Review Article

LOCAL ORTHONORMAL BASIS OF THE FIBRE OF QUASI-HEMI-SLANT RIEMANNIAN SUBMERSION

ABSTRACT. We recall the notions of semi-invariant, semi-slant, hemi-slant and quasi-hemi-slant Riemannian submersions from almost Hermitian manifolds to a Riemannian manifolds. In this paper we find some local orthonormal Basaes for the fibre of quasi-Hemi-Slant Riemannian submersion which is a generalizes hemi-slant, semi-slant and semi-invariant Riemanian submersions from almost Hermitian manifold to a Riemannian manifold.

1. INTRODUCTION

The importance of Immersions and Submersions in Riemannian geometry and other aspects of the differential geometry cannot be over emphasized. Riemannian Submersion was initiated by O'Neill [1] and Gray [2]. Later submersions have been studied widely in differential geometry; for details, see [3]. Riemannian submersions between Riemannian manifolds equipped with an additional structure of almost complex type was first studied by Watson [4]. He defined almost Hermitian submersions between almost Hermitian manifolds and showed that in most cases the base manifold and each fiber have the same kind of structure as the total space. Some Geometers have come up with several other kinds of submersions according to to certain conditions associated with the submersions. Examples are; invariant submersion [5], anti-invariant submersion [6], semi-invariant submersion [7], Submersion of Semi-Invariant Submanifolds of Trans-Sasakian Manifold [8], Geometry of slant submaniflods [9], slant submersions from almost Hermitian manifolds [10], semi-slant submersion [11], Riemannian submersions from almost contact metric manifolds [12], quaternionic submersion [13], almost h-slant submersion and h-slant submersion [14], Hemi-slant submersions [15] and and the one we introduced; On quasi-hemi-slant Riemannian submersion [22].

Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([16], [17]), Kaluza-Klein theory ([18], [19]), Supergravity and superstring theories [20], [21]).

In [22] we introduced a new class of Riemannian submersion; quasi-Hemi-Slant Riemannian Submersion, which is a generalization of Hemi-slant, semi-invariant and semi-slant submersions defined on almost Hermitian manifolds. In this paper

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we introduce local Orthonormal Basis of the fibre for quasi-Hemi-Slant defined on almost Hermitian manifolds.

We organize the work as follows; In Section 2, we recall some basic definitions and important results of some Riemannian submersions. In Section 3 we recall the new Riemannan submersion [22] and introduce orthogornal base for the class. Finally in section 4 we relate the geometry of the fibre to the fundamental tensors.

2. PRELIMINERIES

Here we give definition of Almost Hermitian Manifolds, and recall the necessary background and definition of Riemannian Submersions from an almost hermitian manifold to a Riemannian manifold.

Let (M, g) be an $n-$dimensional Riemannian manifold that admits a tensor field J of type $(1, 1)$ on (M, g) such that, $\forall X, Y \in \Gamma(TM)$

$$J^2 = -I \quad g(JX, JY) = g(X, Y)$$

Then (M, g) is called an almost hermitian manifold, J is called an almost complex structure.

If the almost complex structure J satisfies $(\bar{\nabla}_X J)Y = 0$, for every $X, Y \in \Gamma(TM)$ where $\bar{\nabla}$ is the Levi-Civita connection on M, then M is said to be a Kaehler manifold. The covariant derivative of the complex structure J is defined as;

$$\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y.$$

Thus, if M is a Kaehler manifold then $\bar{\nabla}_X JY = J\bar{\nabla}_X Y$.See [4] for details.

Definition 2.1. [3]*Let* (M, g) *be an* $m-$*dimensional Riemannian manifold and* (N, g) *be an* $n-$*dimensional Riemannian manifolds, where* $dim(M) > dim(N)$. *A surjective map* $\pi : (M, g) \to (N, g)$ *is called a Riemannian submersion if:*

(1) π *has maximal rank,*
(2) π_* *restricted to* $(ker\pi_*)^{\perp}$ *is a linear isometry.*

Here we have for each $y \in N$, $\pi^{-1}(y)$ is a $k-$dimensional submanifold of M called a fiber, where $k = dim(M) - dim(N)$. A vector field on M is called vertical if it is tangent to the fibers, and is horizontal if it is orthogonal to fibers.

