and $G_{He,b+\beta,\epsilon}(t,0)$ be given by Equation (5.10). Then Theorem 2.3 shows that $\tilde{y}_f(t) = {}_RD_t^{\beta+1}G_{He,\beta,\epsilon}(t,0)$ and $\tilde{v}_f(t) = {}_RD_t^{-1/2}\tilde{y}_f(t)$ for $b=0$, given by

$$\tilde{y}_f(t) = {}_RD_t^{\beta+1}G_{He,\beta,\epsilon}(t,0) = \sum_{k=0}^{\infty}\frac{(\frac{1-\beta}{2})_k(2a)^k}{\Gamma(2k+1-\beta+\epsilon)}t^{2k-\beta+\epsilon}H(t), \tag{5.26}$$

$$\tilde{v}_f(t) = {}_RD_t^{-1/2}\tilde{y}_f(t) = \sum_{k=0}^{\infty}\frac{(\frac{1-\beta}{2})_k(2a)^k}{\Gamma(2k+3/2-\beta+\epsilon)}t^{2k+1/2-\beta+\epsilon}H(t), \tag{5.27}$$

are particular solutions of Equations (4.3) and (4.4), respectively.

We see that when $\beta = 0$, Equation (5.26) agrees with Equation (4.7).

Corollary 5.9. *Let $m \in \mathbb{Z}_{>-1}$. and $\beta = 2m$. Then by using Equations* (5.26)*,* (5.23) *and* (5.25)*, we have*

$$\tilde{y}_f(t) = {}_RD_t^{2m+1} G_{He,2m,\epsilon}(t,0) = (\frac{1}{2} - m)_m (2a)^m G_{L,\epsilon}(t,0) + O(\epsilon). \tag{5.28}$$

Corollary 5.10. *Let $m \in \mathbb{Z}_{>-1}$, $\beta = 2m+1$, and $\tilde{y}_f(t)$ be given by* (5.26)*. Then by using Equations* (5.26)*,* (5.4)*, and* (3.26)*, we have*

$$\tilde{y}_f(t) = \sum_{k=0}^{m} \frac{(-m)_k (2a)^k}{\Gamma(2k-2m-1+\epsilon)} t^{2k-2m-1+\epsilon} H(t) = O(\epsilon). \tag{5.29}$$

5.2 Solutions of Equations (5.8) and (5.1) Satisfying Condition 1.1(i) or (ii)

The differential equation satisfied by the Green's function $G_{He,b+\beta,\epsilon}(t,\tau)$ for Equation (5.1) is given by (5.9).

Lemma 5.11. *Let $0 < \tau < t$, $he_{b,0}(t)$ and $he_{b,1}(t)$ be given by Equations* (5.2) *and* (5.3)*, respectively, and $G_{He,b,0}(t,\tau)$ be given by*

$$G_{He,b,0}(t,\tau) = \frac{1}{\tau_{He}\psi'_\tau(\tau)} \psi_\tau(t) H(t-\tau) = \frac{1}{\tau_{He}\psi'_\tau(\tau)} \sum_{k=1}^{\infty} \frac{1}{k!} \psi_\tau^{(k)}(\tau)(t-\tau)^k H(t-\tau), \tag{5.30}$$

where $\tau_{He} = 1$ and $\psi_\tau(t) = he_{b,0}(\tau) he_{b,1}(t) - he_{b,1}(\tau) he_{b,0}(t)$. Then $G_{He,b,\epsilon}(t,\tau)$, given by $G_{He,b,\epsilon}(t,\tau) = {}_RD_t^{-\epsilon} G_{He,b,0}(t,\tau)$, satisfies Equation (5.9)*.*

Proof. Proof of Lemma 3.9 is regarded as a proof of this lemma, if we replace (1.17) by (5.1), $\tilde{G}_{H,0}$ by $\tilde{G}_{He,b,0}$, $G_{H,0}$ by $G_{He,b,0}$, $\tilde{G}_{H,\epsilon}$ by $\tilde{G}_{He,b,\epsilon}$, by (5.9), $\tilde{p}_{H,\epsilon}$ by $\tilde{p}_{He,b+\epsilon}$, $\tau_H a_1$ in (3.30) by $\tau_{He} a_1$, and $a_1 = \frac{1}{\tau_H}$ by $a_1 = \frac{1}{\tau_{He}}$. □

Lemma 5.12. *Let $he_{b+\beta,1}(t)$ be given by Equation* (5.3)*, in which b is replaced by $b+\beta$. Then Remark 5.3 shows that the lemma, which is obtained from Lemma 5.11, by replacing b by $b+\beta$,* (5.9) *by* (5.20)*, and* (5.1) *by* (5.18)*, holds.*

Theorem 5.3. Let Condition 1.1(i), in which $\tilde{f}(t)$ is expressed by

$$\tilde{f}(t) = {}_RD_t^{-\epsilon} f(t) + c_\epsilon \delta_\epsilon(t) + c_\epsilon^{(2)} \frac{d}{dt} \delta_\epsilon(t), \tag{5.31}$$

where $c_\epsilon^{(2)}$ is a constant, be satisfied, $G_{He,b,\epsilon}(t,\tau)$, $G_{He,b,\epsilon}(t,0)$, $G'_{He,b+1,\epsilon}(t,0)$, $G_{He,b,0}(t,0)$ and $G'_{He,b+1,0}(t,0)$ be given in Lemmas 5.11 and 5.2, and $\tilde{u}_f(t)$ and $u_f(t)$ be given by

$$\tilde{u}_f(t) = \int_{-\infty}^{\infty} G_{He,b,\epsilon}(t,\tau) f(\tau) d\tau + c_\epsilon G_{He,b,\epsilon}(t,0) + c_\epsilon^{(2)} G'_{He,b+1,\epsilon}(t,0), \tag{5.32}$$

$$u_f(t) = {}_RD_t^{\epsilon} \tilde{u}_f(t) = \int_{-\infty}^{\infty} G_{He,b,0}(t,\tau) f(\tau) d\tau + c_\epsilon G_{He,b,0}(t,0) + c_\epsilon^{(2)} G'_{He,b+1,0}(t,0), \tag{5.33}$$

