Ishikawa-Collocation Method for NonLinear Fredholm Equations with Non-Separable Kernels

Abstract

A fixed point method is developed on a mesh for the solution of nonlinear Fredholm equation. First, the problem is collocated at mesh points and a second order quadrature rule is used to approximate the nonlinear integral. Under the assumption of nonexpansivity of self-map, we construct an Ishikawa iteration to linearize the resulting system and approximate the solution at the mesh points. Four numerical examples are given to verify the accuracy and practicability of the method. The results show that indeed the method converges with second order of accuracy. One important lesson from this study is that the results support the claim, in previous studies, that fixed point iterations can provide reliable means of solving several nonlinear problems.

Keywords: Fredholm Integral equations, Ishikawa iteration, Discrete Fixed-Point Algorithms, Trapezoidal Rule; Collocation Method, Experimental Order of Convergence.

2020 Mathematics Subject Classifications: 65R20, 45D05.

1 Introduction

An equation in which the unknown solution appears under an integral is known as an integral equation. This type of equations occur in several applications including physical sciences [1, 2, 3], economics and optimal control [4, 5, 6, 7], epidemiology [8, 9], heat transfer [10], antenna wire modelling [11, 12], neuron transport [13] and applied mathematical methods [14, 15]. More applications can be seen in [16, 17, 18, 19, 20].

For practical applications these equations are highly nonlinear and exact analytic solutions cannot be found. This necessitates the development of numerical methods for nonlinear integral equations. In this study we are interested in Fredhold equations which have applications in many areas including antenna wire modelling [11, 12]. Specifically, we are interested in approximating the solution of the following problem:

$$u(x) = g(x) + \int_{y=a}^{b} k(x, y, u(y))\, dy, \qquad x \in [a, b] \subset \mathbb{R}, \tag{1}$$

where $C[a, b] \ni g : \mathbb{R} \to \mathbb{R}, k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and k is nonlinear in u and may or may not be separable in x and y. Further, we assume that k is Lipschitz continuous with respect to its third argument, namely there exists α_k such that

$$|k(x, y, u) - k(x, y, v)| \leq \alpha_k |u - v| \quad \forall \quad u, v.$$

In general, closed form analytical solutions do not exist hence numerical techniques [21, 22, 23, 24, 25, 26] are being developed to tackle the most general cases that may arise from practical applications. Linear Fredholm integral equations have been discussed in [27, 28, 29] whereas the nonlinear Fredholm integral equation poses more challenging problem. The use of Haar Wavelet method has been applied for nonlinear Fredholm equation equations [30], however a brief study of the papers cited in [31] have shown various complications and its usage not generally effective. In other studies [30, 32] investigated another approach for solving the nonlinear second kind of Fredholm equations, while Hes homotopy perturbation method is used to find the solution of non-linear Fredholm equations in [33].

The existence and uniqueness of solution of functional Volterra equations is investigated in [34]. The solution was approximated by using nonlinear solver of Newton-Raphson type. The idea is extended to mixed Volterra-Fredholm equations by Nwaigwe and Benedict [21]. Nwaigwe [35] further extended the work to nonlinear functional mixed equations. Some remarkable contributions of the work of Nwaigwe [35] are the proof of existence and uniqueness of solution without any contraction assumption, and the propounding of new Grownwall inequalities which were used to prove the convergence of their proposed numerical scheme.

The challenge and downside of the above works are the use of nonlinear solvers which presents difficulties in computation, programming and mathematical theories. Hence, Nwaigwe [36] adopted the idea of fixed point iterations [15, 37] to propose a discrete Kranoselkij algorithm for functional Voltera equations without solving nonlinear systems. The results show that such algorithms produce very accurate solutions and are fast. A very recent study in using iterative methods to solve integral equations can be found in [38]. This has motivated us to look into approaching problem (1) through the use of fixed point methods. Obviously, fixed point methods have been widely investigated [39, 40, 41] and applied to variety of problems. The present study is to join the efforts of [15, 37, 36, 38], among others, to extend the application of fixed point methods to solve nonlinear integral equations. In particular, the main essence of the present study is to develop a discrete Ishikawa iteration to solve nonlinear Fredholm equation of the second kind.