Definition 2.2. [5, p. 84] Let π be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J_M) onto a Riemannian manifold (N, g_N). Then we say that π is an *invariant Riemannian submersion* if the vertical distribution is invariant with respect to the complex structure J_M, i.e,

$$J_M \ker \pi_* = \ker \pi_*.$$

From this, we have the following.

Theorem 2.3. [5] *Let* π *be a Riemannian submersion from an almost Hermitian manifold* (M, g_M, J_M) *onto an almost Hermitian manifold* (N, g_N, J_N). *Then the horizontal distribution is invariant with respect to* J_M.

Watson defined almost Hermitian submersions between almost Hermitian manifolds and showed that in most cases, the base manifold and each fiber have the same kind of structure as the total space [4].

Definition 2.4. [6] Let M be a complex finite-dimensional almost Hermitian manifold with Hermitian metric g_M and almost complex structure J and N be a Riemannian manifold with Riemannian metric g_N. Suppose that there exists a Riemannian submersion $\pi : M \longrightarrow N$ such that, $J \ker \pi_* \subseteq (\ker \pi_*)^{\perp}$. Then we say that π is an *anti-invariant Riemannian submersion.*

First of all, since the distribution $\ker \pi_*$ is integrable, the above definition implies that the integral manifold $\pi^{-1}(q)$, $q \in N$ of $\ker \pi_*$ is a totally submanifold of M. From Definition 2.4, we have $J(\ker \pi_*)^{\perp} \cap \ker \pi_* \neq \{0\}$. We denote the complementary orthogonal subbundle to $J \ker \pi_*$ in $(\ker \pi_*)^{\perp}$ by ν. Then we have that;

$$(\ker \pi_*)^{\perp} = J \ker \pi_* \oplus \nu.$$

We observe that ν is an invariant subbundle of $(\ker \pi_*)^{\perp}$ with respect to the complex structure J. This implies for any $Z \in \Gamma(ker\pi_*)^{\perp}$,

$$JZ = BZ + CZ, \tag{2.1}$$

where $BZ \in \Gamma(\ker \pi_*)$ and $CZ \in \Gamma(\nu)$. If $\nu = \{0\}$, then π is called a *Langragian submersion.*

Definition 2.5. [7] Let π be a Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then we say that π is a semi-invariant Riemannian map if there is a distribution $\mathcal{D}_1 \subseteq \ker \pi_*$ such that

$$\ker \pi_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \tag{2.2}$$

and

$$J\mathcal{D}_1 = \mathcal{D}_1, \quad J\mathcal{D}_2 \subseteq (\ker \pi_*)^{\perp}, \tag{2.3}$$

where $\mathcal{D}_2$ is orthogonal complementary to $\mathcal{D}_1$ in $\ker \pi_*$.

Let ν be the complementary orthogonal subbundle to $J \ker \pi_*$ in $(\ker \pi_*)^{\perp}$. Then we have

$$(\ker \pi_*)^{\perp} = J\mathcal{D}_2 \oplus \nu.$$

It is obvious that ν is an invariant subbundle of $(\ker \pi_*)^{\perp}$ with respect to the complex structure J.

Definition 2.6. [10] Let π be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). If for any non-zero vector $X \in \ker \pi_{*p}$, $p \in M$, the angle $\theta(X)$ between JX and the space $\ker \pi_{*p}$ is constant, that is θ is independent of the choice of the point $p \in M$ and choice of the tangent vector X in $\ker \pi_{*p}$. Then we say that π is a *slant submersion.*

In this case, the angle θ is called the *slant angle* of the slant submersion. It follows from the above definition that the fibers of a slant submersion are slant submanifold of M. If the slant angle is $0 < \theta < \frac{\pi}{2}$ then the the submersion is called a proper slant submersion.

If π is a slant submersion from a almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N), then for any $X \in \Gamma(\ker \pi_*)$, we write

$$JX = \phi X + \omega X,$$

where ϕX and ωX are vertical and horizontal parts of JX, respectively.

Also for any $Z \in \Gamma((\ker \pi_*)^{\perp})$, we have

$$JZ = BZ + CZ,$$

where BZ, and CZ are vertical and horizontal components of JZ, respectively (see [10] for more detail).

