Original Research Article

Some fixed point techniques using ψ-contraction mapping on the C*-algebra valued metric space

Abstract

In this article, we introduced some fixed point theorems using ψ-contraction mapping on C^*-algebra valued metric space. In particular, we proved Banach fixed point theorem and some extensions and generalizations of Banach fixed point theorem in C^*-algebras valued metric spaces. Moreover, in order to illustrate the current results, some basic examples are presented and we give an application, existence and uniqueness system linear operator equation.

Keywords: Fixed point theorems; C^*-algebra valued metric space; ψ-contraction mapping.

Mathematics Subject Classification: 47H10, 46L07, 54H25.

1. Introduction

Banach contraction principle (BCP) is one of the most important result in fixed point theory and approximation theory. It plays a central role in solving many important problems in pure and applied mathematics and physics. There is a vast literature dealing with technical extensions and generalizations of BCP. Besides, this classical theorem gives an iteration process through which we can obtain better approximation to the fixed point. It becomes a key role in solving systems of linear algebraic equations involving iteration process. Iteration procedures nearly are applying in every branch of applied mathematics, convergence proof and also in estimating the process of errors, very often by an application of Banach's fixed point theorem [1,8]. In 1966, Jungck initially investgated common fixed points for commuting mappings in metric spaces [6]. In this paper, we introduce some fixed point theorems by using ψ-contraction mapping on C^*-algebra valued metric space.

Suppose that $\mathbb{A}$ is a unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate linear map $x \to x^*$ on $\mathbb{A}$ such that $x^{**} = x$ and $(xy)^* = y^*x^*$ for all $x, y \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a *-algebra. A Banach *-algebra is a *-algebra $\mathbb{A}$ together with a complete submultiplicative norm, $\|xy\| \leq \|x\|\|y\|$ such that $\|x^*\| = \|x\|$. C^*-algebra is a *-Banach algebra such that $\|x^*x\| = \|x\|^2$ (see [2]), when $\mathbb{A}$ is a unital C^*-algebra, then for any $x \in \mathbb{A}_+$ we have $x \preceq I \Leftrightarrow \|x\| \leq 1$ (see [9]).

2. Preliminaries

Definition 2.1. Let $\mathbb{A}$ be a C^*-algebra and then $x \in \mathbb{A}$ is called positive element if $x = x^*$ and $\sigma(x) \subseteq \mathbb{R}^+$ this can be written as $x \succcurlyeq \theta$. The set of all positive elements we denote by $\mathbb{A}^+$. That is, $\mathbb{A}^+ = \{x \in \mathbb{A} : x \succcurlyeq \theta\}$. When θ is zero a element in $\mathbb{A}$ and $\preccurlyeq$ is a partially order relation on $\mathbb{A}$ such that $x \succcurlyeq y$ *or* $x - y \succcurlyeq \theta$.

Definition 2.2. [9] Let X be a nonempty set. Suppose that mapping $d : X \times X \to \mathbb{A}$ satisfies:

(i) $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space .

Definition 2.3. Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and $\{x_n\} \subseteq X$ is a sequence in X. If $x \in X$ and $\epsilon \succ \theta$ there is N such that for all $n > N, d(x_n, x) \prec \epsilon$, then $\{x_n\}$ is called a convergent sequence in X to x and write $\lim_{n\to\infty} x_n = x$. Moreover, for any $\epsilon \succ \theta$, there is N such that for all $n, m > N$, $d(x_n, x_m) \prec \epsilon$ then x_n is called a Cauchy sequence in X.

Definition 2.4. If every Cauchy sequence is converges, then $(X, \mathbb{A}, d)$ is called a completed C^*-algebra valued metric space.

Example 2.5. Let $X = \mathbb{R}$ and $A = M_2(\mathbb{R})$. Define $d(x, y) = diag(|x - y|, k|x - y|)$, where $x, y \in \mathbb{R}$ and $k \geq 0$ is constant. then, d is a C^*-algebra valued metric space and $(X, M_2(\mathbb{R}), d)$ is a complete C^*-algebra valued metric space by the completeness of $\mathbb{R}$.

