

Local Orthonormal Basis of the Fibre of Quasi-Hemi-Slant Riemannian Submersion.

Abstract

In this paper we recall some different Riemannian Submersions such as; Semi-Invariant, Semi-Slant, Hemi-Slant and Quasi-Hemi-Slant Submersion from Almost Hermitian manifold to Riemannian manifold. Our goal in this work is to introduce local orthonormal bases for the fibers of Quasi-Hemi-Slant Riemannian submersion which generalizes Hemi-Slant, Semi-Slant and Semi-Invariant submersion from Almost Hermitian manifold to Riemannian manifold.

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1 Introduction

The importance of Immersions and Submersions in geometry and other parts of differential geometry cannot be over emphasized. O'Neil and Gray first introduced Riemannian Submersion. See [17] [1]. The topic became an important area of study by other geometers. see [[8], [3]]. Riemannian Submersion between Riemannian manifold equipped with some additional structures of almost complex type was first of all studied by Watson. He defined Almost Hermitian submersion between Almost Hermitian manifold and proved that the base manifold and all fibers have similar kind of structure as the total space.

Many other Geometers have introduced several other kinds of submersions base on some conditions associated with the submersion. Some of these well known submersions are; Invariant submersion [7], Semi-Invariant submersion [22], Anti-invariant Submersion [21], Submersion of Semi-Invariant Submanifolds of Trans-Sasakian Manifold [23], Geometry of slant Submanifolds [5], slant submersion from Almost Hermitian manifold [6], Semi-Slant submersion [18], quaternionic submersion [25], Riemannian Submersion from Almost contact metric manifolds [24], almost h-slant submersion and h-slant submersion [12], Hemi-Slant submersion [28] and the one introduced; Quasi-Hemi-Slant Riemannian Submersion [15].

Riemannian Submersion plays a very important roll in some topics in physics, It has applications in Kaluza Klein theory ([11], [26]), Yang Mills theory ([10], [4]), Supergravity and superstring theories. See ([27], [16]) for details.

A new class of Riemannian Submersion called Quasi-Hemi-Slant Riemannian Submersion was introduced. See [15]. This class of submersion generalizes Semi-Invariant, Hemi-Slant and Semi-Slant submersions defined on Almost Hermitian manifolds. This work introduces local Orthonormal Basis of the fibers for Quasi-Hemi-Slant submersion defined on Almost Hermitian manifolds.

The work is organized as follows; Section 2, we give some of the basic definitions and some important results of some Riemannian Submersion. Section 3, we recall the new Riemannian Submersion by Longwap at el [15] and introduce orthonormal bases for the fibers. Finally, in section 4 we relate the geometry of the fibre to the fundamental tensors.

2 Prelimineries

In this section we give some basic definition of Almost Hermitian Manifolds, and recall the necessary background and definition of Riemannian Submersion from an Almost Hermitian manifold to a Riemannian manifold.

Suppose that $(M; g)$ is an n -dimensional Riemannian manifold with a tensor field J of type $(1; 1)$.

If for every, $X, Y \in \Gamma(TM)$

$$g(JX; JY) = g(X; Y); J^2 = -I;$$

then $(M; g)$ is called an Almost Hermitian manifold, J is an almost complex structure in $(M; g)$.

If the structure J and the connection ∇

satisfy the equation $(\nabla_X J)Y = 0$; for all $X, Y \in \Gamma(TM)$,

then M is called a Kaehler manifold. ∇

is the Levi-Civita connection on M . We define the covariant

derivative of the complex structure J as;

$\nabla_X J$

$$(\nabla_X J)Y = \nabla_X (JY) - J(\nabla_X Y)$$

$$= \nabla_X (JY) - J(\nabla_X Y)$$

$\nabla_X Y$:

Thus, if M is a Kaehler manifold then $\nabla_X J = 0$

$$\nabla_X J = 0$$

$\nabla_X Y = \nabla_X Y$. See [3] for details.