Definition 2.7. [11] Let (M, g_M, J) be an almost Hermitian manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\pi : (M, g_M, J) \longrightarrow (N, g_N)$ is called a *semi-slant submersion* if there is a distribution $\mathcal{D}_1 \subset \ker \pi_*$ such that

$$\ker \pi_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for non-zero $X \in \mathcal{D}_2$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker \pi_*$.

Let $\pi : (M, g_M, J) \longrightarrow (N, g_N)$ be a semi-slant submersion. Then for any $X \in \Gamma(\ker \pi_*)$, we have

$$X = PX + QX,$$

where $PX \in \Gamma(\mathcal{D}_1)$ and $QX \in \Gamma(\mathcal{D}_2)$. Applying J to this equation yields

$$JX = \phi X + \omega X,$$

where $\phi X = JPX \in \Gamma(\ker \pi_*)$ and $\omega X = JQX \in \Gamma((\ker \pi_*)^{\perp})$. For any $U \in \Gamma((\ker \pi_*)^{\perp})$, we write

$$U = \mathcal{V}U + \mathcal{H}U,$$

where $\mathcal{V}U \in \Gamma(\ker \pi_*)$ and $\mathcal{H}U \in \Gamma((\ker \pi_*)^{\perp})$. Then we have

$$(\ker \pi_*)^{\perp} = \omega \mathcal{D}_2 \oplus \mu,$$

where μ is the orthogonal complement of $\omega \mathcal{D}_2$ in $(\ker \pi_*)^{\perp}$ and is invariant under J. For more details on semi-slant submersion see [11] and references therein.

Definition 2.8. [15] Let M be a $2m-$dimensional almost Hermitian manifold with Hermitian metric g and almost complex structure J, and N be a Riemannian manifold with Riemannian metric g_N. A Riemannian submersion $\pi : (M, g, J) \rightarrow (N, g_N)$ is called a hemi-slant submersion if the vertical distribution $\ker \pi_*$ of π admits two orthogonal complementary distributions D^{θ} and $D^{\perp}$ such that D^{θ} is slant and $D^{\perp}$ is anti-invariant, i.e, we have

$$\ker \pi_*) = D^{\theta} \oplus D^{\perp},$$

where J

In this case, the angle θ is called the hemi-slant angle of the submersion.

3. QUASI-HEMI-SLANT SUBMERSIONS

In this section we racall the modified submersion introduced in [22] which we refered to as the generalization of hemi-slant, semi-slant and semi-invariant Riemanian submersions from almost Hermitian manifold to a Riemannian manifold.

Theorem 3.1. [22] *Let M be a $2m$-dimensional almost Hermitian manifold with g_M a Riemannian metric on M and almost complex structure J, and N be a Riemannian manifold with Riemannian metric g_N. Then there is a Riemannian submersion $\pi : (M, g_M, J) \longrightarrow (N, g_N)$ such that its vertical distribution $\ker \pi_*$ admits three orthogonal distributions $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ which are invariant, slant and anti-invariant respectively, i.e.*

$$\ker \pi_* = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \mathcal{D}^\perp, \tag{3.1}$$

with $J\mathcal{D} = \mathcal{D}$, the angle θ between $J\mathcal{D}^\theta$ and $\mathcal{D}^\theta$ being constant and $J\mathcal{D}^\perp \subseteq (\ker \pi_*)^\perp$.

If we denote the dimension of $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by m_1, m_2 and m_3, respectively, then we easily see the following particular cases:

(a) If $m_1 = 0$, then M is a Hemi-slant submersion [].
(b) If $m_2 = 0$, then M is a semi-invariant submersion [].
(c) If $m_3 = 0$, then M is a semi-slant submersion [].

The submersion in Theorem 3.1 is called *Quasi-Hemi-Slant (SHS) submersion* and the angle θ is called the *quasi-hemi-slant angle* of the submersion. This means that a quasi-hemi-slant submersion is a generalization of hemi-slant, semi-invariant and semi-slant submersions.

