Some fixed point techniques using ψ-contraction mapping on the C^*-algebra valued metric space

Abstract

In this article, we introduced ana focused our attention to some fixed point theorems using ψ-contraction mapping on C^*-algebra valued metric space. In particular, we established some Banach fixed point theorem as well as several extensions and generalizations of this theorem in C^*-algebras valued metric spaces. Moreover, in order to illustrate the current results, some basic examples are presented and we gave an application on system linear operator equation by investigating the existence and uniqueness to the solution of this equation.

Keywords: Fixed point theorems; C^*-algebra valued metric space; ψ-contraction mapping; system linear operator equation.

Mathematics Subject Classification: 47H10, 46L07, 54H25.

1. Introduction

Fixed point theory is a substantial concept in nonlinear analysis and branches of modern mathematics. This theory is the most important mathematical and has a lot application in biology, chemistry, and physics [2]. One of the key findings in fixed point theory and approximation theory is the Banach contraction principle (BCP). It is essential to the resolution of several significant problems in both practical and pure mathematics and physics. The technical expansions and generalizations of (BCP) are the subject of a large body of literature. Additionally, this classical theorem provides an iteration procedure that allows us to improve our approximations of the fixed point. It becomes crucial while recursively solving systems of linear algebraic equations. Iteration techniques are almost often used in practical mathematics, including convergence proof and error process estimation, which is frequently done by using Banach's fixed point theorem [1, 9]. Jungck first looked at common fixed points for commuting mappings in metric spaces in 1966 [7].

Suppose that $\mathbb{A}$ is a unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate linear map $u \to u^*$ on $\mathbb{A}$ such that $u^{**} = u$ and $(uv)^* = v^*u^*$ for all $u, v \in \mathbb{A}$. The pair

$(\mathbb{A}, *)$ is called a *-algebra. A Banach *-algebra is a *-algebra $\mathbb{A}$ together with a complete submultiplicative norm, $\|uv\| \leq \|u\|\|v\|$ such that $\|u^*\| = \|u\|$. C^*-algebra is a *-Banach algebra such that $\|u^*u\| = \|u\|^2$ (see [3]), when $\mathbb{A}$ is a unital C^*-algebra, then for any $u \in \mathbb{A}_+$ we have $u \preceq I \Leftrightarrow \|u\| \leq 1$ (see [10]).

The aim of our article is to attract the attention to some fixed point results using ψ-contraction mapping on C^*-algebra valued metric spaces (C^*-algebra VMSs) with an application. In Section 1 (Sec. 1), we give identificative introduction about fixed point theory, BCP, and C^*-algebra. In Sec. 2, we take a look on some Basic facts, nontrivial examples and primilinaries that will be useful in our discussion. In Sec. 3, we display our main results to show the existence and uniqueness of fixed point and common fixed point. In Sec. 4, we present an application on system linear operator equation in Hillbert space. In Sec. 5, we state our conclusion in this article.

2. Preliminaries

Definition 2.1. [3] Let $\mathbb{A}$ be a C^*-algebra and then $x \in \mathbb{A}$ is called positive element if $u = u^*$ and $\sigma(u) \subseteq \mathbb{R}^+$ this can be written as $u \succcurlyeq \theta$.
We denote by $\mathbb{A}^+$, the set of all positive elements. That is, $\mathbb{A}^+ = \{u \in \mathbb{A} : u \succcurlyeq \theta\}$. When θ is a zero element in $\mathbb{A}$ and $\preccurlyeq$ is a partially order relation on $\mathbb{A}$ such that $u \succcurlyeq v$ *or* $u - v \succcurlyeq \theta$.

Definition 2.2. [10] A C^*-algebra valued metric is a nonempty set X and let a mapping $d : X \times X \to \mathbb{A}$ satisfies:

(i) $d(u, v) \succcurlyeq \theta$ and $d(u, v) = \theta$ if and only if $u = v$ for all $u, v \in X$,

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$,

(iii) $d(u, v) \preccurlyeq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

This C^*-algebra valued metric is denoted by $(X, \mathbb{A}, d)$ and d is called a C^*-algebra valued metric on X.