Definition 2.6. [9] Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. We call a mapping $T : X \to X$ is a C^*-algebra valued contractive mapping on X, if there exists an $a \in \mathbb{A}$ with $\|a\| < 1$ such that

$$d(Tx, Ty) \preccurlyeq a^* d(x, y) a, \text{ for all } x, y \in X.$$

Theorem 2.7. [9] *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and T is a contractive mapping, then there exists a unique fixed point in X.*

Definition 2.8. The two mappings T and S on a C^*-algebra valued metric space $(X, \mathbb{A}, d)$ is said to be compatible, if for arbitrary sequence x_n in X, such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t \in X$, then $d(TSx_n, STx_n) \to \theta$ as $n \to \infty$.

Theorem 2.9. [6] *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space. Suppose that two mapping $T, S : X \to X$ satisfy*

$$d(Tx, Sy) \preccurlyeq a^* d(x, y)a,$$

for any $x, y \in X$, where $a \in \mathbb{A}$ with $\|a\| \leq 1$. Then T and S have a unique common fixed point in X.

Lemma 2.10. [6] *If $\{b_n\}_{n=1}^{\infty} \subseteq \mathbb{A}$ and $\lim_{n\to\infty} b_n = \theta$, then for any $a \in \mathbb{A}$, $\lim_{n\to\infty} a^* b_n a = \theta$.*

Lemma 2.11. [6] *Let T and S be weakly compatible mappings of a set X. If T and S have a unique point of coincidence, then it is unique common fixed point of T and S.*

3. Main results

Lemma 3.1. [3] *let ψ is mapping from $\psi : A_+ \to A_+$ with conditions*

(i) $\psi(a^*) = (\psi(a))^*$;

(ii) $\psi(ab) = \psi(a)\psi(b)$;

(iii) $\psi(a + b) = \psi(a) + \psi(b)$;

(iv) $\lim_{n\to\infty} \psi^n(a) = \theta$ *for all* $a \succ \theta$ *where* $\psi^n(a) = \psi^{n-1} \circ \psi(a)$;

(v) $\psi(a) = \theta$ *if* $a = \theta$.

Theorem 3.2. If $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space and T is a contraction mapping, satisfy the following condition,

$$d(Tx, Ty) \preccurlyeq \psi(a^* \, d(x, y) \, a),$$

then T has unique fixed point in X.

Proof. Choose $x_0 \in X$ and $x_{n+1} = Tx_n = T^{n+1}x_0$ $,n = 1, 2, ...$

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\
&\preccurlyeq \psi[a^* \ d(x_n, x_{n-1}) \ a] \\
&= \psi(a^*) \ \psi(d(x_n, x_{n-1})) \ \psi(a) \\
&= \psi(a^*) \ \psi(d(x_n, x_{n-1})) \ \psi(a) \\
&\preccurlyeq \psi(a^*) \ \psi(\psi(a^* d(x_{n-1}, x_{n-2})a))\psi(a) \\
&= \psi(a^*) \ \psi(\psi(a^*) \ \psi(d(x_{n-1}, x_{n-2})) \ \psi(a)) \ \psi(a) \\
&= \psi^3(a^*) \ \psi^2(d(x_{n-1}, x_{n-2})) \ \psi^3(a)
\end{aligned}$$

$$\begin{aligned}
&\preccurlyeq \psi^3(a^*) \ \psi^2(\psi(a^* d(x_{n-2}, x_{n-3})a)) \ \psi^3(a) \\
&= \psi^6(a^*) \ \psi^3(d(x_{n-2}, x_{n-3})) \ \psi^6(a) \\
&\preccurlyeq \psi^{10}(a^*) \ \psi^4(d(x_{n-3}, x_{n-4})) \ \psi^{10}(a) \\
&\vdots \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\psi^n(d(x_1, x_0))\psi^{\frac{n(n+1)}{2}}(a) \ .
\end{aligned}$$

