Original Research Article

Quantum Physics in the Context of Countable and Uncountable Infinite Sets from the Perspective of Real Analysis

ABSTRACT

This study aims to investigate countable and uncountable infinite sets from the perspective of real analysis. Key theorems and definitions related to this topic are presented, along with some specific applications in quantum physics, such as the quantization of energy, the relationships between the discrete and the continuous, and the hypothesis of the linearity of the Schrödinger wave equation.

Keywords: quantum physics; infinite sets; countable sets; uncountable sets; real analysis.

1. INTRODUCTION

Infinite sets can be categorized into two types: countable and uncountable. Georg Cantor was the first mathematician to recognize that there are different types of infinite sets, a concept he formalized in his theory of cardinal numbers [1]. The concept of enumerability of infinite sets has practical applications in Statistical Physics, particularly in the study of random and probabilistic phenomena. In this paper, we demonstrate that a hypothesis regarding the wave equation in quantum mechanics implies that the set of solutions for the wave function $\Psi(x,t)$ is infinite and uncountable. Numerous physical problems have infinite sets of solutions, and it is interesting to know whether these sets are countable or uncountable; this is one of the reasons to convince the reader of the importance of this topic. Additionally, the topological structure of the metric space in which every physical phenomenon is modeled relies on a metric defined over an infinite set. Sections 2 and 3 present some key definitions and theorems on this subject, and Section 4 illustrates practical applications in the study of physical phenomena. The final section, Section 5, is reserved for conclusions and final remarks.

2. ENUMERABILITY IN REAL ANALYSIS

The starting point for studying enumerability in real analysis [2] is the set of natural numbers $(\mathbb{N})$, which is defined through the Peano Axioms:

Axiom (1)
An injective function $s: \mathbb{N} \to \mathbb{N}$ exist such that for all $n \in \mathbb{N}$, $s(n)$ is defined as the successor of n, where $s(n) \in \mathbb{N}$. Note: $s(n) = n + 1$.

Axiom (2)
A unique natural number $1 \in \mathbb{N}$ exists such that it is not the successor of any other natural number according to the function $s: \mathbb{N} \to \mathbb{N}$. In symbols: $\exists!\, 1 \in \mathbb{N};\ 1 \notin s(\mathbb{N})$. This means that

the successor function $s: \mathbb{N} \to \mathbb{N}$ is not surjective because $s(\mathbb{N}) = \mathbb{N} - \{1\}$, and therefore $s(\mathbb{N}) \neq \mathbb{N}$.

Axiom (3)
(Principle of Induction) Let $X \subset \mathbb{N}$. If $1 \in X$ and $s(X) \subset X$, then $X = \mathbb{N}$. Note: $s(X) \subset X$ indicates that $s(n) \in X$ for all $n \in X$. In other words, when $X \subset \mathbb{N}$, if the natural number 1 belongs to X and for each element n in X, its successor $s(n)$also belongs to X, then X is the set of natural numbers ($X = \mathbb{N}$).

Peano's Axiom (3) is referred to as the principle of induction in $\mathbb{N}$ and will be used in many proofs of the theorems that follow. Essentially, to prove that a given property *P* holds for every natural number $n \in \mathbb{N}$, we must first show that Pholds for$n = 1$, and then we must prove that Pholds for $s(n) = n + 1$ assuming, utilizing the induction hypothesis, that Pholds for n. In logical terms, demonstrating that a property Pholds for every natural number $n \in \mathbb{N}$means proving that:

$$P(1) \text{ is true and } P(n) \Rightarrow P(s(n)), \forall n \in \mathbb{N}$$

$P(n)$ is true for all $n \in \mathbb{N}$, where the truth of $P(n)$ is the induction hypothesis.

Introducing the well-ordering principle in $\mathbb{N}$ is necessitated for the proof of Theorem 1 below. This is referred to as the Second Principle of Induction. The Well-Ordering Principle states that every non-empty subset $A \subset \mathbb{N}$ has the smallest element.

We will now formally define an *infinite set*. For this purpose, we will assume, without proof, that the set of natural numbers, denoted as $\mathbb{N}$, is infinite. For further information, the reader can refer to the proofs presented [2]. An infinite set, denoted as X, can be defined as a set for which there exists an injective function, denoted as $f: \mathbb{N} \to X$. This means that if X is infinite, then the cardinality of X, denoted as $\mathrm{card}(X)$, is greater than or equal to the cardinality of $\mathbb{N}$, or $\mathrm{card}(X) \geq \mathrm{card}(\mathbb{N})$. The cardinality of a set is a function that assigns a natural number to each set, indicating the number of elements in the set. It should be noted that the injective function $f: \mathbb{N} \to X$ is defined through induction on$n \in \mathbb{N}$. The function is initially defined for $f(1) \in X$. For every $k \in \mathbb{N}$, we choose $f(k) \in A_k = X - \{f(1), f(2), \ldots, f(k-1)\}$. The induction hypothesis assumes that $f(1), f(2), \ldots, f(n)$ are defined and we let $A_{n+1} = X - \{f(1), f(2), \ldots, f(n)\}$. A_{n+1} is a non-empty subset of Xbecause $X$is infinite. Therefore, we can choose $f(n+1) \in A_{n+1}$. This completes the definition of $f: \mathbb{N} \to X$. The injectivity of f follows from the fact that for any $m, n \in \mathbb{N}$ with $m < n, f(m) \in \{f(1), f(2), \ldots, f(n-1)\}$ and $f(n) \in X - \{f(1), f(2), \ldots, f(n-1)\}$; thus, $f(m) \neq f(n)$.