The paper is organized as follows. In section 2 we recall the Ishikawa iteration, formulate the numerical algorithm and give the pseudocode. The results are presented and discussed in section 3 and we give the concluding remarks in section 4.

2 The Numerical Method

We start this section by recalling the fixed point method of Ishikawa [42].

Lemma 2.1 (see Ishikawa [42]). *Let T be a contractive self-map, then the sequence, $\{u_n\}_{n\geq 0}$, generated by*

$$u_{n+1}(x) = (1-a_n)u_n(x) + a_n T v_n(x) \tag{2}$$

$$v_n(x) = (1-b_n)u_n(x) + b_n T u_n(x) \tag{3}$$

for a suitable u_0, where (a_n) and (b_n) are in (0, 1), converges to the fixed point of T.

In this work, we set $a_n = b_n = n^{-1/2}$ [42] and define the map T by

$$(Tu)(x) := g(x) + \int_{y=a}^{b} k(x, y, u(y))\, dy. \tag{4}$$

Hence, the Fredholm equation (1) can be written in the operator form:

$$u(x) = (Tu)(x), \qquad \text{for } x \in [a, b]. \tag{5}$$

We assume that the function k is Lipchitz continuous in $[a, b]$ with Lipchitz constant α_k and

$$\alpha_k(b-a) \leq 1. \tag{6}$$

Notice the equality sign included in the inequality above, it is because the Ishikawa iteration, like the Krasnoselskij iteration still converge for non-expansive maps.

Let $N \in \mathbb{Z}^+$ with $N > 1$. Define the $h = \frac{b-a}{N}, x_i = a + ih : i = 0, 1, \cdots, N$, and let $u_{n,i}$ be the approximation of $u(x_i)$ after $n = 0, 1, 2, \cdots,$ Ishikawa iterations. By collocating at the grid points, we have

$$u(x_i) = (Tu)(x_i), \qquad \text{for } i = 0, 1, \cdots, N. \tag{7}$$

Applying the Ishikawa iteration, we obtain

$$u_{n+1,i} = (1 - a_n)u_{n,i} + a_n T v_{n,i}, \tag{8}$$

$$v_{n,i} = (1 - b_n)u_{n,i} + b_n T u_{n,i}, \quad n \geq 0, \tag{9}$$

where

$$Tv_{n,i} = g(x_i) + I_{n,i}, \text{ where } I_{n,i} = \int_a^b k(x_i, y, u_n(y))\, dy. \tag{10}$$

The integral is approximated by the use of the quadrature rule (see [43, 21]):

$$I_{u,n,i} = \frac{h}{2} \sum_{j=0}^{N-1} \left(k(x_i, x_j, u_{n,j} + k(x_i, x_{j+1}, u_{n,j+1})\right). \tag{11}$$

The algorithm is initialize with

$$u_{0,i} = g(x_i) + \frac{h}{2} \sum_{j=0}^{N-1} \left(k(x_i, x_j, g(x_j) + k(x_i, x_{j+1}, g(x_{j+1})\right) \forall i. \tag{12}$$

2.1 Pseudocode for the Algorithm

(1) Define the grid, $\Omega_h = \{x_i, x_i = a + ih, h = \frac{b-a}{N}, \mathbb{Z} \ni N > 1\}$.

(2) For each i compute $u_{0,i}$ using (12).

(3) Set n = 0:

(4) For each i, do the following:

   (a) Evaluate the integral using (11).
   (b) Compute $Tu_{n,i}$ using (10).
   (c) Compute $u_{n+1,i}$ using (8).
   (d) If $i < N$, set $i = i + 1$ and go to step 4a, otherwise go to the next step.
   (e) If $\|u_{n+1} - u_n\| < TOL$ output u_{n+1} and stop. Otherwise, set $n = n + 1$ and go to step 4.