Let $n, m \in N$ and $n > m$,

$$\begin{aligned}
d(x_{n+1}, x_m) &\preccurlyeq d(x_{n+1}, x_n) + d(x_n, x_m) \\
&\preccurlyeq d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + ... + d(x_{m+1}, x_m) \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}\psi^n(d(x_1, x_0))\psi^{\frac{n(n+1)}{2}}(a) + ... + \psi^{\frac{m(m+1)}{2}}\psi^m(d(x_1, x_0))\psi^{\frac{m(m+1)}{2}}(a) \\
&= \sum_{k=m}^{n} \psi^{\frac{k(k+1)}{2}}(a^*)\psi^k(d(x_1, x_0))\psi^{\frac{k(k+1)}{2}}(a) \\
&= \sum_{k=m}^{n} (\psi^{\frac{k(k+1)}{2}}(a))^*\psi^{\frac{k}{2}}(d(x_1, x_0))\psi^{\frac{k}{2}}(d(x_1, x_0))\psi^{\frac{k(k+1)}{2}}(a) \\
&= \sum_{k=m}^{n} (\psi^{\frac{k(k+1)}{2}}(a)\psi^{\frac{k}{2}}(d(x_1, x_0)))^*(\psi^{\frac{k}{2}}(d(x_1, x_0))\psi^{\frac{k(k+1)}{2}}(a)) \\
&= \sum_{k=m}^{n} |\psi^{\frac{k}{2}}(d(x_1, x_0))\psi^{\frac{k(k+1)}{2}}(a)|^2 \\
&\preccurlyeq \sum_{k=m}^{n} \|\psi^{\frac{k}{2}}(d(x_1, x_0))\psi^{\frac{k(k+1)}{2}}(a)\|.I \\
&\preccurlyeq \sum_{k=m}^{n} \|\psi^{\frac{k}{2}}(d(x_1, x_0))\|\|\psi^{\frac{k(k+1)}{2}}(a)\|.I \longrightarrow \theta \ as \ n \longrightarrow \infty
\end{aligned}$$

then $\{x_n\}$ is a Cauchy sequence in $\mathbb{A}$.

Since

$$\begin{aligned} d(T^{n+1}x, T^n x) &\preccurlyeq \psi(a^* d(T^n x, T^{n-1}x)a) \\ &\preccurlyeq \psi(a^*)\psi(d(T^n x, T^{n-1}x))\psi(a) \\ &\vdots \\ &\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\psi^n(d(Tx, x))\psi^{\frac{n(n+1)}{2}}(a) \to \theta \text{ as } n \to \infty \end{aligned}$$

hence , $T^{n+1}x = T^n x$,so $Tx = x$ i.e x is a fixed point of T.
Now we show the fixed point is unique suppose that $y \neq x$ is another fixed point of T, since

$$\begin{aligned} \theta &\preccurlyeq d(x, y) = d(T^n x, T^n y) \\ &\preccurlyeq \psi(a^* d(T^{n-1}x, T^{n-1}y)\ a) \\ &= \psi(a^*)\psi[d(T^{n-1}x, T^{n-1}y)]\psi(a) \\ &\vdots \\ &\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\ \psi^n(d(x, y))\ \psi^{\frac{n(n+1)}{2}}(a) \to \theta \text{ as } n \to \infty \end{aligned}$$

□

Theorem 3.3. *If $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space and T is a contraction mapping, satisfy the following condition:*

$$d(Tx, Ty) \preccurlyeq \psi(a^* \ \frac{d(x, y) + d(y, Ty)}{2}\ a),$$

then, T has unique fixed point.