Infinite sets can be classified as countable or uncountable. By definition, every finite set is countable. However, in this study, we will focus on infinite sets and specifically what makes an infinite set countable. An infinite set X is countable if there exists a bijection $f: \mathbb{N} \to X$. This means that if X is infinite and countable, then $\mathrm{card}(X) = \mathrm{card}(\mathbb{N})$. In other words, countable infinite sets are, in a sense, the 'smallest infinites'. In mathematics and physics, there are 'infinities greater than others'. An infinite set X is uncountable if there does not exist a surjection $f: \mathbb{N} \to X$, that is, $f(\mathbb{N}) \neq X$ for every function $f: \mathbb{N} \to X$. This means that if X is infinite and uncountable, then $\mathrm{card}(X) \geq \mathrm{card}(\mathbb{N})$.

Therefore, every countable set has an enumeration of the form $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. We can set $f(1) = x_1$, $f(2) = x_2$, ..., $f(n) = x_n$, ... utilizing the bijection $f: \mathbb{N} \to X$. In other words, an infinite set X is countable if it is possible to define x_{k+1} for every element $x_k \in X$. We say that x_{k+1} is the next element of X after x_k. This notion can be used to intuitively verify that the set of real numbers, denoted by $\mathbb{R}$, is uncountable. Given any real number $x \in \mathbb{R}$, it is impossible to determine what the next real number is. For example, given $1.001 \in \mathbb{R}$, what is the next real number? Is it 1.00101? 1.001001? 1.0010001? ... ? $1.001000\ldots0001$? There is no way to know. The set of natural numbers, denoted $\mathbb{N}$, is obviously countable because there exists the trivial bijection $f: \mathbb{N} \to X$, given by $f(n) = n$ for all$n \in \mathbb{N}$, which is the identity function. Another way to see that $\mathbb{N}$ is countable is to consider Peano's Axiom (3), which states that the successor $s(n)$ of$n \in \mathbb{N}$, defined as the injective function $\mathbb{N} \to \mathbb{N}$, is also an element of $\mathbb{N}$. This means that for every$n \in \mathbb{N}$, $s(n) = n + 1 \in \mathbb{N}$. Therefore, it is possible to define an enumeration of the set of natural numbers, namely $\mathbb{N} = \{n_1, n_2, \ldots, n_k, \ldots\} = \{1, s(1), s(s(1)), \ldots, s^k(1), \ldots\}$. In other words, the set of natural numbers is countable because its elements can be counted. This is not the case for $\mathbb{R}$.

In the following section, we will present theorems related to infinite countable sets, which are pertinent to the applications discussed in Section 4. These theorems are important in the fields of statistical physics and quantum mechanics, where random variables and probabilistic conditions often play a role in the phenomena under investigation.

3. THEOREMS RELATED TO INFINITE COUNTABLE SETS

Theorem 1: Every infinite subset $X \subset \mathbb{N}$ is countable.

Proof: We define an enumeration of the infinite subset $X \subset \mathbb{N}$ through induction and use the well-ordering principle in $\mathbb{N}$. We start by taking $x_1 = \min(X)$. We then define $A_1 \subset X$ such that $A_1 = X - \{x_1\}$. We take $x_2 = \min(A1)$ and define $A_2 \subset X$ such that $A_2 = X - \{x_1, x_2\}$. Proceeding inductively, we assume that $A_n \subset X$ is defined such that $A_n = X - \{x_1, x_2, \ldots, x_n\}$. We then take $x_{n+1} = \min(A_n)$. This process gives us an enumeration of the infinite subset $X \subset \mathbb{N}$ given by $X = \{x_1, x_2, \ldots, x_n, \ldots\}$, with $x_1 = \min(X)$ and $x_{k+1} = \min(\mathrm{A}_k)$ for all $k \in \mathbb{N}$, where $A_k = X - \{x_1, x_2, \ldots, x_k\}$.

Theorem 2: Let X, Y be infinite sets, and $f: X \to Y$ be an injective function. IfY is countable, then so is X.

Proof: If Y is infinite and countable, then there exists a bijection $g: \mathbb{N} \to Y$. Furthermore, if $f: X \to Y$is injective, then $f(X) \subset Y$. We can then obtain a bijection $f|_{f(X)}: X \to f(X)$ by restricting the range of the original function $f: X \to Y$to the subset $f(X) \subset Y$. This is similar to how $g: \mathbb{N} \to Y$is bijective and there exists a subset $A \subset \mathbb{N}$ such that $g|_{f(X)}: A \to f(X)$ is also a bijection. Based on Theorem 1, $A \subset \mathbb{N}$ is countable, and since $g|_{f(X)}: A \to f(X)$ is a bijection, it follows that $f(X)$ is countable. In turn, the bijection $f|_{f(X)}: X \to f(X)$implies that X is countable.

Theorem 3: Let X, Y be infinite sets, and $f: X \to Y$ be a surjective function. If X is countable, then so is Y.