Definition 2.1. [8] Suppose that $(M; g)$ is m -dimensional Riemannian manifold, and $(N; g)$ is n -dimensional Riemannian manifold, and that $\dim(M) > \dim(N)$. A surjective map $\pi : (M; g) \rightarrow (N; g)$ is a Riemannian submersion if the following conditions hold.

1. the map π has maximal rank,
 2. if π is restricted to $(\ker \pi)_p$, then it is linear isometry map.
- Here we have that for any $y \in N$; $\pi^{-1}(y)$ is a k -dimensional Riemannian submanifold of M . It is called the fiber, and that $\dim(M) - \dim(N) = k$: Any vector field on M that is tangent to the fibers is a vertical vector, if the vector is orthogonal to the fibers then it is a horizontal vector.

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Definition 2.2. [7, p. 84] If π is a Riemannian Submersion from an Almost Hermitian manifold $(M; g_M; J_M)$ to a Riemannian manifold $(N; g_N)$. Then π is said to be Invariant Riemannian Submersion if the vertical distribution is invariant with respect to the complex structure J_M . That is,

$$J_M \ker \pi = \ker \pi;$$

This follows that;

Theorem 2.1. [7] If $\pi : (M; g_M; J_M) \rightarrow (N; g_N; J_N)$ is a Riemannian Submersion from Almost Hermitian manifold onto an Almost Hermitian manifold. Then the horizontal distribution is invariant with respect to J_M .

Almost Hermitian submersion that exists between Almost Hermitian manifolds was define by Watson. He showed that the base manifold and the fibers of such submersion have similar structures with the total space.[3].

Definition 2.3. [21] Suppose that M is an Almost Hermitian manifold with a hermitian metric g_M and almost complex structure J . Let N be a Riemannian manifold with a Riemannian metric g_N . If there is a Riemannian Submersion $\pi : M \rightarrow N$ with the property that, $J \ker \pi \perp (\ker \pi)_p$. Then π is called Anti-Invariant Riemannian Submersion.

The distribution $\ker \pi$ is integrable, it means that the manifold $\pi^{-1}(q)$, $q \in N$ of $\ker \pi$ is a totally submanifold of M . From Definition 2.3, $J(\ker \pi) \perp \ker \pi = f \circ g$. If we represent the complementary orthogonal distribution to $J \ker \pi$ in $(\ker \pi)_p$ by π . We obtain;

$$(\ker \pi)_p = J \ker \pi \oplus \pi;$$

Obviously π is an invariant distribution of $(\ker \pi)_p$ with respect to the complex structure J . That is for any $F \in \pi(\ker \pi)_p$,

$$JF = GF + CF; \quad (2.1)$$

where $GF \in \pi(\ker \pi)$ and $CF \in \pi(\pi)$. If does not exist, then π is called a Lagrangian submersion.

Definition 2.4. [22] Let $\pi : (M; g_M; J) \rightarrow (N; g_N)$ be a Riemannian map from an Almost Hermitian manifold to a Riemannian manifold. Then π is called a Semi-Invariant Riemannian map if there are distributions $D_1, D_2 \perp \ker \pi$ such that

$$\ker \pi = D_1 \oplus D_2; \quad (2.2)$$

and

$$JD_1 = D_1; JD_2 \perp (\ker \pi)_p; \quad (2.3)$$

D_2 is orthogonal complementary to D_1 in $\ker \pi$.

If we let π be the complementary orthogonal distribution to $J \ker \pi$ in $(\ker \pi)_p$. Then

$$(\ker \pi)_p = JD_2 \oplus \pi;$$

It is obvious that π is an invariant distribution of $(\ker \pi)_p$, with respect to the complex structure J .

Definition 2.5. [6] Let $\pi : (M; g_M; J) \rightarrow (N; g_N)$ be a Riemannian Submersion from an Almost Hermitian manifold onto a Riemannian manifold. If the angle $\theta(X)$ between the space $\ker \pi_p$ and JX is a constant for any non-zero vector $X \in \ker \pi_p$, $p \in M$, that is if θ is independent of the choice of the point $p \in M$ and choice of the tangent vector X in $\ker \pi_p$. Then π is a slant-submersion.