Definition 2.3. Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and $\{u_n\} \subseteq X$ is a sequence in X. If $u \in X$ and $\epsilon \succ \theta$ there is N such that for all $n > N$, $d(u_n, u) \prec \epsilon$, then $\{u_n\}$ is called a convergent sequence in X to u and write $\lim_{n\to\infty} u_n = u$. Moreover, for any $\epsilon \succ \theta$, there is N such that for all $n, m > N$, $d(u_n, u_m) \prec \epsilon$ then u_n is called a Cauchy sequence in X.

Definition 2.4. If every Cauchy sequence is converges, then $(X, \mathbb{A}, d)$ is called a completed C^*-algebra valued metric space.

Example 2.5. Let $X = \mathbb{R}$ and $A = M_2(\mathbb{R})$. Define $d(u,v) = diag(|u-v|, k|u-v|)$, where $u, v \in \mathbb{R}$ and $k \geq 0$ is constant. Then, d is a C^*-algebra valued metric space and $(X, M_2(\mathbb{R}), d)$ is a complete C^*-algebra valued metric space by the completeness of $\mathbb{R}$.

Definition 2.6. [10] Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. We call a mapping $T : X \to X$ is a C^*-algebra valued contractive mapping on X, if there exists an $a \in \mathbb{A}$ with $\|a\| < 1$ such that

$$d(Tu, Tv) \preccurlyeq a^* d(u,v) a, \text{ for all } u, v \in X.$$

Theorem 2.7. [10] Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and T is a contractive mapping, Then there exists a unique fixed point in X.

Definition 2.8. The two mappings T and S on a C^*-algebra valued metric space $(X, \mathbb{A}, d)$ is said to be compatible, if for arbitrary sequence u_n in X, such that $\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Su_n = t \in X$, then $d(TSu_n, STu_n) \to \theta$ as $n \to \infty$.

Theorem 2.9. [7] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space. Suppose that two mapping $T, S : X \to X$ satisfy

$$d(Tu, Sv) \preccurlyeq a^* d(u,v) a,$$

for any $u, v \in X$, where $a \in \mathbb{A}$ with $\|a\| \leq 1$. Then T and S have a unique common fixed point in X.

Lemma 2.10. [7] If $\{b_n\}_{n=1}^{\infty} \subseteq \mathbb{A}$ and $\lim_{n\to\infty} b_n = \theta$, then for any $a \in \mathbb{A}$, $\lim_{n\to\infty} a^* b_n a = \theta$.

Lemma 2.11. [7] Let T and S be weakly compatible mappings of a set X. If T and S have a unique point of coincidence, then it is unique common fixed point of T and S.

3. Main results

Lemma 3.1. [4] Let ψ be a mapping from $\psi : A_+ \to A_+$ with conditions

(i) $\psi(a^*) = (\psi(a))^*$

(ii) $\psi(ab) = \psi(a)\psi(b)$

(iii) $\psi(a+b) = \psi(a) + \psi(b)$

(iv) $\lim_{n\to\infty} \psi^n(a) = \theta$ for all $a \succ \theta$ where $\psi^n(a) = \psi^{n-1} \circ \psi(a)$

(v) $\psi(a) = \theta$ if $a = \theta$.

Theorem 3.2. If $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space and T is a contraction mapping, satisfy the following condition,

$$d(Tu, Tv) \preccurlyeq \psi(a^* \ d(u, v) \ a),$$

then T has unique fixed point in X.

Proof. Choose $u_0 \in X$ and $u_{n+1} = Tu_n = T^{n+1}u_0,\ n = 1, 2, ...$

$$\begin{aligned} d(u_{n+1}, u_n) &= d(Tu_n, Tu_{n-1}) \\ &\preccurlyeq \psi[a^* \ d(u_n, u_{n-1}) \ a] \\ &= \psi(a^*) \ \psi(d(u_n, u_{n-1})) \ \psi(a) \\ &= \psi(a^*) \ \psi(d(u_n, u_{n-1})) \ \psi(a) \\ &\preccurlyeq \psi(a^*) \ \psi(\psi(a^* d(u_{n-1}, u_{n-2})a))\psi(a) \\ &= \psi(a^*) \ \psi(\psi(a^*) \ \psi(d(u_{n-1}, u_{n-2})) \ \psi(a)) \ \psi(a) \\ &= \psi^3(a^*) \ \psi^2(d(u_{n-1}, u_{n-2})) \ \psi^3(a) \\ &\preccurlyeq \psi^3(a^*) \ \psi^2(\psi(a^* d(u_{n-2}, u_{n-3})a)) \ \psi^3(a) \\ &= \psi^6(a^*) \ \psi^3(d(u_{n-2}, u_{n-3})) \ \psi^6(a) \\ &\preccurlyeq \psi^{10}(a^*) \ \psi^4(d(u_{n-3}, u_{n-4})) \ \psi^{10}(a) \\ &\vdots \\ &\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\psi^n(d(u_1, u_0))\psi^{\frac{n(n+1)}{2}}(a) \ . \end{aligned}$$