Proof: For every $y \in Y$, we can choose an element $g(y) \in X$ and define a function $g: Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$. The function $g: Y \to X$, thus defined, is injective. To see this, consider two generic elements $g(y_1) \neq g(y_2) \in X$. We have $f(g(y_1)) \neq f(g(y_2)) \Rightarrow y_1 \neq y_2$. The function g is the right inverse of f. Based on Theorem 2, if $g: Y \to X$ is injective and X is countable (by hypothesis), then so is Y.

Theorem 4: Let X, Y be infinite and countable sets. The Cartesian product $X \times Y$ is also countable.

Proof: Let X and Y be infinite, countable sets. By definition, there exist bijections $f: \mathbb{N} \to X$ and $g: \mathbb{N} \to Y$. In particular, we can consider that there exist surjections $f: \mathbb{N} \to X$ and $g: \mathbb{N} \to Y$. We can then define a surjective function$F: \mathbb{N} \times \mathbb{N} \to X \times Y$by setting $F(m, n) = (f(m), g(n))$ for all $m, n \in \mathbb{N}$. Based on Theorem 3, it suffices to prove that $\mathbb{N} \times \mathbb{N}$ is countable. Indeed, taking the function $\Psi: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $\Psi(m, n) = 2^m . 3^n$, we demonstrate that Ψ is injective due to the uniqueness of the prime factorization of natural numbers, which is ensured by the fundamental theorem of arithmetic [1]. Since the function $\Psi: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is injective, and $\mathbb{N}$ is countable, $\mathbb{N} \times \mathbb{N}$ is countable based on Theorem 2.

Theorem 5: The collection of a countable family of countable sets is countable, that is, if $(X_\lambda)_{\lambda \in L}$ is a countable family whose elements are countable sets, then the union$\bigcup_{\lambda \in L} X_\lambda$ is also countable.

Proof: Consider a countable family $(X_\lambda)_{\lambda \in L}$ whose elements are countable sets X_1, X_2, ..., X_n,... By definition, bijections $f_1: \mathbb{N} \to X_1$, $f_2: \mathbb{N} \to X_2$, ..., $f_n: \mathbb{N} \to X_n$,... exist. In particular, we can consider that surjections $f_1: \mathbb{N} \to X_1$, $f_2: \mathbb{N} \to X_2$, ..., $f_n: \mathbb{N} \to X_n$,... exist. The union $\bigcup_{n=1}^{\infty} X_n$ is the collection of all elements of $(X_\lambda)_{\lambda \in L}$. We can then define a surjective function $F: \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} X_n$ by setting $F(m, n) = f_n(m)$ for all $m, n \in \mathbb{N}$, that is, by setting $F(m, n)$ equal to the n-th function, f_n, applied to the natural number m. As already proven in Theorem 4, $\mathbb{N} \times \mathbb{N}$ is countable. Based on Theorem 3, if $F: \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} X_n$ is surjective, then the union $\bigcup_{n=1}^{\infty} X_n = \bigcup_{\lambda \in L} X_\lambda$ is countable.

Notably, the notation $A \equiv B$ indicates that sets A and B are equipotent, that is, they have the same number of elements. Thus, stating that $A \equiv B$ is equivalent to writing $\mathrm{card}(A) = \mathrm{card}(B)$, which implies that two sets are equipotent if and only if they have the same cardinality. Specifically, if the sets A and B are equipotent, there exists a one-to-one correspondence between them, given by the bijective function $f: A \to B$.

Theorem 6: Every infinite set contains a countably infinite subset.

Proof: Let X be an infinite set. First, consider the case where X is countable. In this case, we can consider that $A = X$. Then, A is a subset of X, and A is infinite countable. Now, consider the case where X is uncountable. By definition, a surjection $\mathbb{N} \to X$ does not exist. However, as X is infinite, there must exist an injective function $h: \mathbb{N} \to X$. This implies that $\mathrm{card}(X) > \mathrm{card}(\mathbb{N})$ and $h(\mathbb{N}) \subset X$. We can then obtain a bijection $h|_{h(N)}: \mathbb{N} \to h(\mathbb{N})$ by restricting the range of the original function $h: \mathbb{N} \to X$ to the subset $h(\mathbb{N}) \subset X$. If $h|_{h(N)}: \mathbb{N} \to h(\mathbb{N})$is a bijection, then $\mathrm{card}(\mathbb{N}) = \mathrm{card}(h(\mathbb{N}))$, which means that $N \equiv h(\mathbb{N})$. Thus, $h(\mathbb{N}) \subset X$ is infinite countable.

The result of Theorem 6 is significant for the investigation of probabilistic physical phenomena whose domains are typically uncountable infinite sets of random variables. It is useful to analyze an infinite countable subset of the original domain, the existence of which is guaranteed by Theorem 6, to derive a mathematical law for these variables and find a deterministic solution for the physical phenomenon being researched. For example, when a researcher is modeling a purely probabilistic phenomenon, the domain of which is represented by an uncountable infinite set of random variables, it may not be possible to determine the set of solutions analytically but only numerically. The researcher will be able to solve the problem using numerical methods but will not be able to establish a mathematical law that governs the studied phenomenon for the entire domain. One potential issue in this scenario may be the non-enumerability of the domain of random variables that describe the parameters of the investigated phenomenon.