3 Numerical Experiments

We now present some examples to demonstrate the accuracy and practicality of the proposed method. The results are computed on 16 Gigabyte RAM quadcore machine running on a Linux operating system. These examples are derived via the method of Manufactured solutions [44, 45, 46, 47, 48, 49, 50].

Example 1

The first example is the problem:

$$u(x) = \frac{\frac{-x}{6} + (x^2+1)cos^{-1}(x^2)}{1+x^2} + \int_0^1 \frac{xy^3cos(u)}{1+x^2}dy,$$

The exact solution of this problem is $u(x) = cos^{-1}(x^2)$. The numerical solution of this problem is computed on a sequence of grids and their errors and experimental order of convergence (EOC) computed. The CPU time to compute all the EOCs is also recorded. The results are recorded in Table 1, it can be seen that proposed method is convergent and has second order of accuracy which is the theoretically expected order of accuracy. The CPU time taken to complete the EOC calculations of this problem 1 is 42.97808265686035.

Further, the numerical solution computed on different grids with 5, 10, 30 and 100 mesh points are plotted and compared with the exact solution in Figure 1. We see that even with 30 mesh points the computed solution is already very accurate.

Table 1: Results for Problem 1, N = number of sub-intervals.

N	Error	EOC
2	0.16333911635262954	-1
4	0.021646255808007164	2.915680921934721
8	0.003967379363779777	2.4478592401930115
16	0.0008612525713960623	2.203678065343175
32	0.00020136113788714327	2.0966511109303756
64	4.872781396353387e-05	2.046967867635893
128	1.1988354706990445e-05	2.0231118260012755
256	2.9733981802698706e-06	2.0114490968584344
512	7.404221460394211e-07	2.005692721549541
1024	1.8474197853446395e-07	2.002836322731808

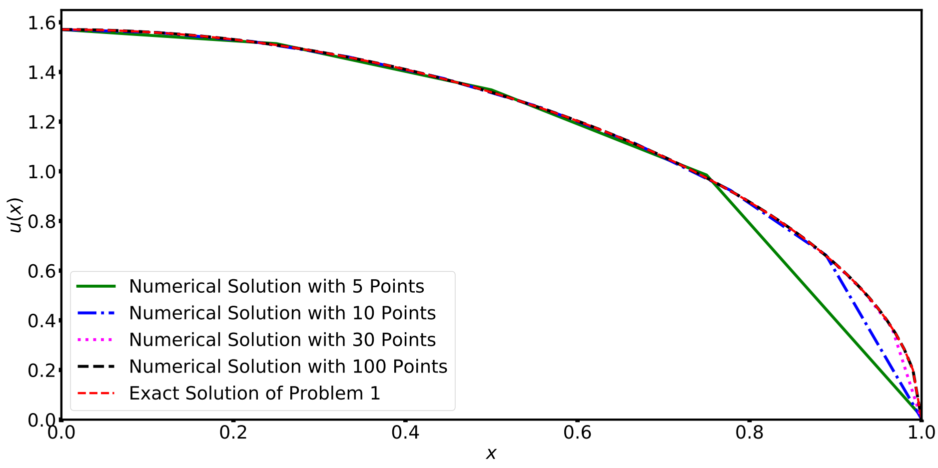

Figure 1: Plot of Exact Solution and Numerical Solution for problem 1

Example 2

The second example is

$$u(x) = x^2\left(x - \frac{\pi\sqrt{3}}{54}\right) + \int_0^1 \frac{x^2y^2u}{1+u^2+u^4}dy$$

The exact solution is $u(x) = x^3$

The solution of this problem is computed on different grids and the results are tabulated in Table 2. Just like in example 1, we can see that the proposed method converges to the exact solution at the expected order of convergence, 2. The CPU time taken to complete problem 2 is 22.412165880203247. Figure 2 presents the plots of the exact solution and the numerical solution computed of different grids. We also see that the numerical solution converges very fast to the exact solution.

Table 2: Results for Problem 2, N = number of sub-intervals.