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θ is called the slant angle of the slant submersion. Following the definition, the fibers of the slant submersion are slant submanifold of M . If $0 < \theta < \frac{\pi}{2}$

then the slant submersion is called a proper

slant submersion.

If $\pi : (M; g_M; J) \rightarrow (N; g_N)$ is a slant submersion from a Almost Hermitian manifold onto a Riemannian manifold, then for any $X \in \pi(\ker \pi)$, we have;

$$JX = \pi X + !X;$$

πX is a vertical part and $!X$ is the horizontal parts of JX .

Similarly, given any $F \in \pi((\ker \pi)_p)$,

$$JF = GF + CF;$$

GF is the vertical component and CF is the horizontal component of JF . (see [6] for more detail).

Definition 2.6. [18] A Riemannian Submersion $\pi : (M; g_M; J) \rightarrow (N; g_N)$ from an Almost Hermitian manifold to a Riemannian manifold is called a Semi-Slant submersion if there are some distributions

$D_1; D_2 \perp \ker \pi$ with the property that

$$\ker \pi = D_1 \oplus D_2; J(D_1) = D_1;$$

and the angle $\theta = \angle(X)$ between the spaces D_2 and JX is constant for any non-zero vector $X \in D_2$.

D_2 is the orthogonal complement of D_1 in $\ker \pi$.

Definition 2.7. [28] Let $(M; g_M; J)$ be an even dimensional Almost Hermitian manifold whose Hermitian metric is g_M and almost complex structure J_M , and $(N; g_N)$ be a Riemannian manifold with Riemannian metric g_N . A Riemannian Submersion $\pi : (M; g_M; J) \rightarrow (N; g_N)$ is a Hemi-Slant submersion if the vertical distribution $\ker \pi$ of π has two orthogonal complementary distributions D_\perp and D_\parallel such that D_\perp is a slant distribution and D_\parallel is anti-invariant distribution, i.e

$$\ker \pi = D_\perp \oplus D_\parallel;$$

The angle θ in the slant distribution is called the Hemi-Slant angle of the submersion.

3 Quasi-Hemi-Slant Submersion

Here we recall the modified submersion introduced by S.Longwap at el, which is the generalization of Hemi-Slant, Semi-Slant and Semi-Invariant Riemannian Submersion from Almost Hermitian manifold to a Riemannian manifold [15]. This type of Riemannian Submersion has been extended to other manifolds. See [19], [9],[14], [20], [2] and [13] for more details.

Theorem 3.1. [15] Suppose that $(M; g_M; J)$ is a $2m$ -dimensional Almost Hermitian manifold, J is the complex structure and g_M is the metric in M . Let $(N; g_N)$ be a Riemannian manifold with its Riemannian metric g_N . There exists a Riemannian Submersion $\pi : (M; g_M; J) \rightarrow (N; g_N)$, where the vertical distribution $\ker \pi$ is a direct sum of three orthogonal distributions D , D_\perp and D_\parallel , such that D is invariant, D_\perp is slant and D_\parallel is anti-invariant. That is;

$$\ker \pi = D \oplus D_\perp \oplus D_\parallel; \quad (3.1)$$

with $JD = D$, the angle θ between JD_\perp and D_\perp is a constant and $JD_\parallel \perp (\ker \pi)$.