In this situation, it is important to emphasize the significance of mathematical studies related to the perturbation of domains of physical phenomena. Simply put, domain perturbation is the intentional modification of the set of elements that make up the domain of the studied phenomenon. For example, constraining a generic function $f: X \to Y$ to a subset $X^{'} \subset X$ of its original domain, resulting in the restriction $f|_{X'}: X^{'} \to Y$, implies a perturbation of the domain of the function $f: X \to Y$. Returning to the scenario from earlier, if the researcher is interested in analytically solving a physical problem with a random or probabilistic component, even if this is not feasible for the entire set of variables in the initial problem, they can determine whether there is a perturbation of the domain that permits the development of a mathematical law capable of modeling the physical phenomenon limited to the variables of the new subset obtained after the perturbation was made. For a more in-depth analysis, consider the generic function $f: X \to Y$ that models a certain physical phenomenon with an uncountable infinite domain X. The researcher can explore the possibility of perturbing the domain of $f: X \to Y$ to obtain an infinite subset $A \subset X$ that is countable. The existence of such a subset $A \subset X$ in the domain of $f: X \to Y$ is guaranteed by Theorem 6. Therefore, the problem involves finding a way to perturb the original domain of the studied physical phenomenon to ensure that a countable subset is obtained.

In summary, it is generally easier to establish a mathematical pattern for an infinite countable set than for an infinite uncountable set, even if this results in a law that only applies to a portion of the studied physical phenomenon, that is, a law that holds only for a subset of the domain. This is because an infinite countable set has a bijection with the set of natural numbers $\mathbb{N}$, which means that it can be counted; this, in turn, leads to a mathematical pattern that allows one to determine, for any element in the set, which element comes next (see the definition of enumerability in Section 2). Additionally, obtaining an analytical solution for the studied physical phenomenon by restricting it to a certain countable subset of variables is still valuable, even if it is not a complete solution.

In certain cases, the method previously described, which is guaranteed to be valid by Theorem 6, may not be sufficient to establish a mathematical law that can provide analytical solutions, even in a limited manner. In such situations, the researcher still has another tool at their disposal. It is possible to divide the infinitely countable subset $A \subset X$ of the domain of the function $f: X \to Y$, obtained from the previous method, into countable parts $A_n \subset A$, $n \in \mathbb{N}$, resulting in a partition. Thus, the subset $A \subset X$ can be analyzed as a collection of countable parts $A_n \subset A$, $n \in \mathbb{N}$. In other words, we have $(A_n)_{n \in \mathbb{N}}$, where $\bigcup_{n=1}^{\infty} A_n = A$. Using this process, the researcher can limit their problem to investigating mathematical laws that explain the physical phenomenon being studied for each countable part $A_n \subset A$, $n \in \mathbb{N}$. From there, it is possible to eliminate unwanted parts of $(A_n)_{n \in \mathbb{N}}$ and only retain those that can be modeled analytically. The researcher is interested in the countable parts $A_\lambda \subset A, \lambda \in L$, which can be gathered to obtain a new subset $\bigcup_{\lambda \in L} A_\lambda = \bar{A}$, where $\bar{A} \subset A$. According to Theorem 5, we know that the union $\bigcup_{\lambda \in L} A_\lambda = \bar{A}$ is countable, because $(A_\lambda)_{\lambda \in L}$ is a countable family of countable parts $A_\lambda \subset A$, $\lambda \in L$.

In the field of mathematical physics known as dynamical systems, the study of functions that describe the evolution of systems over time in topological spaces [3] is a major focus. As a result, the concepts of infinite sets and countability are of great significance for the study of systems that change over time. These systems are often described using partial differential equations, and are particularly relevant in the study of physical phenomena in modern celestial mechanics, mainly the evolution of galaxies, which can be mathematically modeled as dynamical systems.

Some examples and applications are mentioned in the following section.