N	Error	EOC
2	0.05637482608295197	-1
4	0.0037625705430824885	3.905260541833301
8	0.00061178604074974	2.620619536587277
16	0.00012800706701432407	2.2568037282629883
32	2.9747578815797482e-05	2.1053793047664726
64	7.1905011685968745e-06	2.0486080176014885
128	1.7687111649733467e-06	2.0233938566428216
256	4.386726273342134e-07	2.0114818797252187
512	1.0923665838546981e-07	2.0056875996706625
1024	2.7255633971989823e-08	2.0028307037917417

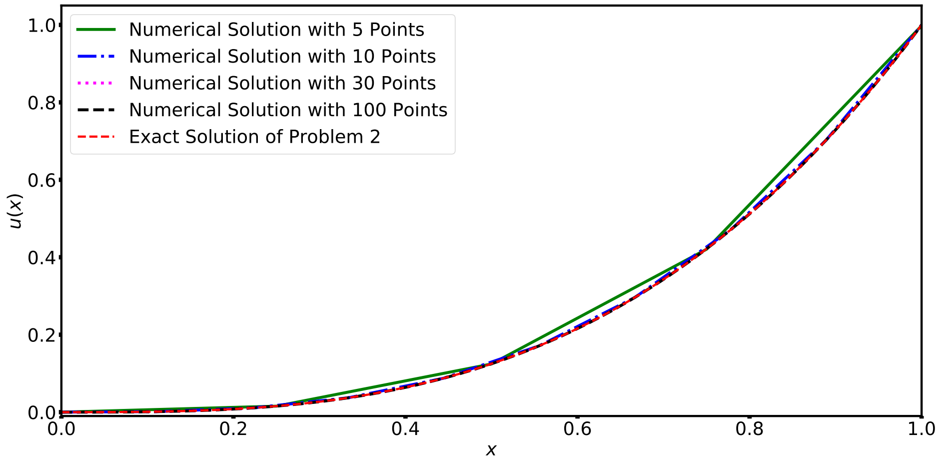

Figure 2: Plot of Exact Solution and Numerical Solution for problem 2

Example 3

Our third problem is

$$u(x) = -\frac{x^2}{12} + sin(x) - \frac{sin(2)}{48} - \frac{1}{54} - \frac{cos(3)}{432} + \frac{cos(1)}{48} + \frac{1}{12}\int_0^1 \left(x^2 + y + \frac{u^3}{3} - u^2\right)\, dy.$$

The exact solution is $u(x) = sin(x)$. Table 3 tabulates the results computed for this problem using the proposed method on a sequence of grids. We also see that the numerical solution converges to the exact solution. This is also seen in Figure 3 which displays the plots of the exact solution and numerical solution on several grids. The CPU time taken to compute the EOCs for problem 3 is 23.757.

Table 3: Results for Problem 3, N = number of sub-intervals.

N	Error	EOC
2	0.0033401995652793115	-1
4	0.0003850850798006178	3.1166851694545703
8	7.08155916803177e-05	2.4430382838362124
16	1.542482707894699e-05	2.1988127217211035
32	3.611565474237466e-06	2.0945580856224573
64	8.744631218027266e-07	2.046154876652523
128	2.151870582833837e-07	2.02280622917233
256	5.337573516506211e-08	2.011335373573074
512	1.329181165132809e-08	2.00564628078104
1024	3.316511398843147e-09	2.0027993653605405