Let $n, m \in \mathbb{N}$ and $n > m$,

$$\begin{aligned} d(u_{n+1}, u_m) &\preccurlyeq d(u_{n+1}, u_n) + d(u_n, u_m) \\ &\preccurlyeq d(u_{n+1}, u_n) + d(u_{n,n-1}) + ... + d(_{m+1}, u_m) \\ &\preccurlyeq \psi^{\frac{n(n+1)}{2}}\psi^n(d(u_1, u_0))\psi^{\frac{n(n+1)}{2}}(a) + ... + \psi^{\frac{m(m+1)}{2}}\psi^m(d(u_1, u_0))\psi^{\frac{m(m+1)}{2}}(a) \\ &= \sum_{k=m}^{n} \psi^{\frac{k(k+1)}{2}}(a^*)\psi^k(d(x_1, x_0))\psi^{\frac{k(k+1)}{2}}(a) \\ &= \sum_{k=m}^{n} (\psi^{\frac{k(k+1)}{2}}(a))^*\psi^{\frac{k}{2}}(d(x_1, x_0))\psi^{\frac{k}{2}}(d(u_1, u_0))\psi^{\frac{k(k+1)}{2}}(a) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=m}^{n} (\psi^{\frac{k(k+1)}{2}}(a)\psi^{\frac{k}{2}}(d(u_1,u_0)))^*(\psi^{\frac{k}{2}}(d(u_1,u_0))\psi^{\frac{k(k+1)}{2}}(a)) \\
&= \sum_{k=m}^{n} |\psi^{\frac{k}{2}}(d(u_1,u_0))\psi^{\frac{k(k+1)}{2}}(a)|^2 \\
&\preccurlyeq \sum_{k=m}^{n} \|\psi^{\frac{k}{2}}(d(u_1,u_0))\psi^{\frac{k(k+1)}{2}}(a)\|.I \\
&\preccurlyeq \sum_{k=m}^{n} \|\psi^{\frac{k}{2}}(d(u_1,u_0))\|\|\psi^{\frac{k(k+1)}{2}}(a)\|.I \longrightarrow \theta, \quad (\text{as } n \longrightarrow \infty).
\end{aligned}
$$

Then, $\{u_n\}$ is a Cauchy sequence in $\mathbb{A}$.
Since,

$$
\begin{aligned}
d(T^{n+1}u, T^n u) &\preccurlyeq \psi(a^* d(T^n u, T^{n-1}u)a) \\
&\preccurlyeq \psi(a^*)\psi(d(T^n u, T^{n-1}u))\psi(a) \\
&\vdots \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\psi^n(d(Tu,u))\psi^{\frac{n(n+1)}{2}}(a) \to \theta, \quad (\text{as } \ n \to \infty).
\end{aligned}
$$

Hence, $T^{n+1}u = T^n u$, so $Tu = u$. i.e., u is a fixed point of T.

Now, we show the fixed point is unique. Suppose that $vnequ$ is another fixed point of T. Since,

$$
\begin{aligned}
\theta &\preccurlyeq d(u,v) = d(T^n u, T^n v) \\
&\preccurlyeq \psi(a^* d(T^{n-1}u, T^{n-1}v)\ a) \\
&= \psi(a^*)\psi[d(T^{n-1}u, T^{n-1}v)]\psi(a) \\
&\vdots \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\ \psi^n(d(u,v))\ \psi^{\frac{n(n+1)}{2}}(a) \to \theta \quad (\text{as } \ n \to \infty).
\end{aligned}
$$

This implies that $u = v$ and this completes the proof. □

Theorem 3.3. If $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space and T is a contraction mapping, satisfy the following condition:

$$
d(Tu, Tv) \preccurlyeq \psi(a^* \ \frac{d(u,v) + d(v,Tv)}{2}\ a),
$$

then, T has unique fixed point.