Proof. Let $x_0 \in X$ and $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 1, 2, \dots$.

$$\begin{aligned}
d(x_{n+1},x_n)=d(Tx_n,Tx_{n-1}) &\preccurlyeq \psi(a^* \ \frac{d(x_n,x_{n-1})+d(Tx_{n-1},x_{n-1})}{2}\ a)\\
&= \psi(a^* \ d(x_n,x_{n-1})\ a)\\
&= \psi(a^*)\ \psi(d(x_n,x_{n-1}))`\psi(a)\\
&\preccurlyeq \psi(a^*)\ \psi(\psi[a^* \ (\frac{d(x_{n-1},x_{n-2}+d(Tx_{n-2},x_{n-2}))}{2})\ a]\ \psi(a))\\
&= \psi(a^*)\ \psi(\psi(a^* \ d(x_{n-1},x_{n-2})\ a))\ \psi(a)\\
&= \psi(a^*)\ \psi(\psi(a^*)\ \psi(d(x_{n-1},x_{n-2}))\ \psi(a))\ \psi(a)\\
&= \psi^3(a^*)\ \psi^2(d(x_{n-1},x_{n-2}))\ \psi^3(a)\\
&\preccurlyeq \psi^3(a^*)\ \psi^2(\psi(a^* \ (\frac{d(x_{n-2},x_{n-3})+d(Tx_{n-2},x_{n-3})}{2})a))\ \psi^3(a)\\
&= \psi^3(a^*)\ \psi^2(\psi(a^* \ d(x_{n-2},x_{n-3})\ a))\ \psi^3(a)\\
&= \psi^6(a^*)\ \psi^3(d(x_{n-2},x_{n-3}))\ \psi^6(a)\\
&\vdots\\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\ \psi^n(d(x_1,x_0))\ \psi^{\frac{n(n+1)}{2}}(a).
\end{aligned}$$

We show $\{x_n\}$is a Cauchy sequence.
We follow the same process in theorem 1 to prove that $Tx = x$, so x is a fixed point on X. Now, we show that the fixed point is unique, suppose that $y \neq x$ is another fixed point of T.

$$\begin{aligned}
d(x,y)=d(T^n x,T^n y) &\preccurlyeq \psi(a^* \ \frac{d(T^{n-1}x,T^{n-1}y+d(T^n y,T^{n-1}y))}{2}\ a)\\
&= \psi(a^*)\ \psi(\frac{d(T^{n-1}x,T^{n-1}y)}{2})\ \psi(a)\\
&\preccurlyeq \psi(a^*)\ \psi(\psi(a^* \ \frac{d(T^{n-2}x,T^{n-2}y)}{2}\ a))\ \psi(a)\\
&= \frac{1}{2}\ \psi(a^*)\ \psi(\psi(a^*)\ \psi(\frac{d(T^{n-2}x,T^{n-2}y)}{2}\ \psi(a))\ \psi(a)\\
&= \frac{1}{2^2}\ \psi^3(a^*)\ \psi^2(d(T^{n-2}x,T^{n-2}y))\ \psi^3(a)\\
&\preccurlyeq \frac{1}{2^3}\ \psi^6(a^*)\ \psi^3(d(T^{n-3}x,T^{n-3}y))\ \psi^6(a)\\
&\vdots\\
&\preccurlyeq \frac{1}{2^n}\ \psi^{\frac{n(n+1)}{2}}(a^*)\ \psi^n(d(x,y))\ \psi^{\frac{n(n+1)}{2}}(a) \to \theta \text{ as } n \to \infty
\end{aligned}$$

so that $d(x,y) = \theta$, and $x = y$. □

Corollary 3.4. *If $\psi = I$, we get theorem 1.8*

Theorem 3.5. *If $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space and let $T, S : X \to X$, satisfy the following condition,*

$$d(Tx, Sy) \preccurlyeq \psi(a^* \ d(x,y) \ a),$$

theh T and S have a unique common fixed point in X.