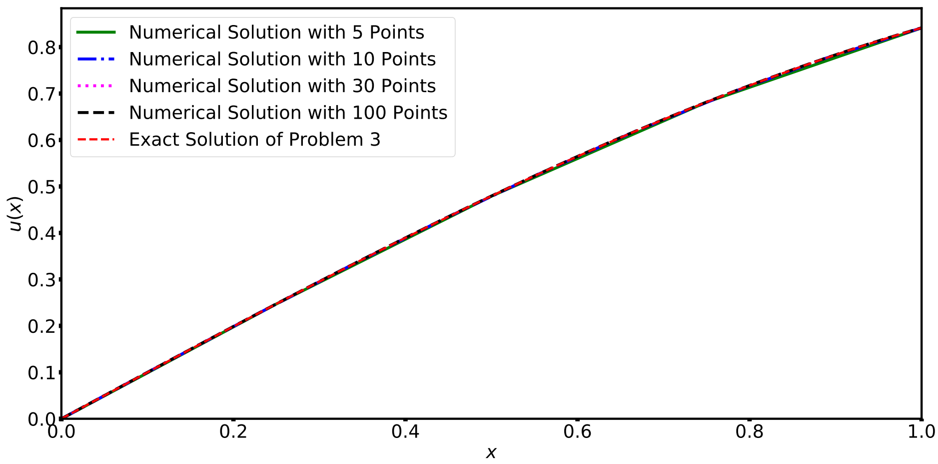

Figure 3: Plot of Exact Solution and Numerical Solution for problem 3

Example 4

Our last example is

$$u(x) = e^{\frac{(x-1)}{2}} - \frac{1}{2}e^{x-e^2} + e^{-x} + \int_0^1 ue^{x-y-u^2}dy.$$

The exact solution is $u(x) = e^{-x}$.

The results for this problem are displayed in Table 4 and Figure 4. One can also see that the numerical solution agrees with the exact solution as we refine the mesh. The CPU time taken to compute the EOCs for this problem is 41.409.

Table 4: Results for Problem 4, N = number of sub-intervals.

N	**Error**	**EOC**
2	0.031120318528157l1	-1
4	0.005963255741055273	2.383684712094301
8	0.0011218506673921214	2.410219562070775
16	0.0002452206271312818	2.193728403766116
32	5.745801591572341 5e-05	2.0935002576668724
64	1.3914595630604154e-05	2.045909192594972
128	3.4242276554241613e-06	2.0227484576080528
256	8.49363330335251e-07	2.0113248911773134
512	2.1150962614679 97e-07	2.005658500148919
1024	5.277262404845828e-08	2.0028616977181466

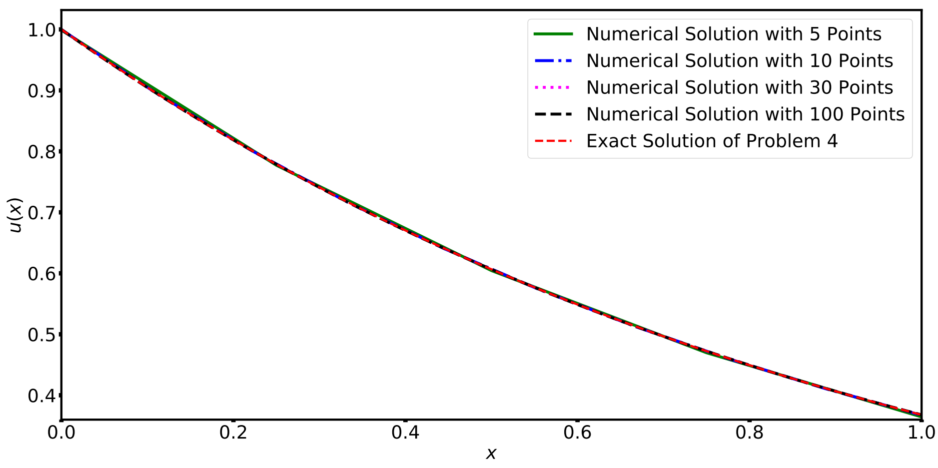

Figure 4: Plot of Exact Solution and Numerical Solution for problem 4

4 Conclusion

A numerical method based on Ishikawa iteration has been proposed and implemented for solving nonlinear Fredholm integral equations with non-separable kernels. The algorithm is verified with four test problems with known exact solutions. The results show that

(i) the scheme converges to the exact solution at the correct theoretical order of convergence,

(ii) the scheme is fast and easy to program, and

(iii) the quality of solution does not degrade as the mesh is refined.

We, therefore, conclude that provided the hypotheses of fixed point theorems are not violated, iterative schemes like the Ishikawa iteration can lead to a very reliable method for solving nonlinear integral equations.

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