If the dimension of D is m_1 ; dimension of D_\perp is m_2 and the dimension of D_\parallel is m_3 , then we observe the following:

(a) If $m_1 = 0$, then $\ker \pi = D_\perp \oplus D_\parallel$ and M is a Hemi-Slant submersion

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(b) If $m_2 = 0$, then $\ker \pi = D \oplus D_\parallel$; M is a Semi-Invariant submersion

(c) If $m_3 = 0$, then $\ker \pi = D \oplus D_\perp$; M is a Semi-Slant submersion

The Riemannian submersion in Theorem 3.1 is called Quasi-Hemi-Slant (QHSS) submersion and the angle θ is called the Quasi-Hemi-Slant angle (QHSSA) of the submersion. The Hemi-Slant, Semi-Invariant and Semi-Slant Riemannian submersions are special cases of the Quasi-Hemi-Slant submersion(QHSS). The Quasi-Hemi-Slant submersion $\pi : (M; g_M; J) \rightarrow (N; g_N)$ is proper Quasi-Hemi-Slant submersion if

$$\dim(D) \neq 0, \dim(D_\parallel) \neq 0, \dim(D_\perp) \neq 0 \text{ and } 0 < \theta < \frac{\pi}{2}.$$

We observe that, Hemi-Slant submersions, Semi-Invariant submersions, and Semi-Slant submersions are all special cases of Quasi-Hemi-Slant submersion.

Let $\pi : (M; g_M; J) \rightarrow (N; g_N)$ be a Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold. Then we know that the tangent bundle is a direct sum of its vertical distribution $\ker \pi$ and its horizontal distribution $(\ker \pi)^\perp$ that is,

$$TM = \ker \pi \oplus (\ker \pi)^\perp; \quad (3.2)$$

If we define the projections P and Q on the tangent bundle TM of M by $P : TM \rightarrow \ker \pi$ and

$Q : TM \rightarrow (\ker \pi)^\perp$, respectively, then for all $X \in TM$,

$$X = PX + QX; \quad (3.3)$$

in this equation $PX \in \ker \pi$ and $QX \in (\ker \pi)^\perp$.

For any vector field $X \in (\ker \pi)^\perp$, it can be written in the form

$$X = PX + QX + RX; \quad (3.4)$$

in this equation P, Q, R are projections of $\ker \pi$ onto D, D_\perp and D_\parallel respectively. Since we know that

$$JX = \pi X + !X; \quad (3.5)$$

for $X \in (\ker \pi)^\perp$ and $!X \in (\ker \pi)^\perp$. From (3.4) and (3.5), we obtain the equation

$$JX = \pi PX + !PX + \pi QX + !QX + \pi RX + !RX;$$

Since $JD = D$ and $JD_\parallel \perp (\ker \pi)$, we have $!PX = 0$ and $\pi RX = 0$, and so

$$JX = \pi PX + \pi QX + !QX + !RX; \quad (3.6)$$

This shows that

$$J \ker \pi = D \oplus D_\perp \oplus !D_\parallel JD_\parallel; \quad (3.7)$$

Since $!D_\perp \perp (\ker \pi)$ and $JD_\parallel \perp (\ker \pi)$, we obtain

$$(\ker \pi)^\perp = !D_\parallel JD_\parallel; \quad (3.8)$$

π is the orthogonal complement of $!D_\parallel JD_\parallel$ in $(\ker \pi)^\perp$. The distribution π is invariant with respect to complex structure J . Similarly, given any vector $F \in (\ker \pi)^\perp$, we can write

$$JF = GF + CF: (3.9)$$

where $GF \in \Gamma(D_\perp)$ and that $CF \in \Gamma(D_\perp)$.

Therefore, from the equations (3.4), (3.8) and (3.9), we get the following.

Lemma 3.2. Let $\pi : (M; g_M; J) \rightarrow (N; g_N)$ be a Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold. Then, the following equations hold;

$$D_\perp = D; \quad D_\perp = f_0g; \quad G!D_\perp = D; \quad G!D_\perp = D:$$

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On the other hand, if we compare the normal and the tangential components of equations (3.4), (3.9) together with the fact that $J^2 = -I$, the following results are obtain

Theorem 3.3. Let π and J , G and C be endomorphisms in the tangent bundle of the manifold $(M; g_M; J)$. Then the following equations hold;

$$(i) \quad D_\perp + G! = -I, \quad (ii) \quad D_\perp + C! = 0,$$

$$(iii) \quad !G + C_2 = -I, \quad (iv) \quad D_\perp G + GC = 0,$$

in these equations, I represents the identity operator on the total space of π .