Proof. Let $u_0 \in X$ and $u_{n+1} = Tu_n = T^{n+1}u_0$, $n = 1, 2, \ldots$.

$$\begin{aligned}
d(u_{n+1},u_n)=d(Tu_n,Tu_{n-1}) &\preccurlyeq \psi(a^* \ \frac{d(u_n,u_{n-1})+d(Tu_{n-1},u_{n-1})}{2} \ a) \\
&= \psi(a^* \ d(u_n,u_{n-1}) \ a) \\
&= \psi(a^*) \ \psi(d(u_n,u_{n-1}))`\psi(a) \\
&\preccurlyeq \psi(a^*) \ \psi(\psi[a^* \ (\frac{d(u_{n-1},u_{n-2}+d(Tu_{n-2},u_{n-2}))}{2}) \ a] \ \psi(a)) \\
&= \psi(a^*) \ \psi(\psi(a^* \ d(u_{n-1},u_{n-2}) \ a)) \ \psi(a) \\
&= \psi(a^*) \ \psi(\psi(a^*) \ \psi(d(u_{n-1},u_{n-2})) \ \psi(a)) \ \psi(a) \\
&= \psi^3(a^*) \ \psi^2(d(u_{n-1},u_{n-2})) \ \psi^3(a) \\
&\preccurlyeq \psi^3(a^*) \ \psi^2(\psi(a^* \ (\frac{d(u_{n-2},u_{n-3})+d(Tu_{n-2},u_{n-3})}{2})a)) \ \psi^3(a) \\
&= \psi^3(a^*) \ \psi^2(\psi(a^* \ d(u_{n-2},u_{n-3}) \ a)) \ \psi^3(a) \\
&= \psi^6(a^*) \ \psi^3(d(u_{n-2},u_{n-3})) \ \psi^6(a) \\
&\vdots \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \ \psi^n(d(u_1,u_0)) \ \psi^{\frac{n(n+1)}{2}}(a).
\end{aligned}$$

Thus, $\{u_n\}$is a Cauchy sequence. Using the same process in Theorem 3.2, we can prove that $Tu = u$, so u is a fixed point on X.

Now, we show that the fixed point is unique. Suppose that $v \neq u$ is another fixed point of T.

$$\begin{aligned}
d(u,v)=d(T^n u,T^n v) &\preccurlyeq \psi(a^* \ \frac{d(T^{n-1}u,T^{n-1}v)+d(T^n v,T^{n-1}v)}{2} \ a) \\
&= \psi(a^*) \ \psi(\frac{d(T^{n-1}u,T^{n-1}v)}{2}) \ \psi(a) \\
&\preccurlyeq \psi(a^*) \ \psi(\psi(a^* \ \frac{d(T^{n-2}u,T^{n-2}v)}{2} \ a)) \ \psi(a) \\
&= \frac{1}{2} \ \psi(a^*) \ \psi(\psi(a^*) \ \psi(\frac{d(T^{n-2}u,T^{n-2}v)}{2} \ \psi(a)) \ \psi(a) \\
&= \frac{1}{2^2} \ \psi^3(a^*) \ \psi^2(d(T^{n-2}u,T^{n-2}v)) \ \psi^3(a) \\
&\preccurlyeq \frac{1}{2^3} \ \psi^6(a^*) \ \psi^3(d(T^{n-3}u,T^{n-3}v)) \ \psi^6(a) \\
&\vdots \\
&\preccurlyeq \frac{1}{2^n} \ \psi^{\frac{n(n+1)}{2}}(a^*) \ \psi^n(d(u,v)) \ \psi^{\frac{n(n+1)}{2}}(a) \to \theta \quad (\ n \to \infty).
\end{aligned}$$

Implies that $d(u,v) = \theta$, and $u = v$. i.e., the proof is complete. □

Theorem 3.4. If $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space and let $T, S : X \to X$, satisfy the following condition,

$$d(Tu, Sv) \preccurlyeq \psi(a^* \ d(u, v) \ a),$$

then T and S have a unique common fixed point in X.