4. APPLICATIONS AND EXAMPLES

4.1 Georg Cantor's diagonal method

In Section 2, a method was described that allows one to obtain infinite countable subsets from any uncountable set based on the results demonstrated in Theorems 5 and 6 in Section 3. In this section, we present a method that takes the inverse approach: the diagonal method, developed by German mathematician Georg Cantor in 1891, which aims at obtaining uncountable sets from infinite countable sets [1]. It was through this method that Cantor proved the existence of infinite sets of distinct natures. Let X be an infinite countable set and Ybe a set containing at least two elements. Let $F(X;Y)$ be the set $F(X;Y) = \{X, Y \subset \mathbb{R};\ \text{there exists}\ f: X \to Y\}$, whose elements are all possible functions $f: X \to y$. Cantor's method states that no function $\varphi: X \to F(X;Y)$ is surjective. Initially, Cantor's argument for the diagonal method was stated for the specific case of the function $\varphi: \mathbb{N} \to F(\mathbb{N};\{0,1\})$, where $X = \mathbb{N}$ and $Y = \{0,1\}$. It was later demonstrated for the more general case, using the same argument, and was established as a theorem. Here, we will only consider the proof for the specific case to define our method of interest. To show that $\mathbb{N} \to \mathrm{F}(\mathbb{N};\{0,1\})$ is not surjective, we can define inductively $\varphi(1) = s_1$, $\varphi(2) = s_2$, $\varphi(N) = s_n$, …, where s_1, s_2, …, s_n, … are sequences whose terms are elements of the set $\{0,1\}$. Let s_{m_n} be the n-th term of the sequence s_m. Therefore, it will always be possible to obtain a new sequence s^* different from all previous ones by taking $s_n^* = 0$ if $s_{n_n} = 1$or $s_n^* = 1$ if $s_{n_n} = 0$. This means that no countable list can exhaust all functions in the set $F(\mathbb{N};\{0,1\})$. This result will be crucial for the following proof. Let $X \subset \mathbb{R}$ be an infinite and countable set. Consider the set $P(X) = \{A \subset \mathbb{R} | A\ \text{is a subset of}\ X\}$ of the parts of X. We will prove that there exists a function $\xi: P(X) \to F(X;\{0,1\})$ that is bijective, using the Cantor Diagonal Method with $Y = \{0,1\}$. For each subset $A \subset X$, that is, for each element of $P(X)$, we define a restricted function $\xi|_A: X \to \{0,1\}$ such that for all $x \in X$, $\xi|_A(x) = 1$ if $x \in A$, and $\xi|_A(x) = 0$ if $x \notin A$. Thus, we obtain the bijection $\xi: P(X) \to F(X;\{0,1\})$ that relates $A \mapsto \xi|_A$for all $A \in P(X)$. We have seen that the function $\varphi: X \to F(X;\{0,1\})$ cannot be surjective. Therefore, the composite function $\xi^{-1} \circ \varphi: X \to P(X)$ must also not be surjective. However, there is a trivial injective function $\psi: X \to P(X)$ given by $\psi(x) = \{x\}$ for all $x \in X$. Therefore, we have $\mathrm{card}(X) < \mathrm{card}(P(X))$. By assumption, we take an arbitrary infinite set $X \subset \mathbb{R}$ that is countable. Then, we have $\mathrm{card}(X) = \mathrm{card}(\mathbb{N})$. As a result, we obtain $\mathrm{card}(\mathbb{N}) < \mathrm{card}(P(X))$. By definition, this implies that the set of parts of X given by $P(X) = \{A \subset \mathbb{R}; A\ \text{is a subset of}\ X\}$ is uncountable, regardless of the countable infinite set $X \subset \mathbb{R}$ considered. In summary, Georg Cantor's diagonal method allows us to obtain an uncountable set $P(X)$ from any countably infinite set $X \subset \mathbb{R}$. To do this, we simply define $P(X)$ as the set of parts of X. Cantor's diagonal method will be used in Example 4.4 to prove an important result in theoretical physics: every physical phenomenon whose nature is discrete can be interpreted as a continuous set of solutions, which may or may not have practical validity.

4.2 Linearity hypothesis of $\Psi(x,t)$ in the quantum mechanics wave equation

The fundamental equation of quantum mechanics, known as Schrödinger's wave equation, was presented by Austrian physicist Erwin Schrödinger in 1925. This second-order partial differential equation, whose solutions are referred to as wave functions and denoted by $\Psi(x,t)$ [4], is a landmark in the history of modern science. These functions describe the behavior of atomic particles and serve as the foundation of quantum knowledge. The one-dimensional wave equation proposed by Schrödinger is given by the following expression:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t},$$

where m is the mass of the particle, $\hbar$ is the reduced Planck constant, $i = \sqrt{-1}$is the imaginary number, and $V(x,t)$ is the potential energy of the particle. Schrödinger was motivated to derive this equation by De Broglie's hypothesis about the dual nature of matter. It is also possible to derive the Schrödinger wave equation from four hypotheses, or axioms, that justify its validity. One of these axioms is the linearity of $\Psi(x,t)$. This assumption states that the wave equation must be linear in $\Psi(x,t)$. This suggests that if $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are two different solutions of the wave equation for a given potential energy $V(x,t)$ of the particle, then any arbitrary linear combination $\Psi(x,t) = \alpha_1\Psi_1(x,t) + \alpha_2\Psi_2(x,t)$ $\Psi(x,t)$ is also a solution, where α_1 and α_2 are constants.

The set of solutions of Schrödinger's wave equation is infinite, owing to the arbitrary nature of linear combinations. In other words, there are an infinite number of wave functions $\Psi(x,t)$ that satisfy this equation. Let Ω be the set of all these functions $\Psi(x,t)$. We will now prove that Ω is uncountable. Given two solutions $\Psi_1(x,t)$ and $\Psi_2(x,t)$ of the Schrödinger equation, we can define the subset $\Omega' \subset \Omega$ as $\Omega' = \{\Psi(x,t) \in \Omega | \Psi(x,t) = \alpha_1\Psi_1(x,t) + \alpha_2\Psi_2(x,t) \wedge \alpha_1, \alpha_2 \in \mathbb{R}\}$, where $\wedge$ represents the logical conjunction operator, which is equivalent to the 'and' connective in grammar. As $\Omega' \subset \Omega$, it follows that $\mathrm{card}(\Omega') \leq \mathrm{card}(\Omega)$. Therefore, it is sufficient to show that Ω' is uncountable. We can define the bijection $\mathcal{F}: \mathbb{R} \times \mathbb{R} \to \Omega'$ as $\mathcal{F}(\alpha_1, \alpha_2) = \alpha_1\Psi_1(x,t) + \alpha_2\Psi_2(x,t)$. We know that the set $\mathbb{R}$ of real numbers is uncountable. Additionally, it follows that $\mathrm{card}(\mathbb{R}) \leq \mathrm{card}(\mathbb{R} \times \mathbb{R})$. By definition, it follows that $\mathbb{R} \times \mathbb{R}$ is also uncountable. Since$\mathcal{F}$is a bijection between $\mathbb{R} \times \mathbb{R}$ and Ω', we demonstrate that $\Omega' \subset \Omega$ is uncountable. In conclusion, it has been proven that the set of solutions Ω of Schrödinger's wave equation is uncountable. Therefore, the following conclusions are valid:

1. The set Ω is not bijective with $\mathbb{N}$. Therefore, $\mathrm{card}(\Omega) > \mathrm{card}(\mathbb{N})$.
2. All quantum mechanical phenomena that satisfy the Schrödinger equation are continuous, as the uncountable nature of the set Ω makes it impossible to quantize them for the full range of wavefunctions$\Psi(x,t) \in \Omega$.
3. According to the result proven in Theorem 6, there exists at least one infinite subset of Ω, which is countable. This subset is denoted as $\Omega'' \subset \Omega$ and can be defined from the auxiliary set Ω' used in the previous test by limiting the coefficients of the linear combinations of Ω' to natural numbers. Specifically, Ω'' is defined as: $\Omega'' = \{\Psi(x,t) \in \Omega; \Psi(x,t) = n_1\Psi_1(x,t) + n_2\Psi_2(x,t) \wedge n_1, n_2 \in \mathbb{N}\}$. To show that $\Omega'' \subset \Omega$ is countable, we consider the bijection $F: \mathbb{N} \times \mathbb{N} \to \Omega''$ defined as $F(n_1, n_2) = n_1\Psi_1(x,t) + n_2\Psi_2(x,t)$. As we proved in Theorem 4, $\mathbb{N} \times \mathbb{N}$ is countable, so it follows that the subset $\Omega'' \subset \Omega$ is also countable.
4. An interesting way to interpret the previous conclusion is that any quantum phenomenon that satisfies the Schrödinger wave equation can be quantified within a certain restricted domain. To do this, it is sufficient to restrict the linearity of the wave equation in $\Psi(x,t)$ to ensure that the allowed coefficients for the linear combinations are natural numbers. In general, for any uncountable set of solutions that describe continuous physical phenomena, exists an infinite, countable subset of these solutions that correspond to the quantifiable, discrete particular cases of the phenomena. This idea is also supported by Theorem 6.

4.3 Quantization of energy, photons of light, and Planck's equation (1900)

In 1900, German physicist Max Planck published his theory on the quantization of light energy to explain the problem of blackbody radiation, which had been a subject of much discussion since it was first observed by Gustav Kirchhoff in 1860. Planck's theory, known as the Planck equation, was able to successfully describe the phenomenon of blackbody radiation emission through a revolutionary approach [5]. According to Planck, the energy E of standing electromagnetic waves that oscillate sinusoidally with time is a discrete quantity rather than a continuous one. Thus, Planck suggests that

$$E = nh\nu, n \in \mathbb{N}$$

where h is a constant, later referred to as Planck's constant, and ν is the frequency of the corresponding electromagnetic wave. By fixing the frequency ν, we consider the set of allowed energies, $\Gamma(E) = \{nh\nu; n \in \mathbb{N}\}$, for the electromagnetic waves of this radiation. It is evident that this set is infinite, as there exists an energy value associated with each natural multiple $= 1, 2, 3, \dots$. To prove that the set $\Gamma(E)$ is countable, we can define the bijection $\mathcal{F}: \mathbb{N} \to \Gamma(E)$ such that $\mathcal{F}(n) = nh\nu$ for all $n \in \mathbb{N}$. This mathematically proves that the quantum phenomena that satisfy Planck's equation are discrete. Notably, in physics, continuous phenomena are always associated with uncountable sets, while discrete phenomena are associated with infinite countable sets, directly related to the existence (or lack thereof) of a restriction of these phenomena to the set of natural numbers $\mathbb{N}$ or to any other set equipotent to it, for example, $\mathbb{N} \times \mathbb{N}$.

4.4 Relationships between the discrete and the continuous in physics

We conducted a general discussion about physical phenomena of continuous nature and showed that they are associated with uncountable sets. As an example, we highlighted the wave equation of quantum mechanics. The result proved in Theorem 6 showed us that there are specific cases of these continuous phenomena, described by countable sets. This defines a quantifiable condition of the original phenomenon for a given restriction of its domain, giving it a discrete interpretation. While this result remains within the realm of theoretical physics, it is interesting to note the relationship between the nature of physical phenomena and the nature of infinite sets.