We say that the quasi-hemi-slant submersion $\pi : (M, g_M, J) \longrightarrow (N, g_N)$ is *proper* if $\mathcal{D} \neq \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$. From the above items, hemi-slant submersions, semi-invariant submersions, and semi-slant submersions are all examples of quasi-hemi-slant submersions.

Let π be a quasi-hemi-slant submersion from an almost hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we have

$$TM = \ker \pi_* \oplus (\ker \pi_*)^\perp. \tag{3.2}$$

Let us define the projections P and Q on the tangent vectors TM of M by $P : TM \longrightarrow \ker \pi_*$ and $Q : TM \longrightarrow (\ker \pi_*)^\perp$, respectively. Now, for any $X \in \Gamma(TM)$,

$$X = PX + QX, \tag{3.3}$$

where $PX \in \Gamma(\ker \pi_*)$ and $QX \in \Gamma((\ker \pi_*)^\perp)$.

Now, for any $X \in \Gamma(\ker \pi_*)$, the vector field X can be written as

$$X = \mathcal{P}X + \mathcal{Q}X + \mathcal{R}X, \tag{3.4}$$

where $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ are projections of $\ker \pi_*$ onto $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ respectively. Put

$$JX = \phi X + \omega X, \tag{3.5}$$

where $\phi X \in \Gamma(\ker \pi_*)$ and $\omega X \in \Gamma((\ker \pi_*)^\perp)$. From (3.4) and (3.5), we have

$$JX = \phi \mathcal{P}X + \omega \mathcal{P}X + \phi \mathcal{Q}X + \omega \mathcal{Q}X + \phi \mathcal{R}X + \omega \mathcal{R}X.$$

Since $J\mathcal{D} = \mathcal{D}$ and $J\mathcal{D}^\perp \subseteq (\ker \pi_*)^\perp$, we have $\omega PX = 0$ and $\phi RX = 0$, and so

$$JX = \phi \mathcal{P}X + \phi \mathcal{Q}X + \omega \mathcal{Q}X + \omega \mathcal{R}X. \tag{3.6}$$

This means that

$$J \ker \pi_* = \mathcal{D} \oplus \phi \mathcal{D}^\theta \oplus \omega \mathcal{D}^\theta \oplus J\mathcal{D}^\perp. \tag{3.7}$$

Since $\omega \mathcal{D}^\theta \subseteq (\ker \pi_*)^\perp$ and $J\mathcal{D}^\perp \subseteq (\ker \pi_*)^\perp$, we obtain

$$(\ker \pi_*)^\perp = \omega \mathcal{D}^\theta \oplus J\mathcal{D}^\perp \oplus \nu, \tag{3.8}$$

where ν is the orthogonal complement of $\omega \mathcal{D}^\theta \oplus J\mathcal{D}^\perp$ in $(\ker \pi_*)^\perp$ and it is invariant with respect to J. Also, for any $Z \in (\ker \pi_*)^\perp$, we have

$$JZ = \mathcal{B}Z + \mathcal{C}Z, \tag{3.9}$$

where $\mathcal{B}Z \in \Gamma(\omega \mathcal{D}^\theta \oplus J\mathcal{D}^\perp)$ and $\mathcal{C}Z \in \Gamma(\nu)$.

Therefore, using (3.4), (3.8) and (3.9), we get the following.

Lemma 3.2. *Let π be a quasi-hemi-slant submersion from an almost hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then, we have*

$$\phi \mathcal{D}^\theta = \mathcal{D}^\theta, \quad \phi \mathcal{D}^\perp = \{0\}, \quad \mathcal{B}\omega \mathcal{D}^\theta = \mathcal{D}^\theta, \quad \mathcal{B}\omega \mathcal{D}^\perp = \mathcal{D}^\perp.$$

On the other hand, comparing tangential and normal components in (3.4), (3.9) and the fact that $J^2 = -\mathbb{I}$, we obtain the following.

Theorem 3.3. *The endomorphisms ϕ and ω, $\mathcal{B}$ and $\mathcal{C}$ in the tangent bundle of (M, g_m, J) satisfy the following identities:*

(i) $\phi^2 + \mathcal{B}\omega = -\mathbb{I}$, *(ii)* $\omega\phi + \mathcal{C}\omega = 0$,
(iii) $\omega\mathcal{B} + \mathcal{C}^2 = -\mathbb{I}$, *(iv)* $\phi\mathcal{B} + \mathcal{B}\mathcal{C} = 0$,

where $\mathbb{I}$ is the identity operator on the total space of π.