Then with the aid of Theorem 2.1, we confirm that $\tilde{u}_f(t)$ is a particular solution of Equation (5.8), and $u_f(t)$ is a sum of a particular solution and two complementary solutions of Equation (5.1).

Theorem 5.4. Let Condition 1.1(ii), in which $\tilde{f}_\beta(t)$ is expressed by

$$\tilde{f}_\beta(t) = {}_RD_t^{-\epsilon} f_\beta(t) + c_{\beta,\epsilon} \delta_\epsilon(t) + c_{\beta,\epsilon}^{(2)} \frac{d}{dt} \delta_\epsilon(t), \tag{5.34}$$

where $c^{(2)}_{\beta,\epsilon}$ is a constant, be satisfied, and $G_{He,b+\beta,\epsilon}(t,\tau)$, $G_{He,b+\beta,\epsilon}(t,0)$ and $G'_{He,b+\beta+1,\epsilon}(t,0)$, be given in Lemmas 5.12 and 5.4. Then with the aid of Theorem 2.2, we confirm that $\tilde{u}_f(t)$ given by

$$\tilde{u}_f(t) = {}_RD_t^{\beta} \int_{-\infty}^{\infty} G_{He,b+\beta,\epsilon}(t,\tau) f_\beta(\tau) d\tau + c_{\beta,\epsilon} \cdot {}_RD_t^{\beta} G_{He,b+\beta,\epsilon}(t,0) + c^{(2)}_{\beta,\epsilon} \cdot {}_RD_t^{\beta+1} G_{He,b+\beta+1,\epsilon}(t,0). \quad (5.35)$$

is a particular solution of Equation (5.8).

Corollary 5.6 gives the expressions of the second and third terms on the righthand side of this equation, for β satisfying $\frac{\beta}{2} \in \mathbb{Z}_{>-1}$.

5.2.1 Solutions of Equations (4.3) and (4.4) Satisfying Condition 1.1(i) or (ii)

Theorem 5.5. Let Condition 1.1(ii) be satisfied. Then Lemma 5.8 and Theorem 5.4 show that particular solutions of Equations (4.3) and (4.4), are given by

$$\tilde{y}_f(t) = {}_RD_t^{\beta+1} \int_{-\infty}^{\infty} G_{He,\beta,\epsilon}(t,\tau) f_\beta(\tau) d\tau + c_{\beta,\epsilon} \cdot {}_RD_t^{\beta+1} G_{He,\beta,\epsilon}(t,0), \quad (5.36)$$

and $\tilde{v}_f(t) = {}_RD_t^{-1/2} \tilde{y}_f(t)$, respectively.

Corollaries 5.9 and 5.10 give the expressions of the second term on the righthand side of Equation (5.36), for $\beta \in \mathbb{Z}_{>-1}$. When β is an even integer, the term is a multiple of $G_{L,\epsilon}(t,0)$ given by (4.7), and when β is an odd integer, it is $O(\epsilon)$.

Remark 5.5. By Remark 1.5, when Condition 1.1(i) is satisfied, Condition 1.1(ii) is satisfied for $\beta = 0$, and Equation (5.36) for $\beta = 0$ holds. By using that equation with Equation (5.25), we confirm Equation (4.11).

6 Conclusion

In [(7)], the problem of obtaining the particular solution of an inhomogeneous ordinary differential equation with polynomial coefficients is discussed in terms of the Green's function, in the framework of distribution theory. It is applied to Kummer's and the hypergeometric differential equation.

In [(9)], a compact recipe is presented, which is applicable to the case of an inhomogeneous fractional differential equation, which is expressed by Equation (1.1). In the recipe, the particular solution is given by Theorems 2.1, 2.2 or 2.3, according as the inhomogeneous part satisfies Condition 1.1(i), (ii) or (iii), in the framework of nonstandard analysis. It is applied to a simple fractional and an ordinary differential equation.

In [(11)], a compact revised recipe in nonstandard analysis is presented, which is more closely related with distribution theory. In this case, the particular solution is given by Theorems 2.1, 2.2, 2.3 or 2.4, according as the inhomogeneous part satisfies Condition 1.1(i), (ii), (iii) or (iv). In that paper, it is applied to Kummer's differential equation.

In the present paper, the revised recipe is applied to the hypergeometric differential equation in Section 3, to the differential equations studied in [(9)], which are a simple ordinary and a fractional differential equation, in Sections 4, 5.1.1 and 5.2.1, and to the Hermite differential equation in Section 5 excluding Sections 5.1.1 and 5.2.1. In these sections, studies are made for the cases where Condition 1.1(i), (ii) or (iii), is satisfied.

In [(8)], an ordinary differential equation is expressed in terms of blocks of classified terms. When the equation is expressed by two blocks of classified terms, the complementary solutions are obtained by using Frobenius' method. In Sections 3.0.1, 3.0.2, 5.0.1 and 5.0.2, some of the Green's functions and complementary solutions are presented by using Frobenius' method.

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