Proof. This result is from Lemma 3.2.

Lemma 3.4. Let $\pi : (M; g_M; J) \rightarrow (N; g_N)$ be a proper Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold. Then we have the following;

$$\cos^2 \theta_X = \cos^2 \theta_X; \quad (3.10)$$

for all vector fields $X, Y \in \Gamma(D_\perp)$, θ is the Quasi-Hemi-Slant angle and π is the endomorphism in theorem 3.3

Proof. We let θ be Quasi-Hemi-Slant angle between JX and πX . Then

$$\cos \theta =$$

$$\frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$; \cos \theta =$$

$$\frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$\text{so these give } \cos^2 \theta =$$

$$\frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

From the properties of the complex structure J it follows that;

$$\cos^2 \theta = \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$; \cos^2 \theta = \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

which shows that

$$\cos^2 \theta = \frac{g(\pi X, \pi X)}{g(JX, JX)} \quad (3.11)$$

Lemma 3.5. Let $\pi : (M; g_M; J_M) \rightarrow (M; g_N)$ be a Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold. If θ is the Quasi-Hemi-Slant angle between JX and πX ; $X \in D_\perp \cap \ker \pi$: Then the following equations hold;

$$1. \quad g(\pi X, \pi X) = \cos^2 \theta g(X, X) \text{ for all } X \in D_\perp$$

$$2. \quad g(!X, !X) = \sin^2 \theta g(X, X) \text{ for all } X \in D_\perp$$

$$3. \quad g(!X, !X) = g(X, X) \text{ for every } X \in D_\perp$$

Proof. We assume θ is the angle between JX and πX . Then,

$$1. \quad \cos \theta =$$

$$\frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$; \cos^2 \theta =$$

$$\frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$= \frac{g(\pi X, \pi X)}{g(JX, JX)}$$

$$\text{so that } g(X, X) \cos^2 \theta = g(\pi X, \pi X)$$

Similarly

$$2. \quad \sin \theta =$$

$$\frac{g(!X, !X)}{g(X, X)}$$

$$= \frac{g(!X, !X)}{g(X, X)}$$

$$; \sin^2 \theta =$$

$$\frac{g(!X, !X)}{g(X, X)}$$

$g(JX; JX) = g(X; X)$
 $g(JX; JX) = g(X; X)$
 and so $g(X; X) = g(JX; JX)$
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3. If $JX = !X$ for every $X \in D_x$, then

$$g(X; X) = g(JX; JX) = g(!X; !X) \quad (3.12)$$

Theorem 3.6. If $\pi : (M; g_M; J_M) \rightarrow (N; g_N)$ is a proper Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold and suppose $e_1; e_2; \dots; e_{m-n}$ is the local orthonormal basis of $\ker \pi$. Then

1. $\{ \sec \pi e_1; \sec \pi e_2; \dots; \sec \pi e_{m-n} \}$ is a local orthonormal basis of $\pi(D_x)$.

2. $\{ \csc \pi e_1; \csc \pi e_2; \dots; \csc \pi e_{m-n} \}$ is a local orthonormal basis of $!D_x$.

Proof. 1. In this proof, we only need to that $g_M(\sec \pi e_i; \sec \pi e_j) = \delta_{ij}$, for all $i; j \in \{1; 2; \dots; m-n\}$. We have from Lemma 3.5 that

$$g_M(\sec \pi e_i; \sec \pi e_j) = \sec^2 \pi g_M(e_i; e_j) = \sec^2 \pi \cos^2 \pi g_M(e_i; e_j) = g_M(e_i; e_j) = \delta_{ij} \quad (3.13)$$

2. in a similarly way we show that $g_M(\csc \pi e_i; \csc \pi e_j) = \delta_{ij}$.

$$g_M(\csc \pi e_i; \csc \pi e_j) = \csc^2 \pi g_M(e_i; e_j) = \csc^2 \pi \sin^2 \pi g_M(e_i; e_j) = g_M(e_i; e_j) = \delta_{ij} \quad (3.14)$$

as required.