Proof. Choose $u_0 \in X$ and $u_{2n+1} = Tu_{2n}, \qquad u_{2n+2} = Su_{2n+1}, \qquad n \in \mathbb{N}$, we get

$$\begin{aligned}
d(u_{2n+2}, u_{2n+1}) &= d(Su_{2n+1}, Tu_{2n}) \\
&\preccurlyeq \psi(a^* \ d(u_{2n+1}, u_{2n}) \ a) \\
&= \psi(a^*) \ \psi(d(u_{2n+1}, u_{2n})) \ \psi(a) \\
&= \psi(a^*) \ \psi(d(Tu_{2n}, Su_{2n-1})) \ \psi(a) \\
&\preccurlyeq \psi(a^*) \ \psi(\psi(a^* d(u_{2n}, u_{2n-1}) a)) \ \psi(a) \\
&= \psi^3(a^*) \ \psi^2(d(u_{2n}, u_{2n-1})) \ \psi^3(a) \\
&\preccurlyeq \psi^6(a^*) \ \psi^3(d(u_{2n-1}, u_{2n-2})) \ \psi^6(a) \\
&\vdots \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \ \psi^n(d(u_1, u_0)) \ \psi^{\frac{n(n+1)}{2}}(a).
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(u_{2n+1}, u_{2n}) &= d(Su_{2n}, Tu_{2n-1}) \\
&\preccurlyeq \psi(a^* d(u_{2n}, u_{2n-1}) a) \\
&= \psi(a^*) \ \psi(d(u_{2n}, u_{2n-1})) \ \psi(a) \\
&\preccurlyeq \psi(a^*) \ \psi(\psi(a^* d(u_{2n-1}, u_{2n-2}) a)) \ \psi(a) \\
&= \psi^3(a^*) \ \psi^2(d(u_{2n-1}, u_{2n-2})) \ \psi^3(a) \\
&\preccurlyeq \psi^6(a^*) \ \psi^3(d(u_{2n-2}, u_{2n-3})) \ \psi^6(a) \\
&\vdots \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \ \psi^n(d(u_1, u_0)) \ \psi^{\frac{n(n+1)}{2}}(a).
\end{aligned}$$

Now, we get for any $n \in \mathbb{N}$

$$d(u_{n+1}, u_n) \preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \psi^n(d(u_1, u_0)) \psi^{\frac{n(n+1)}{2}}(a).$$

Suppose that $n, m \in \mathbb{N}$ and $n > m$. Following the proof of Theorem 3.2, we obtain $\{u_n\}$ is a

Cauchy sequence in $\mathbb{A}$.

$$\begin{aligned} d(u,Su) &\preccurlyeq d(u,u_{2n+1}) + d(u_{2n+1},Su) \\ &= d(u,u_{2n+1}) + d(Tu_{2n},Su) \\ &\preccurlyeq d(u,u_{2n+1}) + \psi(a^* d(u_{2n},u)a) \to \theta, \quad (n \to \infty). \end{aligned}$$

This implies that,

$$d(u,Su) = \theta.$$

Consequently,

$$u = Su.$$

Also,

$$d(Tu,u) \;=\; d(Tu,Su) \;\preccurlyeq\; \psi(a^* d(u,u)a) \;=\; \theta,$$

which is a contradiction. This means that $d(Tu,v) = \theta \quad \Rightarrow \quad Tu = u.$

To prove the uniqueness: let v ($\neq u$) be another fixed point

$$d(u,v) = d(Tu,Sv) \preccurlyeq \psi(a^* d(u,v)a)$$

$$= \psi(a^*)\psi(d(u,v))\psi(a)$$

$$\theta \preccurlyeq \|d(u,v)\| \preccurlyeq \psi(a)\|\psi(d(u,v))\|$$

$$\theta \preccurlyeq (1-\psi)\|d(u,v)\| \preccurlyeq \theta$$

$$\|d(u,v)\| = \theta$$

$$u = v.$$

This implies that the proof is complete. □

Theorem 3.5. Let $(X,\mathbb{A},d)$ be a complete C^*-algebra valued metric space and $T,S : X \to X$ satisfy the following condition,

$$d(Tu,Tv) \preccurlyeq \psi(a^* \; d(Su,Sv) \; a).$$

If $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in X, then $T$and $S$are have a unique point of coincidence in X. If T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof. Choose $u_1 \in X$ with $Su_1 = Tu_0$ and $u_2 \in X$ with $Su_2 = Tu_1$, that is $Su_n = Tu_{n-1}$ for all $n \in \mathbb{N}$. We get