Proof. This is a straight forward calculations from Lemma 3.2 □

We also have the following.

Lemma 3.4. *Let π be a quasi-hemi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then*

$$\cos^2 \theta X = -\phi^2 X, \tag{3.10}$$

for any X, $Y \in \Gamma(\mathcal{D}^\theta)$.

Proof. If θ is the slant angle between JX and ϕX. Then $cos(\theta) = \dfrac{\|\phi X\|}{\|JX\|}$, $\quad cos(\theta) = \dfrac{g(JX, \phi X)}{\|JX\| \|\phi X\|}$

and so $cos^2(\theta) = \dfrac{g(JX, \phi X)}{\|JX\|^2}$ from the properties of J we obtain the equation;

$$cos^2(\theta) = -\frac{g(X,\phi^2 X)}{g(JX,JX)}, \quad cos^2(\theta) = -\phi^2\frac{g(X,X)}{g(X,X)},$$

this implies that

$$cos^2(\theta) = -\phi^2 \tag{3.11}$$

□

Lemma 3.5. *Let π be a quasi-hemi-slant submersion from an almost Hermitian manifold (M, g_m, J_m) onto a Riemannian manifold (M, g_n). Let θ be the angle between JX and ϕX, $X \in \mathcal{D}^\theta \subset \ker \pi_*$. Then the following holds*

1. $g(\phi X, \phi X) = cos^2(\theta) g(X, X)$ *for all* $X \in D^\theta$
2. $g(\omega X, \omega X) = sin^2(\theta) g(X, X)$ *for all* $X \in D^\theta$
3. $g(\omega X, \omega X) = g(X, X)$ *for all* $X \in D^\perp$

Proof. Let θ be the angle between JX and ϕX. Then,

1. $cos(\theta) = \dfrac{\|\phi X\|}{\|JX\|}, \quad cos^2(\theta) = \dfrac{\|\phi X\|^2}{\|JX\|^2} \quad cos^2(\theta) = \dfrac{g(\phi X, \phi X)}{g(JX, JX)}$

   so that $g(X,X)cos^2(\theta) = g(\phi X, \phi X)$

   Similarly
2. $sin(\theta) = \dfrac{\|\omega X\|}{\|JX\|}, \quad sin^2(\theta) = \dfrac{\|\omega X\|^2}{\|JX\|^2} \quad sin^2(\theta) = \dfrac{g(\omega X, \omega X)}{g(JX, JX)}$

   and so $g(X,X)sin^2(\theta) = g(\omega X, \omega X)$
3. If $JX = \omega X$ for all $X \in D^\perp$, then

$$g(X, X) = g(JX, JX) = g(\omega X, \omega X). \tag{3.12}$$

□

Theorem 3.6. *Let π be a proper quasi-hemi-slant submersion from an almost Hermitian manifold (M, g_m, J_m) onto a Riemannian manifold (M, g_n). Let $\{e_1, e_2, ..., e_{m-n}\}$ be a local orthonormal basis of $ker\pi_*$. Then*

1. $\{sec\theta\phi e_1, sec\theta\phi e_2, ..., sec\theta\phi e_{(m-n)}\}$ *is a local orthonormal basis of* $\phi\mathcal{D}^\theta$.
2. $\{\csc\theta\phi e_1, \csc\theta\phi e_2, ..., \csc\theta\phi e_{(m-n)}\}$ *is a local orthonormal basis of* $\omega\mathcal{D}^\theta$.