Corollary 3.7. Let $\pi : (M; g_M; J_M) \rightarrow (N; g_N)$ be a Quasi-Hemi-Slant Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold. Let $e_1; e_2; \dots; e_{m-n}$ be a local orthonormal basis of $\ker \pi$. Then

1. $\{ e_1; \sec \pi e_1; e_2; \sec \pi e_2; \dots; e_{m-n}; \sec \pi e_{m-n} \}$ is a local orthonormal basis of $\pi(\ker \pi)$.

2. $\{ \csc \pi e_1; !e_1; \csc \pi e_2; !e_2; \dots; \csc \pi e_{m-n}; !e_{m-n} \}$ is a local orthonormal basis of $!(\ker \pi)$.

Proof. 1. We are to show that $g_M(\sec \pi e_i; \sec \pi e_j) = \delta_{ij}$. this result comes from theorem 3.6 and already $g_M(e_i; e_j) = \delta_{ij}$

2. Similarly, we show that $g_M(\csc \pi e_i; \csc \pi e_j) = \delta_{ij}$ and $g(!e_i; !e_j) = \delta_{ij}$. This is already in theorem 3.6 and from Lemma 3.5 we have

$$g_M(!e_i; !e_j) = g_M(e_i; e_j) = \delta_{ij}$$

and this ends the proof.

4 Fundamental Tensor fields

Recall that the tangent space TM of a Manifold M is split into horizontal and vertical subspaces.

That is

$$TM = VTM \oplus HTM;$$

HTM are the horizontal vectors and VTM are the vertical vectors of the tangent space TM . Riemannian submersion $\pi : (M; g_M) \rightarrow (N; g_N)$ determines two (1; 2)-tensors fields T and A called the fundamental tensor fields of π : The tensor fields are defined by the horizontal projection $H : TM \rightarrow HTM$ and

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vertical projection $V : TM \rightarrow VTM$ according to the formula:

$$A_E F = H r_H E V F + V r_H E H F;$$

$$T_E F = H r_V E V F + V r_V E H F; \quad (4.1)$$

for any vector fields E, F on M , where r is the Levi-Civita connection of g_M . See [8]. T_E and A_E are skew-symmetric operators on the tangent bundle TM : If the vertical and the horizontal vectors are reversed, then we can now summarize the properties of the tensor fields T and A . Let π_* be vertical and π^* be horizontal vector fields on M , then we have

$$\begin{aligned}
 \pi_* \pi^* &= T \pi^* + V r \pi^*; \\
 \pi_* \pi^* &= T \pi^* + H r \pi^*; \\
 \pi_* \pi^* &= A \pi^* + V r \pi^*; \\
 \pi_* \pi^* &= A \pi^* + H r \pi^* + (4.2)
 \end{aligned}$$

We observe that T acts on the fibers as the second fundamental form, and A acts on the horizontal distribution as measures of the obstruction to integrability of the distribution. For details on fundamental tensor fields, we refer to [17].

Let $(M; g_M; J)$ be hermitian manifold with the complex structure J ; and $(N; g_N)$ be a Riemannian manifold. The complex structure effects the fundamental tensor fields T and A of the submersion.