At this point, we are aware that it is possible to obtain the discrete from the continuous. We extensively discussed the physical significance of this restriction, which characterizes a disturbance of the domain that governs the studied phenomenon. However, we have not yet analyzed the reverse path. Therefore, we pose the following question: can we define a physical phenomenon of continuous nature from a discrete case, that is, quantifiable? Georg Cantor's diagonal method, discussed in Section 4.1, provides the answer. We will prove that it is indeed possible to obtain the continuous from the discrete. To do this, let us consider the case of Planck's equation again. Given any radiation of frequency ν of the electromagnetic spectrum, consider the set of all allowed energies for electromagnetic waves, $\Gamma(E) = \{nh\nu; n \in \mathbb{N}\}$. As we saw in the analysis conducted in Section 4.3, the set $\Gamma(E)$ is infinite and countable. Therefore, the Diagonal Method allows us to easily obtain the uncountable set $P(\Gamma) = \{A \subset \mathbb{R}; A \text{ is a subset of} \Gamma(E)\}$, whose elements are the subsets of $\Gamma(E)$. In physical terms, this set can be interpreted as the sum of all events in nature where Planck's equation is applicable, for some restriction of the natural multiples of $h\nu$ to a subset of $\mathbb{N}$. Proceeding in this way, we obtain a continuous case, that is, uncountable, from the discrete quantum phenomena that follow Planck's equation.

The set $\Gamma(E) = \{nh\nu; n \in \mathbb{N}\}$ is countable because it describes the allowable energies for waves of specific electromagnetic radiation, whose frequency ν is fixed. However, if we consider the set $\Upsilon(E) = \{nh\nu; n \in \mathbb{N}, \nu \in R\}$ of all energies allowed for waves of any radiation in the electromagnetic spectrum, then we have $\Upsilon(E)$ is an uncountable set because there exists a clear bijection $\mathcal{F}: \mathbb{N} \times \mathbb{R} \to \Upsilon(E)$ given by $\mathcal{F}(n, \nu) = nh\nu$ for all $n \in \mathbb{N}$ and $\nu \in \mathbb{R}$, where the Cartesian product $\mathbb{N} \times \mathbb{R}$ is evidently uncountable. Therefore, the uncountable nature of the set $\Upsilon(E)$ allows us to infer that the electromagnetic spectrum as a whole is continuous, which is consistent with the theory of physics concerning electromagnetic waves.

4.5 The principle of induction and the linearity hypothesis of the Schrödinger equation

In Section 4.2, we mentioned that four assumptions, or axioms, were made regarding the quantum mechanical wave equation (also known as the Schrödinger equation). These hypotheses justify its validity and support a possible proof of it. In Section 4.2, we examined the assumption of the linearity of the Schrödinger equation in $\Psi(x,t)$, from which we proved the uncountable nature of the solution set Ω. This hypothesis states that if $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are two distinct solutions of the wave equation for a given potential energy $V(x,t)$ of the particle, then any arbitrary linear combination $\Psi(x,t) = \alpha_1\Psi_1(x,t) + \alpha_2\Psi_2(x,t)$ is also a solution, where $\alpha_1, \alpha_2 \in \mathbb{R}$. Let $\Psi_1(x,t)$, $\Psi_2(x,t)$, ...,$\Psi_n(x,t)$, ... be solutions of the Schrödinger equation. Therefore, we have:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_1(x,t)}{\partial x^2} + V(x,t)\Psi_1(x,t) - i\hbar\frac{\partial\Psi_1(x,t)}{\partial t} = 0$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_2(x,t)}{\partial x^2} + V(x,t)\Psi_2(x,t) - i\hbar\frac{\partial\Psi_2(x,t)}{\partial t} = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_n(x,t)}{\partial x^2} + V(x,t)\Psi_n(x,t) - i\hbar\frac{\partial\Psi_n(x,t)}{\partial t} = 0\,.$$

Induction on $n \in \mathbb{N}$ can prove that the linearity assumption holds for every arbitrary linear combination of these solutions. For $n = 1$, we have $\Psi^{(1)} = \alpha_1\Psi_1(x,t)$. Thus, it follows:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^{(1)}}{\partial x^2} + V\Psi^{(1)} - i\hbar\frac{\partial\Psi^{(1)}}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2(\alpha_1\Psi_1(x,t))}{\partial x^2} + V(\alpha_1\Psi_1(x,t)) - i\hbar\frac{\partial(\alpha_1\Psi_1(x,t))}{\partial t}$$

$$= \alpha_1\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_1(x,t)}{\partial x^2}\right) + \alpha_1(V\Psi_1(x,t)) - \alpha_1\left(i\hbar\frac{\partial\Psi_1(x,t)}{\partial t}\right)$$

$$= \alpha_1\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_1(x,t)}{\partial x^2} + V\Psi_1(x,t) - i\hbar\frac{\partial\Psi_1(x,t)}{\partial t}\right)$$

$$= \alpha_1 \cdot (0) = 0.$$

Thus, the linear combination $\Psi^{(1)} = \alpha_1\Psi_1(x,t)$ is a solution to the Schrödinger Equation. To prove this, we must admit the induction hypothesis for $n \in \mathbb{N}$ and show that the property also holds for $n+1$. Let $\Psi_1(x,t), \Psi_2(x,t), \ldots, \Psi_n(x,t), \Psi_{n+1}(x,t)$ be solutions to the Schrödinger equation.