Proof. (1) It is enough to show that $g_m(sec\theta\phi e_i, sec\theta\phi e_j) = \delta_{ij}^{\mathcal{D}}$, for any $i, j \in \{1, 2, ..., m-n\}$. We have from Lemma 3.5

$$\begin{aligned} g_m(sec(\theta)\phi e_i, sec(\theta)\phi e_j) &= sec^2(\theta)\phi^2 g_m(e_i, e_j) \\ &= sec^2(\theta)cos^2(\theta) g_m(e_i, e_j) = g_m(e_i, e_j) = \delta_{ij} \end{aligned} \tag{3.13}$$

(2) in a similarly way we show that $g_m(csc\theta\omega e_i, csc\theta\omega e_j) = \delta_{ij}$. We have

$$\begin{aligned} g_m(csc(\theta)\omega e_i, csc(\theta)\omega e_j) &= csc^2(\theta)\omega^2 g_1(e_i, e_j) \\ &= csc^2(\theta)sin^2(\theta) g_m(e_i, e_j) = g_m(e_i, e_j) = \delta_{ij} \end{aligned} \tag{3.14}$$

which proves the assertions. □

We have a similar assertion;

Corollary 3.7. *Let π be a quasi-hemi-slant submersion submersion from an almost Hermitian manifold (M, g_m, J_m) onto a Riemannian manifold (M, g_n). Let $\{e_1, e_2, ..., e_{m-n}\}$ be a local orthonormal basis of $ker\pi_*$. Then*

1. $\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, ..., e_{m-n}, \sec\theta\phi e_{m-n}\}$ *is a local orthonormal basis of* $\phi(ker\pi_*)$.
2. $\{csc\theta\omega e_1, \omega e_1, csc\theta\omega e_2, \omega e_2, ..., csc\theta\omega e_{(m-n)}, \omega e_{(m-n)}\}$ *is a local orthonormal basis of* $\omega(ker\pi_*)$.

Proof. (1) We are to show that $g_m(sec\theta\phi e_i, sec\theta\phi e_j) = \delta_{ij}$. this is straight forward from theorem 3.6 and already $g_m(e_i, e_j) = \delta_{ij}$

(2) Similarly, we show that $g_m(csc\theta\phi e_i, csc\theta\phi e_j) = \delta_{ij}$, and $g(\omega e_i, \omega e_j) = \delta_{ij}$. This is already in theorem 3.6 and from Lemma 3.5 we have that

$$\begin{aligned} g_m(\omega e_i, \omega e_j) &= g_m(e_i, e_j) \\ &= \delta_{ij} \end{aligned}$$

which proves the assertions. □

4. FUNDAMENTAL TENSOR FIELDS

Riemannan submersion $\pi : (M, g^M) \to (N, g^N)$ determines two $(1,2)-$tensors fields $\mathcal{T}$ [3] we can Define the tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\begin{aligned} \mathcal{A}_E F &= \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \\ \mathcal{T}_E F &= \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \end{aligned} \tag{4.1}$$

for any vector fields E, F on M, where ∇ is the Levi-Civita connection of g_M. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let U, V be vertical and ξ, η be horizontal vector fields on M, then we have

$$\nabla_U V = \mathcal{T}_U V + \mathcal{V}\nabla_U V, \tag{4.2}$$

$$\nabla_U \xi = \mathcal{T}_U \xi + \mathcal{H}\nabla_U \xi, \tag{4.3}$$

$$\nabla_\xi U = \mathcal{A}_\xi U + \mathcal{V}\nabla_\xi U, \tag{4.4}$$

$$\nabla_\xi \eta = \mathcal{H}\nabla_\xi \eta + \mathcal{A}_\xi \eta, \tag{4.5}$$

where $\mathcal{H}\nabla_V \xi = \mathcal{A}_\xi V$, if ξ is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on Riemannian submersions, we refer to O'Neill's paper [1].

Let (N, g_N) be a Riemannian manifold. We now examine how the Kählerian structure on M effects the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a quasi-hemi-slant submersion $\pi : (M, g, J) \longrightarrow (N, g_N)$.

Proposition 4.1. [22] *Let π be a quasi-hemi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then*

$$\mathcal{V}\nabla_U\phi V + \mathcal{T}_U\omega V = \phi\mathcal{V}\nabla_U V + \mathcal{B}\mathcal{T}_U V, \tag{4.6}$$

$$\mathcal{T}_U\phi V + \mathcal{H}\nabla_U\omega V = \omega\mathcal{V}\nabla_U V + \mathcal{C}\mathcal{T}_U V, \tag{4.7}$$