$$\begin{aligned} d(Su_{n+1}, Su_n) &= d(Tu_n, Tu_{n-1}) \\ &\preccurlyeq \psi(a^* \ d(Su_n, Su_{n-1}) \ a) \\ &= \psi(a^*) \ \psi(d(Su_n, Su_{n-1})) \ \psi(a) \\ &\preccurlyeq \psi^2(a^*) \ \psi^2(d(Su_{n-1}, Su_{n-2})) \ \psi^2(a) \\ &\vdots \\ &\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \ \psi^n(d(u_1, u_0)) \ \psi^{\frac{n(n+1)}{2}}(a). \end{aligned}$$

Now, we show that $\{Su_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $R(S)$, when $R(S)$ is complete in X, there exists $w \in X$ such that $\lim_{n\to\infty} Su_n = Sw$,

$$d(Su_n, Tw) = d(Tu_n, Tw) \preccurlyeq \psi(a^* \ d(Su_{n-1}, Sw) \ a) \to \theta \quad (n \to \infty).$$

Using Lemma 2.10., we get $\lim_{n\to\infty} Su_n = Tw$, then $Tw = Sw$.

Suppose that $r \in X$ such that $Tr = Sr$,

$$d(Sw, Sr) = d(Tw, Tr) \preccurlyeq \psi(a^* \ d(Sw, Sr) \ a) \to \theta \quad (n \to \infty).$$

We obtain $Sw = Sr$, T and S have a unique point of coincidence in X. It follow from Lemma 2.11. that $T$and S have a unique common fixed point in X.

□

4. Application

We give an application of contraction mapping theorem on C^*-algebra valued metric spaces to support Thoerem 3.2.

Assume that H is a Hibrrt space, $L(H)$ is the set of linear operators on H and $U \in L(H)$

$$U - \sum_{n=1}^{\infty} \psi(A^*UA) = Q,$$

where Q is a positive, then the equation has a unique solution in $L(H)$.

Proof. Assume that $\mu = \sum_{n=1}^{\infty} \|a_n\|^2$, $\mu > 0$.

Let Q be a positive operator, $\psi(a) = \frac{a}{2}$ and $K \in L(H)$, for $U, V \in L(H)$, set

$$d(U, V) = \|U - V\|K.$$

It is clear that $d(U,V)$ is a C^*-algebra valued metric and $(L(H),d)$ is a complete, since $L(H)$ is a Banach space.

Take into account the mapping $T : L(H) \to L(H)$ defined by

$$T(X) = \sum_{n=1}^{\infty} \psi(a_n^* U a_n) + Q.$$

Then

$$\begin{aligned} d(T(U),T(V)) &= \|T(U) - T(V)\|K \\ &= \|\sum_{n=1}^{\infty} \psi(a_n^*\ U\ a_n) - \sum_{n=1}^{\infty} \psi(a_n^*\ V\ a_n)\|K \\ &= \|\sum_{n=1}^{\infty} \frac{a_n^*\ U\ a_n}{2} - \sum_{n=1}^{\infty} \frac{a_n^*\ V\ a_n}{2}\|K \\ &\preccurlyeq \sum_{n=1}^{\infty} \frac{1}{2}\|a_n\|^2\|U - V\|K \\ &= \frac{1}{2}\ \mu\ d(U,V) \\ &= \psi((\mu^{\frac{1}{2}}\ I)^*\ d(U,V)\ (\mu^{\frac{1}{2}}\ I)). \end{aligned}$$

Using Theorem 3.2, there exists a unique fixed point of T. □

5. Conclusion

In this article, we applied some fixed point theorems using ψ-contraction mapping on C^*-algebra valued metric spaces. The theorems have been obtained represent a generalization and extension to many theorems in the scientific literature. In one of our main results, we showed the exsitence and uniquness to the solution of system linear operator equation. In the future, our results can be applied in other spaces such as C^*-algebra valued b-metric space (C^*-algebra Vb-MSs), C^*-algebra bipolar metric spaces (C^*-algebra bi-MSs) and C^*-algebra partial valued metric spaces (C^*-algebra pV-MSs). Also, we can apply $(\alpha - \psi)$-contraction mapping on this article and on many spaces as a future work.

6. Declarations

Conflict of interest There are no conflict of interests

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