Proof. Choose $x_0 \in X$ and $x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Sx_{2n+1}, \quad n \in \mathbb{N}$, we get

$$\begin{aligned}
d(x_{2n+2}, x_{2n+1}) &= d(Sx_{2n+1}, Tx_{2n}) \\
&\preccurlyeq \psi(a^* \ d(x_{2n+1}, x_{2n}) \ a) \\
&= \psi(a^*) \ \psi(d(x_{2n+1}, x_{2n})) \ \psi(a) \\
&= \psi(a^*) \ \psi(d(Tx_{2n}, Sx_{2n-1})) \ \psi(a) \\
&\preccurlyeq \psi(a^*) \ \psi(\psi(a^* d(x_{2n}, x_{2n-1}) a)) \ \psi(a) \\
&= \psi^3(a^*) \ \psi^2(d(x_{2n}, x_{2n-1})) \ \psi^3(a) \\
&\preccurlyeq \psi^6(a^*) \ \psi^3(d(x_{2n-1}, x_{2n-2})) \ \psi^6(a) \\
&\vdots \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \ \psi^n(d(x_1, x_0)) \ \psi^{\frac{n(n+1)}{2}}(a)
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(x_{2n+1}, x_{2n}) &= d(Sx_{2n}, Tx_{2n-1}) \\
&\preccurlyeq \psi(a^* d(x_{2n}, x_{2n-1}) a) \\
&= \psi(a^*) \ \psi(d(x_{2n}, x_{2n-1})) \ \psi(a) \\
&\preccurlyeq \psi(a^*) \ \psi(\psi(a^* d(x_{2n-1}, x_{2n-2}) a)) \ \psi(a) \\
&= \psi^3(a^*) \ \psi^2(d(x_{2n-1}, x_{2n-2})) \ \psi^3(a) \\
&\preccurlyeq \psi^6(a^*) \ \psi^3(d(x_{2n-2}, x_{2n-3})) \ \psi^6(a) \\
&\vdots \\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \ \psi^n(d(x_1, x_0)) \ \psi^{\frac{n(n+1)}{2}}(a)
\end{aligned}$$

Now ,we get for any $n \in \mathbb{N}$

$$d(x_{n+1}, x_n) \preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \psi^n(d(x_1, x_0)) \psi^{\frac{n(n+1)}{2}}(a).$$

Suppose that $n, m \in \mathbb{N}$ and $n > m$, follow the proof of theorem 2.2.
We obtain
$\{x_n\}$ is a Cauchy sequence in $\mathbb{A}$

$$\begin{aligned} d(x, Sx) &\preccurlyeq d(x, x_{2n+1}) + d(x_{2n+1}, Sx) \\ &= d(x, x_{2n+1}) + d(Tx_{2n}, Sx) \\ &\preccurlyeq d(x, x_{2n+1}) + \psi(a^* d(x_{2n}, x)a) \to \theta \text{ as } n \to \infty. \end{aligned}$$

$$d(x, Sx) = \theta.$$

Then,

$$x = Sx.$$

$$d(Tx, x) = d(Tx, Sx) \leq \psi(a^* d(x, x)a) = \theta$$

$$Tx = x$$

Let y is another fixed point

$$d(x, y) = d(Tx, Sy) \preccurlyeq \psi(a^* d(x, y)a)$$

$$= \psi(a^*)\psi(d(x, y))\psi(a)$$

$$\theta \preccurlyeq \|d(x, y)\| \preccurlyeq \psi(a)\|\psi(d(x, y))\|$$

$$\theta \preccurlyeq (1 - \psi)\|d(x, y)\| \preccurlyeq \theta$$

$$\|d(x, y)\| = \theta$$

$$x = y.$$

□

Corollary 3.6. *If $\psi = I$, we get theorem 1.10*

Theorem 3.7. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and let $T, S : X \to X$ satisfy the following condition,*

$$d(Tx, Ty) \preccurlyeq \psi(a^* \ d(Sx, Sy) \ a),$$

if $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in X, then $T$and $S$are have a unique point of coincidence in$X$. if T and $S$are weakly compatible, then $T$and S have a unique common fixed point in X.