$$\mathcal{V}\nabla_\xi\eta + \mathcal{A}_\xi\mathcal{C}\eta = \phi\mathcal{A}_\xi\eta + \mathcal{B}\mathcal{H}\nabla_\xi\eta, \tag{4.8}$$

$$\mathcal{A}_\xi\mathcal{B}\eta + \mathcal{H}\nabla_\xi\mathcal{C}\eta = \omega\mathcal{A}_\xi\eta + \mathcal{C}\mathcal{H}\nabla_\xi\eta, \tag{4.9}$$

$$\mathcal{V}\nabla_U\mathcal{B}\xi + \mathcal{T}_U\mathcal{C}\xi = \phi\mathcal{T}_U\xi + \mathcal{B}\mathcal{H}\nabla_U\xi, \tag{4.10}$$

$$\mathcal{T}_U\mathcal{B}\xi + \mathcal{H}\nabla_U\mathcal{C}\xi = \omega\mathcal{T}_U\xi + \mathcal{C}\mathcal{H}\nabla_U\xi, \tag{4.11}$$

where U, $V \in \Gamma(\ker\pi_*)$ and ξ, $\eta \in (\ker\pi_*)^\perp$.

From the proposition 4.1 it follows that; if π is a quasi-hemi-slant submersion from a manifold (M, g_M, J) onto a Riemannian manifold (N, g_N), then

$$(\nabla_X\phi)Y := \mathcal{V}\nabla_X\phi Y - \phi\mathcal{V}\nabla_X Y, \tag{4.12}$$

$$(\nabla_X\omega)Y := \mathcal{H}\nabla_X\omega Y - \omega\mathcal{V}\nabla_X Y, \tag{4.13}$$

$$(\nabla_Z\mathcal{C})W := \mathcal{H}\nabla_Z\mathcal{C}W - \mathcal{C}\mathcal{H}\nabla_Z W, \tag{4.14}$$

$$(\nabla_Z\mathcal{B})W := \mathcal{V}\nabla_Z\mathcal{B}W - \mathcal{B}\mathcal{H}\nabla_Z W, \tag{4.15}$$

for any $X, Y \in \Gamma(\ker\pi_*)$ and $Z, W \in \Gamma((\ker\pi_*)^\perp)$. Thus, we have the following.

Corollary 4.2. *Let π be a quasi-hemi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we have*

(i) $(\nabla_X\phi)Y = \mathcal{B}\mathcal{T}_X Y - \mathcal{T}_X\omega Y$,
(ii) $(\nabla_X\omega)Y = \mathcal{C}\mathcal{T}_X Y - \mathcal{T}_X\phi Y$,
(iii) $(\nabla_Z\mathcal{C})W = \omega\mathcal{A}_Z W - \mathcal{A}_Z\mathcal{B}W$,
(iv) $(\nabla_Z\mathcal{B})W = \phi\mathcal{A}_Z W - \mathcal{A}_Z\mathcal{C}W$,

for any vectors $X, Y \in \Gamma(\ker\pi_*)$ and $Z, W \in \Gamma((\ker\pi_*)^\perp)$.

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on M, respectively, then

$$\mathcal{B}\mathcal{T}_X Y = \mathcal{T}_X\omega Y,$$

and

$$\mathcal{C}\mathcal{T}_X Y = \mathcal{T}_X\phi Y,$$

for any $X, Y \in \Gamma(TM)$.

For the integrability of distributions and decomposition theorem on the quasi-hemi-slant submersion, we refer to [22] for the details.

5. CONCLUSIONS

Let M be a $2m$-dimensional almost Hermitian manifold with g_M a Riemannian metric on M and almost complex structure J, and N be a Riemannian manifold with Riemannian metric g_N. Then there is a Riemannian submersion $\pi : (M, g_M, J) \longrightarrow (N, g_N)$ such that its vertical distribution $\ker\pi_*$ admits three orthogonal distributions $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ which are invariant, slant and anti-invariant respectively.Let π be a quasi-hemi-slant submersion submersion from an almost

Hermitian manifold (M, g_m, J_m) onto a Riemannian manifold (M, g_n). Let $\{e_1, e_2, ..., e_{m-n}\}$ be a local orthonormal basis of $ker\pi_*$. Then

(1) $\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, ..., e_{m-n}, \sec\theta\phi e_{m-n}\}$ is a local orthonormal basis of $\phi(ker\pi_*)$.