The induction hypothesis states that the linear combination

$$\Psi^{(n)}(x,t) = \alpha_1\Psi_1(x,t) + \alpha_2\Psi_2(x,t) + \ldots + \alpha_n\Psi_n(x,t)$$

is a solution to this equation. Based on the induction hypothesis, we need to show that

$$\Psi^{(n+1)} = \alpha_1\Psi_1(x,t) + \alpha_2\Psi_2(x,t) + \ldots + \alpha_n\Psi_n(x,t) + \alpha_{n+1}\Psi_{n+1}(x,t),$$

also satisfies the Schrödinger equation. To simplify the notation, we define

$$\Psi^{(n+1)} = \Psi^{(n)} + \alpha_{n+1}\Psi_{n+1}$$

This yields the following equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^{(n+1)}}{\partial x^2} + V\Psi^{(n+1)} - i\hbar\frac{\partial\Psi^{(n+1)}}{\partial t} =$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\Psi^{(n)} + \alpha_{n+1}\Psi_{n+1}) + V(\Psi^{(n)} + \alpha_{n+1}\Psi_{n+1}) - i\hbar\frac{\partial}{\partial t}(\Psi^{(n)} + \alpha_{n+1}\Psi_{n+1})$$

$$= \quad -\frac{\hbar^2}{2m}\frac{\partial^2\Psi^{(n)}}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2(\alpha_{n+1}\Psi_{n+1})}{\partial x^2} + V\Psi^{(n)} + V(\alpha_{n+1}\Psi_{n+1}) - i\hbar\frac{\partial\Psi^{(n)}}{\partial t} - i\hbar\frac{\partial(\alpha_{n+1}\Psi_{n+1})}{\partial t}$$

$$= \left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^{(n)}}{\partial x^2} + V\Psi^{(n)} - i\hbar\frac{\partial\Psi^{(n)}}{\partial t}\right) + \left(-\frac{\hbar^2}{2m}\frac{\partial^2(\alpha_{n+1}\Psi_{n+1})}{\partial x^2} + V(\alpha_{n+1}\Psi_{n+1}) - i\hbar\frac{\partial(\alpha_{n+1}\Psi_{n+1})}{\partial t}\right)$$

$$= \left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^{(n)}}{\partial x^2} + V\Psi^{(n)} - i\hbar\frac{\partial\Psi^{(n)}}{\partial t}\right) + \alpha_{n+1}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_{n+1}}{\partial x^2} + V\Psi_{n+1} - i\hbar\frac{\partial\Psi_{n+1}}{\partial t}\right)$$

$$= (0) + \alpha_{n+1}(0) = 0,$$

that is, the linear combination $\Psi^{(n+1)} = \Psi^{(n)} + \alpha_{n+1}\Psi_{n+1}(x,t)$ is also a solution to the Schrödinger equation. This completes our proof. Therefore, we have just proved by induction that the hypothesis of the linearity of the Schrödinger equation holds for any solutions $\Psi_1(x,t), \Psi_2(x,t), \ldots, \Psi_n(x,t)$, ... of this equation.

5. conclusion

In this article, we investigate the relationships between the discrete and the continuous, focusing on quantum physics. We demonstrate that the linearity of wave functions $\Psi(x,t)$ in the Schrödinger equation can be proven through the principle of induction, which is the third Peano axiom. This principle is only valid for mathematical events that have a bijective relationship with the set of natural numbers or countable phenomena. In contrast, it was shown that the set Ω of all wave functions permitted by the Schrödinger equation is uncountable, thus implying no bijection between $\mathbb{N}$ and Ω due to the principle of superposition, which states that if $\Psi_n(x,t)$ is a countable sequence of wave functions that solve the Schrödinger equation for $n \in \mathbb{N}$, then the linear combination $\Sigma\alpha_n\Psi_n(x,t)$ is also a solution, where $\alpha_n \in \mathbb{R}$. Therefore, we can conclude that the wave functions $\Psi_n(x,t)$ of quantum mechanics exhibit discrete characteristics, that is, countable, when analyzed separately. However, the set of possibilities for $\Psi_n(x,t)$ is continuous, that is, uncountable. These results reflect a unique aspect of the quantum world: depending on the perspective used to analyze phenomena in this domain, both discrete and continuous behaviors can be observed. In this article, we also demonstrate a similar concept for the case of the quantization of electromagnetic wave energy through Planck's equation. We showed that each frequency range of the electromagnetic spectrum has an infinite, countable set of permissible energy values for an electromagnetic wave, where the frequency of the wave is fixed. This set is given by $\Gamma(E) = \{nh\nu; n \in \mathbb{N}\}$ and has a bijective relationship with $\mathbb{N}$, implying that the energies of each frequency range of the electromagnetic spectrum form a discrete set, that is, are quantized. However, we showed that the set of all permissible energy levels for the electromagnetic waves of these radiations is uncountable when analyzing the entire electromagnetic spectrum, where the frequency of radiation $\nu \in \mathbb{R}$ is variable. This set is given by $\Upsilon(E) = \{nh\nu; n \in \mathbb{N}, \nu \in \mathbb{R}\}$ and does not have a bijective relationship with$\mathbb{N}$. In summary, each frequency band of the electromagnetic spectrum has a countable set of energy levels, meaning these levels are discrete. In contrast, the electromagnetic spectrum in its entirety, with all its possibilities of radiation beams, presents an uncountable set of energy levels, that is, the total energy of the electromagnetic spectrum is continuous, despite it being constituted by infinitely many discrete sets of energy levels, generating a mathematical contradiction with Theorem 5, which states that the assembly of a countable family of countable sets is countable, leading to the question of the continuity of the electromagnetic spectrum. In short, this article allows us to emphasize analytically and mathematically that the main paradigms and contradictions of quantum mechanics originated from the duality between the discrete and the continuous.

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