Proof. Choose $x_1 \in X$ where $Sx_1 = Tx_0$ and $x_2 \in X$ where $Sx_2 = Tx_1$, i.e $Sx_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. We get

$$\begin{aligned} d(Sx_{n+1}, Sx_n) &= d(Tx_n, Tx_{n-1}) \\ &\preccurlyeq \psi(a^* \ d(Sx_n, Sx_{n-1}) \ a) \\ &= \psi(a^*) \ \psi(d(Sx_n, Sx_{n-1})) \ \psi(a) \\ &\preccurlyeq \psi^2(a^*) \ \psi^2(d(Sx_{n-1}, Sx_{n-2})) \ \psi^2(a) \\ &\vdots \\ &\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*) \ \psi^n(d(x_1, x_0)) \ \psi^{\frac{n(n+1)}{2}}(a). \end{aligned}$$

Now, we show that $\{Sx_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $R(S)$,when $R(S)$ is complete in X, there exists $z \in X$ such that $\lim_{n\to\infty} Sx_n = Sz$,

$$d(Sx_n, Tz) = d(Tx_n, Tz) \preccurlyeq \psi(a^* \ d(Sx_{n-1}, Sz) \ a) \to \theta \ as \ n \to \infty.$$

By using lemma 2.10., we get $\lim_{n\to\infty} Sx_n = Tz$, then $Tz = Sz$.
Suppose that r is another point in X such that $Tr = Sr$,

$$d(Sz, Sr) = d(Tz, Tr) \preccurlyeq \psi(a^* \ d(Sz, Sr) \ a) \to \theta \ as \ n \to \theta.$$

We obtain $Sz = Sr$, T and S have a unique point of coincidence in X. It follow from lemma 2.11. that $T$and S have a unique common fixed point in X.

□

4. Application

We give some applications of contractive mapping theorem on C^*-algebra valued metric spaces.
Assume that H is a Hibrrt space, $L(H)$ is the set of linear operators on H and $X \in L(H)$

$$X - \sum_{n=1}^{\infty} \psi(A^* X A) = Q,$$

where Q is a positive, then the equation has a unique solution in $L(H)$.

Proof. Assume $\mu = \sum_{n=1}^{\infty} \|a_n\|^2$, $\mu > 0$.
let Q be a positive operator, $\psi(a) = \frac{a}{2}$ and $K \in L(H)$, for $X, Y \in L(H)$, set

$$d(X, Y) = \|X - Y\|$$

It is clearly that $d(X, Y)$ is a C^*-algebra valued metric and $(L(H), d)$ is a complete since

$L(H)$ is a Banach space.
Take into account the mapping $T : L(H) \to L(H)$ defined by

$$T(X) = \sum_{n=1}^{\infty} \psi(a_n^* X a_n) + Q.$$

Then

$$\begin{aligned}
d(T(X), T(Y)) &= \|T(X) - T(Y)\|K \\
&= \|\sum_{n=1}^{\infty} \psi(a_n^*\ X\ a_n) - \sum_{n=1}^{\infty} \psi(a_n^*\ Y\ a_n)\|K \\
&= \|\sum_{n=1}^{\infty} \frac{a_n^*\ X\ a_n}{2} - \sum_{n=1}^{\infty} \frac{a_n^*\ Y\ a_n}{2}\|K \\
&\preccurlyeq \sum_{n=1}^{\infty} \frac{1}{2} \|a_n\|^2 \|X - Y\|K \\
&= \frac{1}{2}\ \mu\ d(X, Y) \\
&= \psi((\mu^{\frac{1}{2}}\ I)^*\ d(X, Y)\ (\mu^{\frac{1}{2}}\ I)).
\end{aligned}$$

Using Theorem 2.2, there exists a unique fixed point of T. □

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