Proposition 4.1. [15] Let $\pi : (M; g_M; J) \rightarrow (N; g_N)$ be a quasi-hemi-slant submersion from a Kaehler manifold onto a Riemannian manifold. Then we have the following equations:

$$V r \pi^* + T ! \pi^* = \pi_* V r \pi^* + G T \pi^*; \quad (4.3)$$

$$T \pi^* + H r ! \pi^* = ! V r \pi^* + C T \pi^*;$$

$$\begin{aligned} V r_{\perp} + A_{\perp} C_{\perp} &= A_{\perp} + G H r_{\perp}; \\ A_{\perp} G_{\perp} + H r_{\perp} C_{\perp} &= ! A_{\perp} + C H r_{\perp}; \\ V r_{\perp} G_{\perp} + T_{\perp} C_{\perp} &= T_{\perp} + G H r_{\perp}; \\ T_{\perp} G_{\perp} + H r_{\perp} C_{\perp} &= ! T_{\perp} + C H r_{\perp}; \end{aligned} \quad (4.4)$$

where $\perp, \perp \in \ker(\perp)$ and $\perp, \perp \in \ker(\perp)$, G and C are projections of vectors in $\perp(\ker \perp)$ to vertical and horizontal vectors respectively while V and H are projections of vectors in $\perp(\ker \perp)$ to vertical and horizontal vectors respectively.

Corollary 4.1. If \perp is a Quasi-Hemi-Slant submersion from a manifold $(M; g_M; J)$ onto a Riemannian manifold $(N; g_N)$, then

$$\begin{aligned} (r_X)_{\perp} Y &:= V r_X Y \perp V r_X Y; \\ (r_X!)_{\perp} Y &:= H r_X! Y \perp ! V r_X Y; \\ (r_Z C)_{\perp} W &:= H r_Z C W \perp C H r_Z W; \\ (r_Z G)_{\perp} W &:= V r_Z G W \perp G H r_Z W; \end{aligned} \quad (4.5)$$

for all $X; Y \in \perp(\ker \perp)$ and $Z; W \in \perp(\ker \perp)$.

Thus, we have the following:

Corollary 4.2. Let \perp be a quasi-hemi-slant submersion from a Kaehler manifold $(M; g_M; J)$ onto a Riemannian manifold $(N; g_N)$. Then we have

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- (i) $(r_X)_{\perp} Y = G T_X Y \perp T_X! Y$,
- (ii) $(r_X!)_{\perp} Y = C T_X Y \perp T_X Y$,
- (iii) $(r_Z C)_{\perp} W = ! A_Z W \perp A_Z G W$,
- (iv) $(r_Z G)_{\perp} W = A_Z W \perp A_Z C W$,

for all vectors $X; Y \in \perp(\ker \perp)$ and $Z; W \in \perp(\ker \perp)$.

If the tensors \perp and $!$ are parallel with respect to the linear connection r on M , then

$$G T_X Y = T_X! Y;$$

and

$$C T_X Y = T_X Y;$$

for all vectors $X, Y \in \perp(TM)$.

For the integrability of distributions and decomposition theorem on the Quasi-Hemi-Slant submersion, we refer to [15] for the details.

5 CONCLUSIONS

Let $(M; g_M; J)$ be a $2m$ -dimensional Almost Hermitian manifold with g_M a Riemannian metric on M and almost complex structure J , and $(N; g_N)$ be a Riemannian manifold with Riemannian metric g_N . Then there is a Riemannian Submersion $\perp : (M; g_M; J) \rightarrow (N; g_N)$ such that its vertical distribution $\ker \perp$ admits three orthogonal distributions D, D_{\perp} and D_{\perp} which are invariant, slant and anti-invariant respectively. Let \perp be a Quasi-Hemi-Slant submersion from an almost Hermitian manifold $(M; g_M; J)$ onto a Riemannian manifold $(N; g_N)$. Let $e_1; e_2; \dots; e_{m \perp n}$ be a local orthonormal basis of $\ker \perp$. Then

1. $e_1; \sec \perp e_1; e_2; \sec \perp e_2; \dots; e_{m \perp n}; \sec \perp e_{m \perp n}$ is a local orthonormal basis of $\perp(\ker \perp)$.
2. $f_{csc} \perp e_1; ! e_1; csc \perp e_2; ! e_2; \dots; csc \perp e_{m \perp n}; ! e_{m \perp n}$ is a local orthonormal basis of $!(\ker \perp)$.

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