(2) $\{csc\theta\omega e_1, \omega e_1, csc\theta\omega e_2, \omega e_2, ..., csc\theta\omega e_{(m-n)}, \omega e_{(m-n)}\}$ is a local orthonormal basis of $\omega(ker\pi_*)$.

REFERENCES

[1] . ONeill, B The fundamental equations of a submersion. Mich. Math. J. 13, 458469 (1966).
[2] . A Gray, Pseudo-Riemannian almost product manifolds and submersions, J. Math. Mech. 16(1967) , 715-738.
[3] Falcitelli, Maria et al.(2004). Riemannian submersions and related topics. World Scientific.
[4] . B.Watson, Almost Hermitian submersions, J. Differential Geom., 11(1) (1976), 147-165.
[5] . Bayram Sahin. Riemannian submersion fron almost Hermitian manifolds. TAIWANESE journal of mathematic Volume 17 no (2) pp 629-659, appril 2013.
[6] Sahin, Bayram.(2010). Anti-invariant Riemannian submersions from almost Hermitian manifolds.Open Mathematics, 8, no. 3, 437-447.
[7] Sahin, Bayram.(2012). Semi-invariant Riemannian maps fron almost Hermitian manifolds. ndagationes Mathematicae 23 80-94 Elsevier.
[8] Shahid, Mohammad Hasan et al.(2013). Submersion of Semi-Invariant Submanifolds of Trans-Sasakian Manifold. Asian Academy of Management Journal of Accounting & Finance vol 9 no 1.
[9] . B.Y.Chen, Geometry of slant submaniflods, Katholieke Universiteit Leuven, Leuven,1990.
[10] . B.Sahin, slant submersions from almost Hermitian manifolds, Bull. Math. Soc. Sci. Math. Roumanie Tome, 54(102) No. 1, (2011), 93-105.
[11] Park, Kwang-Soon and Prasad, Rajendra.(2013). Semi-slant submersions. Bull. Korean Math. Soc, No. 3, 951-962.
[12] . S.Ianus, A. M. Ionescu, R. Mazzocco and G. E. Vilcu, Riemannian submersions from almost contact metric manifolds, arXiv: 1102.1570v1 [math. DG].
[13] . S.Ianus, R. Mazzocco and G. E. Vilcu, Riemannian submersions from quaternionic manifolds, Acta. Appl. Math., 104 (2008), 83-89.
[14] . K.S.Park, H-slant submersions, Bull. Korean Math. Soc., 49(2) (2012), 329-338.
[15] Tastan, Hakan Mete et al.(2016). Hemi-slant submersions.Mediterr J Math DOI 10, no. 4, s00009–015.
[16] . J. P. Bourguignon and H. B. Lawson, Stability and isolation phenomena for Yang-mills fields, Commum. Math. Phys., 79 (1981), 189-230.
[17] . B. Watson, G, $G'-$Riemannian submersions and nonlinear gauge field equations of general relativity, in: Global Analysis - Analysis on manifolds, T. Rassias, (ed.) dedicated M. Morse. Teubner-Texte Math., 57(1983), 324-349, Teubner, Leipzig.

[18] . J. P. Bourguignon and H. B. Lawson, A mathematicians visit to Kaluza-Klein theory, Rend. Sem. Mat. Univ. Politec. Torino, Special Issue, (1989), 143-163.

[19] . S. Ianus and M. Visinescu, Kaluza-Klein theory with scalar fields and generalized Hopf manifolds, Class. Quantum Gravity, 4 (1987), 1317-1325.

[20] . S. Ianus and M. Visinescu, Space-time compactification and Riemannian submersions, in: The Mathematical Heritage of C. F. Gauss, G. Rassias, (ed.), 1991, pp. 358-371, World Scientific, River Edge.

[21] . M. T. Mustafa, Applications of harmonic morphisms to gravity, J. Math. Phys., 41(10) (2000), 6918-6929.

[22] Longwap, S., F. Massamba, and N. E. Homti. ”On quasi-hemi-slant Riemannian submersion.” Journal of Advances in Mathematics and Computer Science 34.1 (2019): 1-14.